QMI-cv6 Lineární harmonický oscilátor

$\langle 0|1\rangle = 0$
$\langle 0|0\rangle = 1$

ÚLOHA 1

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \ldots \quad \underline{|\alpha|^2+|\beta|^2=1} = \langle\psi|\psi\rangle$

1) α, β zvolit tak, aby $\langle \hat{x}\rangle$ co největší

OPAKOVÁNÍ: $\quad a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a|0\rangle = 0$

$\langle 0|a^+ = 0$

$$\boxed{\hat{x} = \frac{x_0}{\sqrt{2}}(a+a^+)} \qquad \hat{p} = \frac{p_0}{i\sqrt{2}}(\hat{a}-\hat{a}^+)$$

$$\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle = \left(\alpha^*\langle 0| + \beta^*\langle 1|\right)\frac{x_0}{\sqrt{2}}(\hat{a}+\hat{a}^+)\left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$= \frac{x_0}{\sqrt{2}}\left(\alpha^*\langle 0| + \beta^*\langle 1|\right)\left(\beta|0\rangle + \alpha|1\rangle + \cancel{\beta\sqrt{2}|2\rangle}\right) = \frac{x_0}{\sqrt{2}}\left(\alpha^*\beta + \beta^*\alpha\right)$$

$|\alpha|^2+|\beta|^2 = 1 \quad \ldots \quad$ BÚNO $\boxed{\alpha > 0}$

$\alpha = |\alpha|\cancel{e^{i\varphi}} \qquad e^{-i\varphi}$

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$\alpha = \cos\phi \quad = \frac{\sqrt{2}}{2}$

$\beta = \sin\phi\, e^{i\delta} = \frac{\sqrt{2}}{2} \qquad \langle x\rangle = \frac{x_0}{\sqrt{2}}\sin\phi\cos\phi\left(e^{i\delta}+e^{-i\delta}\right) \to 2\cos\delta$

$\sin\phi\cos\phi \to \frac{1}{2}\sin 2\phi \quad - \quad \phi = \frac{\pi}{4} \qquad 2\phi = \frac{\pi}{2} \qquad \delta = 0$

$$\alpha = \beta = \frac{1}{\sqrt{2}} \qquad \left|\, |\psi'\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right.$$

2) $\langle x(t)\rangle = \langle\psi(t)|\hat{x}|\psi(t)\rangle$

možnosti: $\bullet\ |\psi(t)\rangle = \hat{U}(t)|\psi\rangle \qquad \hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t} = \sum_n e^{-\frac{i}{\hbar}E_n t} P_n$

Taylor $\sum_n \frac{1}{n!}\left(-\frac{i}{\hbar}\hat{H}t\right)^n \qquad E_n$

$\bullet\ \langle\psi|\hat{U}^+(t)\hat{x}\,\hat{U}(t)|\psi\rangle$

$\bullet\ |\psi\rangle = \sum_n \psi_n|n\rangle \longrightarrow |\psi(t)\rangle = \sum_n \psi_n e^{-\frac{i}{\hbar}E_n t}|n\rangle$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad E_0 = \frac{\hbar\omega}{2} \qquad E_1 = \frac{3}{2}\hbar\omega \qquad -i\frac{\omega t}{2} \qquad -i\frac{3\omega t}{2}$

$$|\psi(t)\rangle = e^{-i\frac{\omega}{2}t}\left(\alpha|0\rangle + e^{-i\omega t}\beta|1\rangle\right) \qquad \beta \to \beta e^{-i\omega t}$$

$$\langle x\rangle = \frac{x_0}{\sqrt2}\left(\alpha^*\beta + \alpha\beta^*\right) = \frac{x_0}{\sqrt2}\,|\alpha||\beta|\left(e^{-i\omega t} + e^{+i\omega t}\right)$$

$$|\alpha| = \frac{\alpha}{\sqrt2},\quad \cancel{|\beta| = \frac{\alpha}{\sqrt2}} \qquad \cancel{2}\cos\omega t$$

$$\langle x(t)\rangle = \frac{x_0}{\sqrt2}\cos\omega t$$

3) $$\langle n|\hat x|n\rangle = \langle n|\frac{\hat a + \hat a^\dagger}{\sqrt2}x_0|n\rangle = \frac{x_0}{\sqrt2}\langle n|(a+a^\dagger)|n\rangle$$

$$= \frac{x_0}{\sqrt2}\langle n|\left(\sqrt n\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right) = 0$$

4) $$\Delta x^2 = \langle n|(x-\langle x\rangle)^2|n\rangle = \langle n|x^2|n\rangle - \underbrace{\langle x\rangle^2}_{0} = \langle n|x^2|n\rangle$$

$$\frac{x_0^2}{2}\langle n|(\hat a + \hat a^\dagger)^2|n\rangle = \frac{x_0^2}{2}\langle n|\left(\cancel{\hat a^2} + \hat a a^\dagger + a^\dagger a + \cancel{a^{\dagger 2}}\right)|n\rangle$$

$$\hat a a^\dagger \to a^\dagger a + \underbrace{[a,a^\dagger]}_{1}$$

$$\hat N = a^\dagger a \qquad \hat N|n\rangle = n|n\rangle$$

$$= \frac{x_0^2}{2}\langle n|(2\hat N + 1)|n\rangle = \frac{x_0^2}{2}(2n+1) \qquad E_n = \frac{\hbar\omega}{2}(2n+1)$$

$$\Delta x = x_0\sqrt{n+\tfrac12}$$

$\boxed{\text{ÚLOHA 2}}$

$$e^{2xt-t^2} = 1 + \frac{1}{1!}(2xt - t^2) + \frac12(2tx - t^2)^2 + \frac16(\quad)^3 \ldots$$

$$\sum_n \frac{H_n}{n!}t^n$$

t^0 -- $H_0 = 1$

t^1 $H_1 = 2x$

t^2 $H_2 = \left(-1 + \frac12(2x)^2\right)2$

$$H_2 = 4x^2 - 2$$

$H_{-1} = 0$

rekurentní relace $H_{n+1} = 2x H_n - 2n H_{n-1}$ $\qquad H_0 = 1$

$H_1 = 2x\cdot 1 \qquad H_2 = 2x H_1 - 2H_0 = (2x)^2 - 2$

3. $\int e^{-x^2} H_n(x) H_m(x)\,dx \overset{?}{=}$ $S(t,x) = e^{-t^2+2tx} = \sum_n \frac{H_n}{n!} t^n$

$$I(t,u) = \int_{-\infty}^{\infty} e^{-x^2} S(t,x)\, S(u,x)\, dx$$

$$\int_{-\infty}^{\infty} e^{-x^2} e^{-t^2+2xt} e^{-u^2+2xu}\, dx =$$

$$= e^{-t^2-u^2} e^{(t+u)^2} \underbrace{\int e^{(-x^2+2x(t+u)-(t+u)^2)} dx}_{(x-t-u)^2 \ \sqrt{\pi}} = \sqrt{\pi}\, e^{2tu}$$

$$I(t,u) = \sqrt{\pi} e^{2tu} = \sum_{n\,m} \frac{t^n}{n!} \frac{u^m}{m!} \int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx$$

Idea – taylorův rozvoj ve dvou proměnných porovnat člen po členu

členy u $t^n u^m$

$$\boxed{\frac{1}{n!}\frac{1}{m!} \int H_n(x) H_m(x) e^{-x^2} dx}$$

$$\Rightarrow \sqrt{\pi} \sum_n \frac{1}{n!} (2tu)^n = \sqrt{\pi} \sum_n \sum_m \frac{1}{n!} \delta_{nm} 2^n t^n t^m$$

$n \neq m$ 0

$$\delta_{nm} \sqrt{\pi} \frac{1}{n!} 2^n = \frac{1}{n!}\frac{1}{m!} \int H_n H_m e^{-x^2}$$

$$\Rightarrow \int H_n H_m e^{-x^2} dx = \boxed{\delta_{nm} \sqrt{\pi}\, 2^n n!}$$

$\langle n | m \rangle$ -- ortonorm

$$\varphi_n = N H_n e^{-x^2/2}$$

$\langle n | x^k | m \rangle$

$\langle n | p | m \rangle$

$\quad = -i\hbar \frac{d}{dx}$