QM I - 2 Formalismus QM I (diskrét. syst.)

Opakování: ① stavy .. $|\psi\rangle \in \mathcal{H}$ .. LVP

② veličiny ... operátory $A = A^+ = \sum_{a\in\sigma} a P_a$

③ $p_a = \langle\psi|P_a|\psi\rangle \left(= |\langle a|\psi\rangle|^2\right)$ $\quad |\psi\rangle \to P_a|\psi\rangle$ .. redukce vln. fce

- n kvant. teček ... $\mathcal{H} = \mathbb{C}^n$
- spin $\mathcal{H} = \mathbb{C}^2$ ... $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ ... moment hybn. (spinový) $\hat{S}_i = \frac{\hbar}{2}\hat{\sigma}_i$

$\vec{\sigma} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$

reprezentace .. $|\psi\rangle \longleftrightarrow \begin{pmatrix} \langle a_1|\psi\rangle \\ \langle a_2|\psi\rangle \\ \vdots \\ \langle a_n|\psi\rangle \end{pmatrix}$

$A \ldots \sigma_A \ldots \{|a_i\rangle\}$

$\ldots |\psi\rangle = I|\psi\rangle = \sum_a |a\rangle\langle a|\psi\rangle \left(\sum_{a,k} |a,k\rangle\langle a,k|\psi\rangle \right)$

$\hookrightarrow \sum |a\rangle\langle a| = I$

Nekompatibilní pozorovatelné a relace neurčitosti

$\sum_k |ak\rangle\langle ak|$

Momenty pravděpodobnostního rozdělení:

$\hat{A} \quad |\psi\rangle \quad .. \quad a \in \sigma_A \quad \ldots \quad p_a = \sum_k |\langle a,k|\psi\rangle|^2 \equiv \langle\psi|\hat{P}_a|\psi\rangle$

$\langle A\rangle = \sum_{a\in\sigma_A} a\, p_a$ $\qquad \sum_{a\in\sigma} p_a = 1 \quad \ldots \mu_0$

$\sum_a \langle\psi|\hat{P}_a|\psi\rangle = \sum_{ak} \langle\psi|ak\rangle\langle ak|\psi\rangle$

$= \langle\psi|\hat{I}|\psi\rangle = \langle\psi|\psi\rangle = 1$

$\langle A\rangle = \sum_a a\langle\psi|\hat{P}_a|\psi\rangle$

$= \sum_a \langle\psi|a P_a|\psi\rangle = \langle\psi|\underbrace{\sum_a a P_a}_{A}|\psi\rangle$

$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle = \mu_1$

A^n

moment pravd. rozdělení (náhod. proměnné) $\quad \mu_n \equiv \sum_a a^n p_a$

$\langle A^n\rangle = \mu_n =$

$\mu_2 \longleftrightarrow$ rozptyl (měření a .. střední kvadr. odch

variance

ΔA

$$(\Delta A)^2 \equiv \langle\psi|\underbrace{(\hat{A}-\langle A\rangle)^2}_{(A^2-2\langle A\rangle A+\langle A\rangle^2)}|\psi\rangle = \langle\psi|A^2|\psi\rangle - 2\langle A\rangle\overbrace{\langle\psi|A|\psi\rangle}^{\langle a\rangle} + \langle A\rangle^2\underbrace{\langle\psi|\psi\rangle}_{1}$$

$$= \langle\psi|A^2|\psi\rangle - 2\langle A\rangle\langle A\rangle + \langle A\rangle^2 = \langle A^2\rangle - \langle A\rangle^2$$

$$\boxed{\Delta A^2 = \mu_2 - \mu_1^2} \qquad \Delta A = \sqrt{\mu_2 - \mu_1^2}$$

pozn: ve vl. st. $\Delta A = 0$

$|\psi\rangle = |a\rangle$ .. $p_a = 1$ $\quad a' \neq a \quad p_{a'} = 0 \quad \langle a'|a\rangle = 0$

$P_a|a\rangle = |a\rangle$ $\quad \mu_1 = \sum_a a\, p_a = a \qquad \Delta A = 0$

$\mu_2 = a^2 p_a = a^2$

Nekompat. pozorovatelné: $[\hat{A},\hat{B}] \neq 0 \qquad \hat{A}\hat{B} \neq \hat{B}\hat{A}$

$A = A^+ \quad B = B^+ \qquad \hat{A}\hat{B} - \hat{B}\hat{A}$

• $[\hat{A},\hat{B}] = i\hat{C} \neq 0$

RELACE NEURČITOSTI: $(\Delta A)_\psi \cdot (\Delta B)_\psi \geq \frac{1}{2}\langle\psi|\hat{C}|\psi\rangle$

DK: Schwartz ... $\|\phi_1\| \cdot \|\phi_2\| \geq |\langle\phi_1|\phi_2\rangle|$

$|\phi_1\rangle = (\hat{A} - \langle A\rangle)|\psi\rangle \qquad \|\phi_1\| = \sqrt{\langle\psi|(A-\langle A\rangle)(A-\langle A\rangle)|\psi\rangle}$

$|\phi_2\rangle = (\hat{B} - \langle B\rangle)|\psi\rangle \qquad = \Delta A$

$\|\phi_2\| = \Delta B$

$\Delta A \cdot \Delta B = \|\phi_1\| \cdot \|\phi_2\| \geq |\langle\psi|(\hat{A}-\langle A\rangle)(\hat{B}-\langle B\rangle)|\psi\rangle|$

$AB - A\langle B\rangle - B\langle A\rangle + \langle A\rangle\langle B\rangle$

$= |\langle\psi|(\hat{A}\hat{B} - \langle A\rangle\langle B\rangle)|\psi\rangle| = |\langle\psi|\frac{AB+BA}{2}|\psi\rangle - \langle A\rangle\langle B\rangle + \frac{i}{2}\langle C\rangle|$

$\frac{AB-BA}{2} \quad \frac{AB+BA}{2}$

$\geq \frac{1}{2}|\langle C\rangle|$

Pozn.. Důsl. ~~rel.~~ • Heisenberg $\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$ $\qquad [\hat{S}_a, \hat{S}_b] = i\hbar\,\varepsilon_{abc}\hat{S}_c$

• spin $\frac{1}{2}$ $\quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y \qquad \Delta S_x\, \Delta S_z \geq \frac{1}{2}\langle S_y\rangle \qquad \pm\frac{\hbar}{2}$

částice vl. st. $S_z \to \Delta S_z = 0 \implies \langle S_y\rangle = 0 \qquad p_+ = p_- = \frac{1}{2}$

Kompatibilní pozorovatelné

$[\hat A, \hat B] = 0 \qquad \hat A \hat B = \hat B \hat A$

Lemma 1: $[\hat A, \hat B] = 0 \ \wedge \ B|\psi\rangle = b|\psi\rangle \Rightarrow B|\phi\rangle = b|\phi\rangle$

pro $|\phi\rangle \equiv A|\psi\rangle$

… vlastní podprostory $\hat B$ jsou invariantní vůči působení $\hat A$

$$B|\phi\rangle = BA|\psi\rangle = A\underbrace{B|\psi\rangle}_{b} = Ab|\psi\rangle = bA|\psi\rangle = b|\phi\rangle$$

cbd

Lemma 2: $[\hat A, \hat B] = 0 \quad \hat A|a_i\rangle = a_i|a_i\rangle \quad a_1 \neq a_2 \Rightarrow \langle a_1|\hat B|a_2\rangle = 0$

$\langle a_1| \qquad |a_2\rangle$

$d_{a_n} \quad |a_n k\rangle \quad a_1$

$\hat B_{(A)} =$

DK:

$$0 = \langle a_1|\hat A\hat B - \hat B\hat A|a_2\rangle$$

$$= \underbrace{\langle a_1|\hat A}_{a_1\langle a_1|} B|a_2\rangle - \langle a_1|B\underbrace{\hat A|a_2\rangle}_{a_2|a_2\rangle} = a_1\langle a_1|B|a_2\rangle - a_2\langle a_1|B|a_2\rangle$$

$$0 = \langle a_1|B|a_2\rangle(\underbrace{a_1 - a_2}_{\neq 0})$$

c.b.d.

Lemma 3a: $[\hat A, \hat B] = 0 \quad \hat A = \sum_a a\hat P_a \quad \hat B = \sum_b b\hat P_b \Rightarrow [\hat P_a, \hat B] = 0$

DK: $\hat A|a,\alpha\rangle = a|a,\alpha\rangle \quad .. \quad \hat P_a = \sum_\alpha |a,\alpha\rangle\langle a,\alpha|$

$P_a|a_1\alpha_1\rangle = \delta_{aa_1}|a_1\alpha_1\rangle$

pro báze

$$\langle a_1,\alpha_1|[\hat P_a, \hat B]|a_2,\alpha_2\rangle = \qquad P_a = \sum_\alpha |a\alpha\rangle\langle a\alpha|$$

$PB - BP$

$$= \sum_\alpha \underbrace{\langle a_1\alpha_1|a\alpha\rangle}_{\delta_{a_1a}\delta_{\alpha_1\alpha}}\langle a\alpha|B|a_2\alpha_2\rangle - \langle a_1\alpha_1|B|a\alpha\rangle\underbrace{\langle a\alpha|a_2\alpha_2\rangle}_{\delta_{aa_2}\delta_{\alpha\alpha_2}}$$

$$= \underbrace{\delta_{a_1a}}_{=0}\langle a_1\alpha_1|B|a_2\alpha_2\rangle - \delta_{aa_2}\underbrace{\langle a_1\alpha_1|B|a_2\alpha_2\rangle}_{0}$$

$a_1 = a_2$

$a_1 \neq a_2$ — lemma 2

CBD

Lemma 3: $\underbrace{\{A,B\}=0}$ $\Rightarrow$ $\underline{[\hat{P}_a,\hat{P}_b]=0}$ $\forall a,b$

DK: Lemma 3a $\Rightarrow$ $\underbrace{[P_a,B]=0}$ $\Rightarrow$ $\underline{[P_a,P_b]=0}$ ✓

Lema 3a $A\to B$, $B\to P_a$

Důsledek: $\hat{P}_{ab}=\underbrace{\hat{P}_a\hat{P}_b}$ ... je projektor ✓

$$\hat{P}_{ab}=\hat{P}_{ab}^+=P_b^+P_a^+=P_bP_a=P_aP_b=P_{ab}$$

$$\hat{P}_{ab}^2=P_a\underbrace{P_bP_a}_{P_aP_b}P_b=\underset{P_a}{P_a^2}\,\underset{P_b}{P_b^2}=P_aP_b=P_{ab}$$

$A=\sum aP_a$ (A,B)

Věta: A,B samosdr.: $A\,B$ kompat. $\Leftrightarrow$ $\underline{[P_a,P_b]=0}$

$\xrightarrow[\text{Lema 3}]{}$

$$[A,\alpha B_1+\beta B_2]=\alpha[A,B_1]+\beta[A,B_2]$$ $[A,B]=0$

Věta: samosdružené operátory A,B mají spol. bázi z vl. v. $\Leftrightarrow$ $[\hat{A},\hat{B}]=0$

triv.

DK: ~ vaznena bázi $\hat{P}_{ab}$ $\forall a,b$ $P_aP_b=0$

- $\hat{P}_{ab}$ $\lambda=1$ $\to$ $\forall|\psi\rangle$ $\hat{P}_{ab}|\psi\rangle=|\psi\rangle$ $\hookrightarrow \underline{|a,b,k\rangle}$

  $\lambda=0$

$\to$ $\{|a,b,k\rangle,\ k=1,\dots d_{ab}\}$ .. to je spol. báze

$a\neq a'$ $P_{ab}\cdot P_{a'b'}$ $P_aP_{a'}=0$

$\langle a,k|a'k'\rangle=\delta_{aa'}\delta_{kk'}$

$$P_{ab}\cdot P_{a'b'}=\delta_{aa'}\delta_{bb'}P_{ab}$$

$$\hat{P}_{ab}=\underline{\sum_k|a,b,k\rangle\langle a,b,k|}$$

$$\hat{I}=\sum_{ab}\hat{P}_{ab}$$

$$=\sum_{abk}|abk\rangle\langle abk|$$

$$|\psi\rangle=\sum_{abk}\psi_{abk}|abk\rangle$$

$$\hat{I}=\hat{I}_a\cdot\hat{I}_b=\sum_a\hat{P}_a\sum_b\hat{P}_b=\sum_{ab}\hat{P}_{ab}$$

P_{ab} $\quad \hat{P}_a = \underbrace{\sum_b \hat{P}_{ab}}$ $= \sum_b P_a P_b = P_a \sum_b P_b = \hat{P}_a$

$\hat{A}|abk\rangle = a|abk\rangle$ $\qquad P_{a'b'}|abk\rangle = |abk\rangle \, \delta_{aa'} \delta_{bb'}$

$\hat{A} = \sum_a a P_a = \sum_{ab} a P_{ab}$

I_b

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poz.: použití spol. báze: $\boxed{I = \sum_{ab} P_{ab} = \sum_{abk} |abk\rangle\langle abk|}$

- společný spektr. rozk. $\hat{P}_a = \sum_{kb} \hat{P}_{ab} = \sum_{kb} |abk\rangle\langle abk|$

$$\hat{A} = \sum_{ab} a P_{ab} = \sum_{abk} a |a,b,k\rangle\langle a,b,k|$$

$$B = \sum_{ab} b P_{ab} \qquad \hat{A}|abk\rangle = a|abk\rangle$$

- funkce dvou proměnných

$$f(\hat{A},\hat{B}) = \sum_{ab} f(a,b) P_{ab} \qquad BA = AB = \boxed{\frac{AB+BA}{2}}$$

Γab

$\hat{A}_1, \hat{A}_2, \hat{A}_3 \ \dots \ \hat{A}_N \qquad [\hat{A}_i, \hat{A}_j] = 0$