$\boxed{\text{QMI-8}}$ Částice v magnetickém poli

OPAKOVÁNÍ: (1) klasicky: elektromagnetické potenciály:

$$\left.\begin{aligned}\vec{E} &= -\nabla\phi - \partial_t \vec{A} \\ \vec{B} &= \nabla\times\vec{A}\end{aligned}\right\} \to$$

Hamiltonův formalismus pro Lorentzovu sílu $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

$$\to H(\vec{x},\vec{p}) = \frac{1}{2m}\Big[\underbrace{\vec{p} - q\vec{A}(x)}_{m\vec{v}}\Big]^2 + q\,\phi(x)$$

$\vec{p}$ … kanonická hybnost; $m\vec{v}$ .. kinematická hybnost

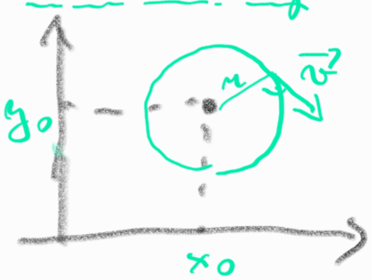
(2) QM: nahrazení $\hat{\vec{p}} = -i\hbar\nabla$ …operátor rychlosti: $\hat{V} = \frac{1}{m}(\hat{p} - q\hat{A})$

.. komutační relace:

$$[\hat{x}_\alpha, \hat{p}_\beta] = i\hbar\delta_{\alpha\beta} \quad \hat{v}_\alpha = \frac{i}{\hbar}[\hat{H}, \hat{x}_\alpha] \quad [\hat{x}_\alpha, \hat{v}_\beta] = \frac{i\hbar}{m}\delta_{\alpha\beta} \quad [\hat{v}_\alpha, \hat{v}_\beta] = \frac{i\hbar q}{m^2}\varepsilon_{\alpha\beta\gamma}\hat{B}_\gamma$$

(3) Pohyb v homogenním mg. poli $\vec{B} = B\vec{e}_z$ …. $\vec{A} = (-yB, 0, 0)$

• **klasicky** .. $\omega_c = \frac{v}{r} = \frac{qB}{m}$



$$x = x_0 + r\cos\omega_c t$$
$$y = y_0 - r\sin\omega_c t$$
$$v_x = -\omega_c r\sin\omega_c t$$
$$v_y = -\omega_c r\cos\omega_c t$$

• **QM - algebraicky**

$$H = \frac{1}{2}m\underbrace{(V_x^2 + V_y^2)}_{} + \frac{1}{2}mV_z^2$$

komutátor konst. .. jako $[x, p]$

$$\to \text{def } \hat{Q} = \frac{m}{\sqrt{|q|B}}\hat{V}_x, \quad \hat{P} = \frac{m}{\sqrt{|q|B}}\hat{V}_y$$

$$\to E = E_{xy} + E_z = \frac{1}{2}mv_z^2 + \hbar\omega_c\left(n + \frac{1}{2}\right)$$ --- LHO

• **QM - v souřadnicové reprezentaci:**

$$H = \frac{1}{2m}\left[(\hat{p}_x + \hat{y}qB)^2 + \hat{p}_y^2 + \hat{p}_z^2\right]$$ .. $\{\hat{p}_x, \hat{p}_z, \hat{H}\}$ .. ÚSKO

$\hat{y}_0$

.. $\psi = e^{i(k_x x + k_z z)}\phi(y)$ .. (SR): $H|\psi\rangle = E|\psi\rangle \to$

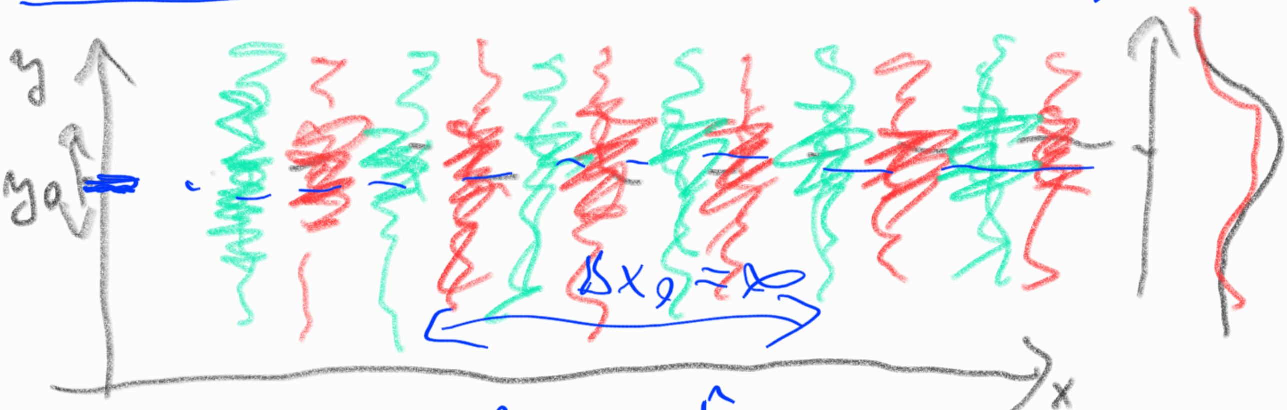
$$\to -\frac{\hbar^2}{2m}\phi''(y) + \frac{m}{2}\omega_c^2(y - y_0)^2\phi(y) = \epsilon\,\phi(y)$$

kde $y_0 = -\frac{\hbar k_x}{qB}$, $\epsilon = E - \frac{\hbar^2 k_z^2}{2m}$

ZÁVĚR: $\boxed{E_n(k_x, k_z) = \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m}}$ $\quad \boxed{\psi = N e^{ik_x x + ik_z z}\underbrace{\phi_n(y)}_{}}$

LHO

Interpretace nalezeného řešení: $H_n(\alpha y)\, e^{-\frac{1}{2}\alpha^2 (y-y_0)^2}$



$\Delta y_0 = 0$

pozorovatelné: $\hat{x}_0 = \hat{x} + \frac{\hat{v}_y}{\omega_c}$

$\hat{y}_0 = \hat{y} - \frac{\hat{v}_x}{\omega_c}$

$\hat{r}^2 = (\hat{x}-\hat{x}_0)^2 + (\hat{y}-\hat{y}_0)^2 = \frac{\hat{v}_x^2 + \hat{v}_y^2}{\omega_c^2}$

platí: $\hat{H}_{xy} = \frac{1}{2} m \omega_c^2 \hat{r}^2 =$

$r = \sqrt{\frac{2E_{xy}}{m\omega_c^2}} =$

$\langle r \rangle_{\psi_n} = \frac{1}{\alpha}\sqrt{(2n+1)}$

x_0, y_0 .. integrály pohybu

$[\hat{H}, \hat{x}_0] = [\hat{H}, \hat{y}_0] = 0$

$[\hat{x}_0, \hat{y}_0] = -\frac{i\hbar}{m\omega_c}$! nekompatibilní !

relace neurč. $\Delta x_0 \cdot \Delta y_0 \geq \frac{\hbar}{2m\omega_c}$

$\hat{y}_0 = \hat{y} - \frac{\hat{v}_x}{\omega_c} = \hat{y} + \frac{1}{\omega_c m}(\hat{p}_x - q\hat{A}_x) = \frac{i\hbar}{m\omega_c}\partial_x \;..\; \hat{p}_x$

$\omega_c = \frac{qB}{m}$, $\hat{p}_x = -i\hbar\partial_x$, $\hat{A}_x = -yB$

nalezené řešení tedy: $\Delta y_0 = 0 \Rightarrow \Delta x_0 = \infty$

lze volit i jinak

4) Kalibrační transformace

$\vec{E} = -\nabla\phi - \partial_t \vec{A}$

$\vec{B} = \nabla \times \vec{A}$

libov. skalární pole

$$\boxed{\begin{aligned} \vec{A} &\to \vec{A}' = \vec{A} + \nabla \chi \\ \Phi &\to \Phi' = \phi - \partial_t \chi \\ \psi &\to \psi' = e^{\frac{i}{\hbar} q \chi}\, \psi \end{aligned}}$$

změní to z elektromag.

je třeba přidat, aby

$\Rightarrow$ (SR) je invariantní

DK: Lemma $(-i\hbar\vec{\nabla}-q\vec{A}')\psi' = e^{\frac{i}{\hbar}q\chi}(-i\hbar\vec{\nabla}-q\vec{A})\psi$

$$(-i\hbar\nabla - q(A+\nabla\chi))e^{\frac{i}{\hbar}q\chi}\psi = e^{\frac{i}{\hbar}q\chi}(-i\hbar\nabla\psi - qA\psi) + \cancel{-i\hbar\, e^{\frac{i}{\hbar}q\chi}\,\frac{i}{\hbar}q\nabla\chi\,\psi}\; \cancel{-q\nabla\chi\, e^{\frac{i}{\hbar}q\chi}\psi}$$

invar SR

$$\frac{1}{2m}(-i\hbar\nabla - q\vec{A}')^2\psi' + q\phi'\psi' - i\hbar\partial_t\psi' = 0$$

$e\psi$ Lemma

$e\psi$

$$= \frac{1}{2m}\cancel{e^{\frac{i}{\hbar}q\chi}}(-i\hbar\nabla - qA)^2\psi + q\phi\cancel{e^{\frac{i}{\hbar}q\chi}}\psi \cancel{-q\partial_t\chi\, e^{\frac{i}{\hbar}q\chi}\psi}$$
$$-i\hbar\cancel{e^{\frac{i}{\hbar}q\chi}}\partial_t\psi \cancel{-i\hbar\frac{i}{\hbar}q\partial_t\chi\, e\,\psi}$$

$$\rightarrow \frac{1}{2m}(-i\hbar\vec{\nabla}-q\vec{A})^2\psi + q\phi\psi - i\hbar\partial_t\psi = 0 \qquad \checkmark$$

$\psi' = e^{\frac{i}{\hbar}q\chi}\psi$ $\qquad \boxed{\langle\psi|\, p,\, A\, |\psi\rangle}$

e^- $\quad e^+$ $\qquad \partial_x \qquad p$

$\langle p\rangle \qquad p = -i\hbar\nabla \quad \sim \quad v = \frac{1}{m}(\hat{p} - q\hat{A})$

$$\hat{v}'\psi' = e^{\frac{i}{\hbar}q\chi}\hat{v}\psi$$

kalibrační invariance dalších pozorovatelných

- $\langle\psi|\hat{v}|\psi\rangle = \langle\psi'|\hat{v}'|\psi'\rangle$ $\quad \psi^*\psi$
- tok hustoty pravd.? $\quad |\psi|^2 = \rho(x)\ \ldots$

$$\partial_t\rho + \operatorname{div}\vec{j} = 0$$

$$\vec{j} = \frac{i\hbar}{2m}\left(\psi \nabla \psi^* - \psi^* \nabla \psi\right) - \frac{q}{m}\vec{A}|\psi|^2$$ kalibr. invariantní

$$= \frac{1}{2m}\left[\psi^* \hat{p}\psi - \psi \hat{p}\psi^* - 2qA|\psi|^2\right]$$

$$= \frac{1}{m}\,\mathrm{Re}\left\{\psi^*(\hat{p} - q\vec{A})\psi\right\} = \mathrm{Re}\left\{\psi^* \hat{\vec{v}}\psi\right\}$$

$$= \mathrm{Re}\left\{\psi'^* \hat{\vec{v}}'\psi'\right\}$$

5) Aharonov–Bohmův efekt

Ballentine

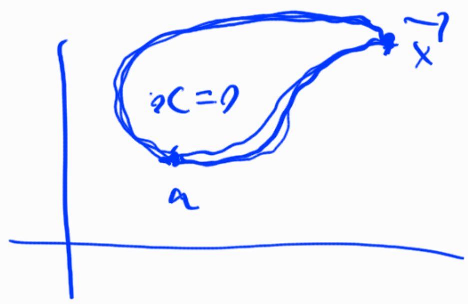
a) volná částice a kalibrační transf.

$\vec{E} = 0$, $\vec{B} = 0$ − lze volit $\vec{A} = 0$, $\phi = 0$

$$\boxed{\vec{A}' = \nabla\chi}$$

$$\psi' = e^{\frac{i}{\hbar}q\chi}\,\psi_0(x,t)$$

$$\boxed{\chi = \int_a^{\vec{x}} \nabla\chi \cdot d\vec{s} = \int_a^{\vec{x}} \vec{A}\cdot d\vec{s}}$$



($\vec{B} = \mathrm{rot}\,\vec{A} = 0$ nezávis. na traj.

tuto transf lze případně $A \neq 0$ odstranit.

b) oblast s $\vec{B} = 0$ ale ne jednoduše souvislá

$$\chi_1 = \int_{a,\,S_1}^{x} \vec{A}\cdot ds$$

$$\chi_2 = \int_{a,\,S_2}^{x} \vec{A}\cdot ds$$

$$\chi_1 \neq \chi_2$$

$$\chi_1 - \chi_2 = \oint_C \vec{A}\cdot d\vec{s} = \int_S \underbrace{\mathrm{rot}\,\vec{A}}_{B}\, dV = \int_S \vec{B}\, dV = \Phi$$

interference s rozdílem fází $\exp\{\frac{i}{\hbar} q \Phi\}$

$e^{\frac{i}{\hbar} q (\chi_1 - \chi_2)}$

klas. mech $\boxed{F = q\vec{v} \times B = 0}$ $\vec{A}$,

QM

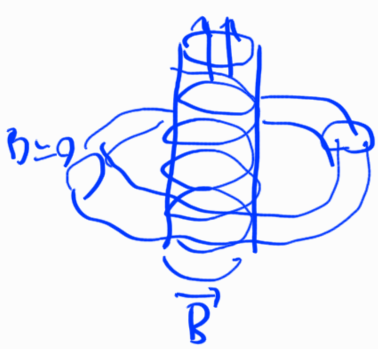
- $\vec{A}$ není pozorovatelné ... $\vec{B}$ je pozorovatelné

ale působí nelokálně

- $\vec{A}$ je pozorovatelné, působí lokálně

  ↪ ale měřitelné jen kalibračně invar veličiny

Pozn.: AB efekt pro vázané stavy



$\psi(\varphi)$ $\varphi \in \langle 0, 2\pi)$

$\Phi = \int \vec{B} \cdot d\vec{S}$

$e^{im\varphi}$ → n. podmínce s.

→ přibude mg. faktor $e^{\frac{i}{\hbar} q \Phi}$

$E = \frac{\hbar^2 m^2}{2I}$