$\boxed{\text{QMI-cv9}}$ – Výpočty komutátorů

ÚLOHA 1: najít vzorečky pro různé výrazy z komutátory

- $[A+B,C] = (A+B)C - C(A+B) = [A,C] + [B,C]$
- $[A,B+C] = [A,B] + [A,C]$
- $[\hat{A},\hat{B}\hat{C}] = ABC - BAC + BAC - BCA$
  $= [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$
- $[AB,C] = A[B,C] + [A,C]B$
- $[A,B]=0 \qquad e^{A+B} = e^A e^B \qquad P_{ab} = P_a P_b$

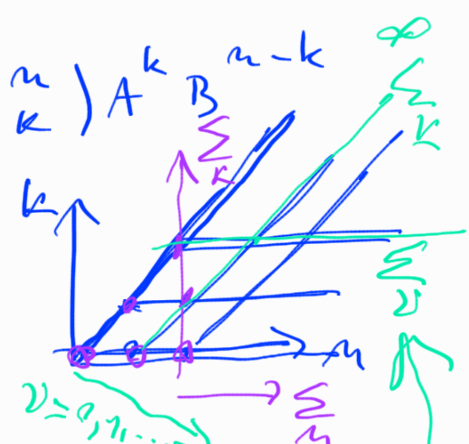
$A = \sum_{ab} a P_{ab} \qquad B = \sum_{ab} b P_{ab}$ ze spektr. rozkl.

$$e^{\hat{A}+\hat{B}} = \sum_{ab} \underbrace{e^{a+b}}_{e^a e^b} \hat{P}_{ab} = \sum_{ab} e^a \hat{P}_a e^b \hat{P}_b$$
$$= \sum_a a^a \hat{P}_a \sum_b e^b \hat{P}_b = e^A e^B$$

- z Taylora $\quad e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{n!}\binom{n}{k} A^k B^{n-k}$$

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$$



$(A+B)(A+B) \cdots (A+B)$

$A^2 B^{n-2} = BAB \ldots A$ komutují!

subst... $\nu = n-k$ ... $\sum_{n=0}^{\infty}\sum_{k=0}^{n} = \sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{1}{n!}\frac{n!}{k!\,\nu!} A^k B^\nu$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \sum_{\nu=0}^{\infty} \frac{1}{\nu!} B^\nu = e^A \cdot e^B \checkmark$$

- Jacobi

$$[AB-BA, C]$$

$$\begin{aligned} &[[A,B],C] + \\ &+[[C,A],B] \\ &+[[B,C],A] \end{aligned} = \begin{matrix} ABC - \overline{BAC} - CAB + \overline{CBA} \\ CAB - CBA - BCA + ACB \\ BCA - ACB - ABC + BAC \\ S \quad L \quad S \quad L \end{matrix} = 0$$

$\boxed{\text{ÚLOHA 2}}$ ... $[A,B] = C \quad [A,C] = [B,C] = 0$

- $n=0 \quad [A, B^0] = 0 \qquad n=1 \quad [A,B] = [A,B]$

$[A, B^n] = n B^{n-1}[AB] \leftarrow \underline{\forall n' \leq n}$

indukce $\underline{[A, B^{n+1}]} = [A, B \cdot B^n] = \underline{B}[A, B^n] + [A,B]B^n$

$$[A, B^n] = n B^{n-1}[A,B]$$

$$= n B^n [A,B] + B^n [A,B]$$

$$= (n+1) B^n [A,B]$$

- $[A^n, B] = n A^{n-1}[A,B]$ — stejně

- $[\hat{A}, f(\hat{B})] = \sum_n \frac{f_n}{n!}[\hat{A}, B^n] = \underbrace{\sum_n \frac{f_n}{n!} n B^{n-1}}[A,B]$

$\hookrightarrow f(\hat{B}) = \sum \frac{f_n}{n!} B^n \qquad f'(B) = f'(B)[AB]$

$\boxed{\text{ÚLOHA 3}}$ -- jednoduché

$\boxed{\text{ÚLOHA 4}}$ $[\hat{x}_\alpha, \hat{p}_\beta] = i\hbar\,\delta_{\alpha\beta}$ ←

$$\boxed{\hat{L}_\alpha = \varepsilon_{\alpha\beta\gamma}\, x_\beta p_\gamma} \quad \cdots \ \Sigma$$

$$[\hat{L}_\alpha, \hat{x}_\delta] = [\varepsilon_{\alpha\beta\gamma} x_\beta p_\gamma, x_\delta]$$

$$= \varepsilon_{\alpha\beta\gamma}\, [\underbrace{x_\beta p_\gamma, x_\delta}_{=0}] = \varepsilon_{\alpha\beta\gamma}\, x_\beta \underbrace{[p_\gamma, x_\delta]}_{-i\hbar\delta_{\gamma\delta}}$$

$$= -i\hbar\, \varepsilon_{\alpha\beta\delta}\, x_\beta = i\hbar\, \varepsilon_{\alpha\delta\beta}\, x_\beta \quad ✓$$

$$[\hat{L}_\alpha, \hat{L}_\beta] = i\hbar\, \varepsilon_{\alpha\beta\gamma} \hat{L}_\gamma \ ? \qquad \text{x.p.}$$

$$= [\varepsilon_{\alpha\gamma_1\gamma_2} x_{\gamma_1} p_{\gamma_2}, \varepsilon_{\beta\delta_1\delta_2} x_{\delta_1} p_{\delta_2}]$$

$$= \varepsilon_{\alpha\gamma_1\gamma_2}\, \varepsilon_{\beta\delta_1\delta_2}\, [x_{\gamma_1} p_{\gamma_2}, x_{\delta_1} p_{\delta_2}]$$

$$= (\varepsilon_{\alpha\gamma_1\gamma_2}\, \varepsilon_{\beta\delta_1\delta_2}) \qquad [xp, x]p + x[xp, p]$$

$$\left(x_{\gamma_1} \underbrace{[p_{\gamma_2}, x_{\delta_1}]}_{-i\hbar\delta_{\gamma_2\delta_1}} p_{\delta_2} + x_{\delta_1} \underbrace{[x_{\gamma_1}, p_{\delta_2}]}_{i\hbar\delta_{\gamma_1\delta_2}} p_{\gamma_2} \right.$$

$$= i\hbar \left(-x_{\gamma_1} p_{\delta_2}\, \varepsilon_{\alpha\gamma_1\gamma}\, \varepsilon_{\beta\gamma\delta_2} + x_{\delta_1} p_{\gamma_2}\, \varepsilon_{\alpha\gamma\gamma_2}\, \varepsilon_{\beta\delta_1\gamma} \right)$$

$$\boxed{\varepsilon_{\alpha_1\beta_1\gamma}\, \varepsilon_{\alpha_2\beta_2\gamma} = \delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2} - \delta_{\alpha_1\beta_2}\delta_{\beta_1\alpha_2}}$$

$$i\hbar\Big[+ \underset{a}{x_{\gamma_1}}\, \underset{b}{p_{\sigma_2}} \left(\delta_{\alpha\beta}\delta_{\gamma_1\sigma_2} - \delta_{\alpha\sigma_2}\delta_{\gamma_1\beta}\right)$$

$$- \underset{a}{x_{\sigma_1}}\, \underset{b}{p_{\gamma_2}} \left(\delta_{\alpha\beta}\delta_{\gamma_2\sigma_1} - \delta_{\alpha\sigma_1}\delta_{\gamma_2\beta}\right)\Big]$$

$$i\hbar\, x_a p_b \left[\cancel{\delta_{\alpha\beta}\delta_{ab}} - \delta_{\alpha b}\delta_{a\beta} - \cancel{\delta_{\alpha\beta}\delta_{ba}} + \delta_{\alpha a}\delta_{\beta b}\right]$$

$$= i\hbar\, x_a p_b \underbrace{\left[\delta_{\alpha a}\delta_{\beta b} - \delta_{\alpha b}\delta_{\beta a}\right]}_{\varepsilon_{ab\gamma}\,\varepsilon_{\alpha\beta\gamma}}$$

$$= i\hbar\, \varepsilon_{\alpha\beta\gamma} \underbrace{\varepsilon_{\gamma ab}\, x_a p_b}_{L_\gamma} = i\hbar L_\gamma$$