CVIČENÍ QMI – 11

stat. $\{|\psi_i\rangle, p_i\}$ -- $\hat\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$ spin 1/2

$|+\rangle$ $|-\rangle$

$\psi = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

ÚLOHA 1

a) $|\psi\rangle = |+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ čistý -- $p=1$

$\begin{pmatrix}0\\1\end{pmatrix}$

$\hat\rho = |\psi\rangle\langle\psi| = \begin{pmatrix}1&0\\0&0\end{pmatrix}$

b) $\{(|+\rangle, \frac12), (|-\rangle, \frac12)\}$ --- $\hat\rho = \frac12\left[|+\rangle\langle+| + |-\rangle\langle-|\right] = \begin{pmatrix}1&0\\0&1\end{pmatrix}\frac12 = \frac{\hat I}{2}$

c) $|\psi\rangle = |x+\rangle = \frac{1}{\sqrt2}(|+\rangle + |-\rangle)$ $\hat\rho = |x+\rangle\langle x+| =$

$= \frac12 (|+\rangle + |-\rangle)(\langle+| + \langle-|)$

$\rho = \frac12 \begin{array}{c c} + & - \\ \begin{pmatrix}1&1\\1&1\end{pmatrix} \end{array} \begin{array}{c}+\\-\end{array} = P_{x+}$

$\begin{pmatrix}1\\1\end{pmatrix}$ $\begin{pmatrix}1\\-1\end{pmatrix}$ $= \frac12 \left(|+\rangle\langle+| + |-\rangle\langle-| + |+\rangle\langle-| + |-\rangle\langle+|\right)$

d) $\{(|x+\rangle, \frac12), (|x-\rangle, \frac12)\}$ $\hat\rho = \frac12 \hat P_{x+} + \frac12 \hat P_{x-} = \frac14 \begin{pmatrix}2&0\\0&2\end{pmatrix} = \frac{\hat I}{2}$

$|x-\rangle = \frac{1}{\sqrt2}(|+\rangle - |-\rangle)$ ---- $P_{x-} = \frac12 \begin{pmatrix}1&-1\\-1&1\end{pmatrix}$

$\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = -1\begin{pmatrix}1\\-1\end{pmatrix}$ $\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

e) čistý $\hat S_n = \frac{\hbar}{2}\vec n\cdot\vec\sigma = \frac{\hbar}{2}(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)\cdot(\sigma_x, \sigma_y, \sigma_z)$

-- vl. č. $\hbar/2$, $-\hbar/2$ -- $|\vec n+\rangle = \begin{pmatrix}\cos\frac\theta2 e^{-i\phi/2}\\ \sin\frac\theta2 e^{+i\phi/2}\end{pmatrix} \equiv \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

$\hat\rho_{\vec n} = |n+\rangle\langle n+| = \begin{pmatrix}|\alpha|^2 & \alpha\beta^*\\ \alpha^*\beta & |\beta|^2\end{pmatrix} = \begin{pmatrix}\cos^2\frac\theta2 & \sin\frac\theta2\cos\frac\theta2 e^{-i\varphi}\\ \sin\frac\theta2\cos\frac\theta2 e^{i\phi} & \sin^2\frac\theta2\end{pmatrix} \equiv P_{\vec n}$

$\mathrm{Tr}\,\hat\rho = \cos^2\frac\theta2 + \sin^2\frac\theta2 = 1$

f) stat. $\{|\vec n+\rangle \;\forall \vec n$ se stej. pravd.$\}$

$\int_{S_2} p(\vec n)\, d\Omega = 1$ $\hookrightarrow p = \frac{1}{4\pi}$

$\hat\rho = \int_{S_2} p(\vec n)\,\rho(\vec n)\, d\Omega = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \begin{pmatrix}\cos^2\frac\theta2 & \sin\frac\theta2\cos\frac\theta2 e^{-i\varphi}\\ * & \sin^2\frac\theta2\end{pmatrix}$

$= \begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{12}^* & \rho_{22}\end{pmatrix}$

$$\rho_{12} = \frac{1}{4\pi}\underbrace{\int_0^{2\pi} e^{-i\varphi}d\varphi}_{\cos\varphi - i\sin\varphi \;\to\; 0} \int_0^{\pi} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\theta\, d\theta = 0$$

$$\cos^2\frac{\theta}{2} = \frac{1 \pm \cos\theta}{2} \quad (\sin)$$

$$\rho_{11} = \frac{1}{2}\int_0^{\pi}\cos^2\frac{\theta}{2}\sin\theta\, d\theta = \frac{1}{4}\int_0^{\pi}(1+\underbrace{\cos\theta}_{z})\underbrace{\sin\theta\, d\theta}_{dz} \qquad z = \cos\theta$$

$$\int_{-1}^{1}(\underbrace{1}_{2} + \underbrace{z}_{0})dz = \frac{2}{4} = \frac{1}{2}$$

$$\rho_{22} = \frac{1}{2} \qquad \hat\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2\end{pmatrix} = \frac{\hat I}{2} = \frac{\hat I}{\dim\mathcal{H}} = \text{stav s max. entropií}$$

$\boxed{\text{ÚLOHA 2}}$ spin $\frac{1}{2}$ — charakterizace $\hat\rho = \begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}$ .. 2×2 $\quad \mathcal{H} = \mathbb{C}^2$

$\mathrm{Tr}\,\hat\rho = 1 \qquad \rho^+ = \rho \qquad \langle v|\rho|v\rangle \geq 0 \qquad \{(|\psi_i\rangle, p_i)\}_{i=1,2}$

$p_1 + p_2 = 1 \qquad p_1 \quad p_2 \qquad p_{1,2} \geq 0$

$\qquad |\psi_1\rangle \quad |\psi_2\rangle$

$\hat I, \hat\sigma_x, \hat\sigma_y, \hat\sigma_z \quad .. \quad \hat\rho = A\hat I + B\hat\sigma_x + C\hat\sigma_y + D\hat\sigma_z$

$A, B, C, D \in \mathbb{R}$

a) $\rho^+ \overset{!}{=} \rho = A^* I + B^*\sigma_x + C^*\sigma_y + D^*\sigma_z$

$\mathrm{Tr}\,\hat\rho = 1 \qquad \mathrm{Tr}\, I = 2 \qquad \mathrm{Tr}\,\sigma_i = 0$

$= A\,\mathrm{Tr}\, I + \underbrace{B\,\mathrm{Tr}\,\sigma_x + \dots}_{0} = 2A = 1 \Rightarrow A = \frac{1}{2}$

$\hat\rho = \frac{1}{2}(\hat I + \vec p\cdot\vec{\hat\sigma}) \quad .. \quad \vec p = (2B, 2C, 2D)$ polarizační vektor $(|\vec p| \in \langle 0, 1\rangle)$

b) $\hat\rho$ je čistý stav $\Leftrightarrow |\vec p| = 1$

$\mathrm{Tr}(\hat\rho^2) = 1 \qquad \boxed{\mathrm{Tr}(\sigma_\alpha\sigma_\beta) = 2\delta_{\alpha\beta}} \qquad \sigma_\alpha^2 = I$

$\mathrm{Tr}(\rho^2) = \frac{1}{4}\mathrm{Tr}\left((\hat I + p_\alpha\hat\sigma_\alpha)(\hat I + p_\beta\hat\sigma_\beta)\right)$

$\frac{1}{4}\left[\mathrm{Tr}\, I + 2p_\alpha\underbrace{\mathrm{Tr}\,\sigma_\alpha}_{0} + p_\alpha p_\beta \underbrace{\mathrm{Tr}(\hat\sigma_\alpha\hat\sigma_\beta)}_{2\delta_{\alpha\beta}}\right]$

$$= \frac{1}{4}\left[2 + 2|\vec{p}|^2\right] = \frac{1}{2}\left[1 + |\vec{p}|^2\right] \leq 1$$

$$|\vec{p}|^2 \leq 1$$

rovnost $\Leftrightarrow$ čistý stav ... $|\vec{p}|=1$ čistý stav.

c) střední hodnota $\bar{S}_\alpha$ ... $\hat{S}_{\vec{n}} = \frac{\hbar}{2}\vec{n}\cdot\vec{\sigma} = \frac{\hbar}{2} n_\alpha \sigma_\alpha$

$$\langle \hat{S}_n \rangle = \operatorname{Tr} \hat{\rho}\hat{S}_n = \operatorname{Tr}\left(\frac{1}{2}(\hat{I} + p_\alpha \hat{\sigma}_\alpha)\frac{\hbar}{2} n_\beta \hat{\sigma}_\beta\right)$$

$$= \frac{\hbar}{4} n_\beta \underbrace{\operatorname{Tr}\sigma_\beta}_{0} + \frac{\hbar}{4} p_\alpha n_\beta \underbrace{\operatorname{Tr}\hat{\sigma}_\alpha\hat{\sigma}_\beta}_{2\delta_{\alpha\beta}} = \frac{\hbar}{2} n_\alpha p_\alpha = \frac{\hbar}{2}\vec{n}\cdot\vec{p}$$

$\vec{n} = \vec{e}_x \quad S_n = S_x = \frac{\hbar}{2}\sigma_x$ ... $\langle S_x \rangle = \frac{\hbar}{2} p_x \quad \langle S_z \rangle = \frac{\hbar}{2} p_z$

$\langle S_y \rangle = \frac{\hbar}{2} p_y$

závěr:

$$\hat{\rho} = \frac{1}{2}(\hat{I} + \vec{p}\cdot\vec{\hat{\sigma}}) \qquad |\vec{p}| \in \langle 0,1\rangle$$

$|p|=1$ ... čistý stav $|\vec{n}+\rangle$ kde $\vec{n} = \vec{p}$

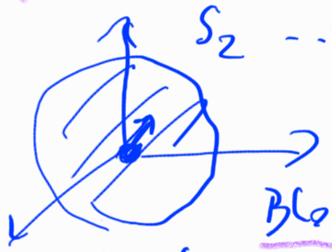
$|p|=0 \qquad \hat{\rho} = \frac{1}{2}\hat{I}$ ... zcela nepolarizovaný stav

největší entropie

$\vec{p}$ určuje směr spinu $\langle \sigma_i \rangle = p_i$

S_z ... $\leftrightarrow \vec{p}$

$\leftrightarrow \hat{\rho}$



Blochova sféra

povrch .. čisté stavy