QMI-4 Bodová částice v 1D

OPAKOVÁNÍ: Formalismus QM

① stavy ... $\mathcal{H}$ $|\psi\rangle$ $\langle\psi|$

symetrické $\langle\psi|\hat{A}|\phi\rangle = \langle\phi|\hat{A}|\psi\rangle^*$

② pozorovatelné $\hat{A}: \mathcal{H} \to \mathcal{H}$ ③ měření $|a,\alpha\rangle$

④ časový vývoj $i\hbar\,\partial_t|\psi\rangle = \hat{H}|\psi\rangle$

kompatibilní pozorovatelné: $[\hat{A},\hat{B}] = 0$ .. báze $|a,b,\alpha\rangle$

$\overline{\mathcal{X}} \supset \mathcal{H} \supset \mathcal{X}$

ÚSKO $A, B, \dots Z$ .. $|a,b,\dots z\rangle$

- $\boxed{\text{ÚSKO}: \hat{X}}$ ... $|x_0\rangle$

-- $|\psi\rangle$ — souřadnicová repr. (x-reprezentace) -- $\psi(x)$ ... $|\psi\rangle = \int \psi(x)|x\rangle dx$

$\mathcal{H} = L^2(R)$ -- $\int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty$ $\xrightarrow[\text{norm.}]{\text{dá se}}$ $\|\psi\| = 1$

- souřadnice $\hat{X}$: $\hat{X}\psi(x) = x\,\psi(x)$

$\sigma(\hat{X}) = \mathbb{R}$

$M_{xx'}$ $\langle x|\hat{X}|x'\rangle = x\,\delta(x-x')$

$$x' \begin{pmatrix} \ddots & & 0 \\ & x & \\ 0 & & \ddots \end{pmatrix} = \delta(x-x')$$

- vl. fce $\phi_{x_0} = \langle x|x_0\rangle = \delta(x-x_0)$
- normování $\langle x_0|x_0'\rangle = \delta(x_0-x_0')$
- $\hat{I} = \int_{-\infty}^{\infty} |x_0\rangle\langle x_0|\,dx_0$

- hybnost $\hat{p}$ $\hat{p}\,\psi(x) = -i\hbar\frac{d}{dx}\psi(x)$

$\langle\phi|\hat{p}|\psi\rangle = \langle\psi|\hat{p}|\phi\rangle^* = \int_{-\infty}^{\infty}\psi^*(-i\hbar)\phi'\,dx^* = i\hbar\int_{-\infty}^{\infty}\psi'(x)\phi(x)dx$

per. part. $\forall \psi \in L^2$

hran. členy vymizí $\psi(x) \to 0$, $|x| \to \infty$

- vlastní fce: $\hat{P}\psi(x) = p\,\psi(x)$ $-i\hbar\,\psi' = p\,\psi(x)$

$|p\rangle$

$$\phi_p(x) = \langle x|p\rangle = \boxed{\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px}} \to N$$

$|\phi|^2 = \frac{1}{2\pi\hbar}$

- normování $\langle p|p'\rangle = \delta(p-p') = \int_{-\infty}^{\infty}\phi_p^*(x)\phi_{p'}(x)dx$

$\sigma(\hat{P}) = \mathbb{R}$ $p \in \mathbb{R}$

$$= |\eta|^2 \int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar} x (p'-p)\right] dx$$

$$\boxed{\delta(x) = \int_{-\infty}^{\infty} e^{2\pi i k x}\, dk} \to \infty$$

subst. → hodnota $\eta = \frac{1}{\sqrt{2\pi\hbar}}$

$\langle p|x'\rangle$

- relace úplnosti: $\hat{I} = \int_{-\infty}^{\infty} |p\rangle\langle p|\, dp$

$$\langle x|\hat{I}|x'\rangle = \delta(x-x') = \int \langle x|p\rangle \langle x'|p\rangle^* dp$$

$$\frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}p(x-x')}\, dp$$

- **Hamiltonián** .. volná částice kinet. energ. $\boxed{H_0 = T = \frac{\hat{p}^2}{2m}}$

$$\sigma(H_0) = (0, \infty)$$

$\boxed{\bullet \text{ USKO } \hat{p}}$ (momentum) -- báze $|p\rangle$ — hybnostní reprezentace

p-repr.

$I = \int |x\rangle\langle x|\, dx$

$$\psi(p) = \langle p|\psi\rangle = \int_{-\infty}^{\infty} \underbrace{\langle p|x\rangle}_{\phi_p^*(x)} \underbrace{\langle x|\psi\rangle}_{\psi(x)} dx = \underbrace{\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}px} \psi(x)\, dx}_{\text{F.T.}}$$

$\phi_{x_0}(x) = \delta(x-x_0)$

pozorovatelné:

- $\hat{p}$ -- $\hat{p}\,\psi(p) = p\,\psi(p)$
- $\hat{x}$ -- $\hat{x}\,\psi(p) = i\hbar \frac{d}{dp}\psi(p)$

$$\langle p|\hat{x}\,\hat{I}_x|\psi\rangle = \int dx\, \langle p|x\rangle\, x\, \langle x|\psi\rangle$$

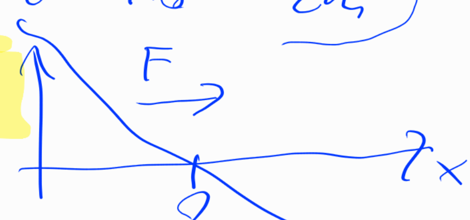
$$x \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}px} = \hbar i \frac{d}{dp} e^{-\frac{i}{\hbar}px}$$

$$= i\hbar \frac{d}{dp} \underbrace{\int \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}px} \psi(x)\, dx}_{\psi(p)}$$

- $\phi_{p_0}(p) = \langle p | p_0 \rangle = \delta(p-p_0)$
- $\phi_{x_0}(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}px_0}$ $\quad |p\rangle$

PŘ: (použití hybnostní repr.) $\quad$ v $H_0 = \frac{p^2}{2m}$

ČÁSTICE V HOMOGENNÍM POLI



$$\hat H = \hat T + \hat V(x)$$
$$\hat H = \frac{\hat p^2}{2m} + V(\hat x) \qquad \boxed{V(x) = -F\hat x}$$

stacionární stavy – $\hat H|\psi\rangle = E|\psi\rangle$

$\hat p = -i\hbar\frac{d}{dx} \qquad p^2 = -\hbar^2\frac{d^2}{dx^2}$

- v x reprezentaci:

$$\left(\frac{\hat p^2}{2m} + V(x)\right)\psi(x) = -\frac{\hbar^2}{2m}\psi''(x) - Fx\,\psi(x) = E\psi(x)$$

→ $\boxed{\text{ODR II. ř.}}$ $\qquad \psi' = 0$

$\hat p\,\psi(p) = p\,\psi(p) \qquad \hat x\,\psi = i\hbar\,\psi'(p)$

- v p reprezentaci:

$$H\psi(p) = \left(\frac{\hat p^2}{2m} - F\hat x\right)\psi(p) = \frac{p^2}{2m}\psi(p) - i\hbar F\psi'(p) = E\psi$$

→ $\boxed{\text{ODR I. ř.}}$

$$(\ln\psi)' = \frac{\psi'(p)}{\psi(p)} = -\frac{E}{i\hbar F} + \frac{p^2}{2m}\frac{1}{i\hbar F} = f(p) \qquad |\psi(x)|^2$$

$$\psi(p) = A\exp\left\{\frac{i}{\hbar}\left(\frac{E}{F}p - \frac{p^3}{6mF}\right)\right\}$$ řešení v p-repr.

transf. zpět do x $\quad \sim \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}px}$

$$\psi(x) \equiv \langle x|\psi\rangle = \int dp\,\langle x|p\rangle\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp\,\exp\left\{\frac{i}{\hbar}\left(px + p\frac{E}{F} - \frac{p^3}{6mF}\right)\right\}$$

$\mathbb{1}\,dp$

řešení v x-repr.

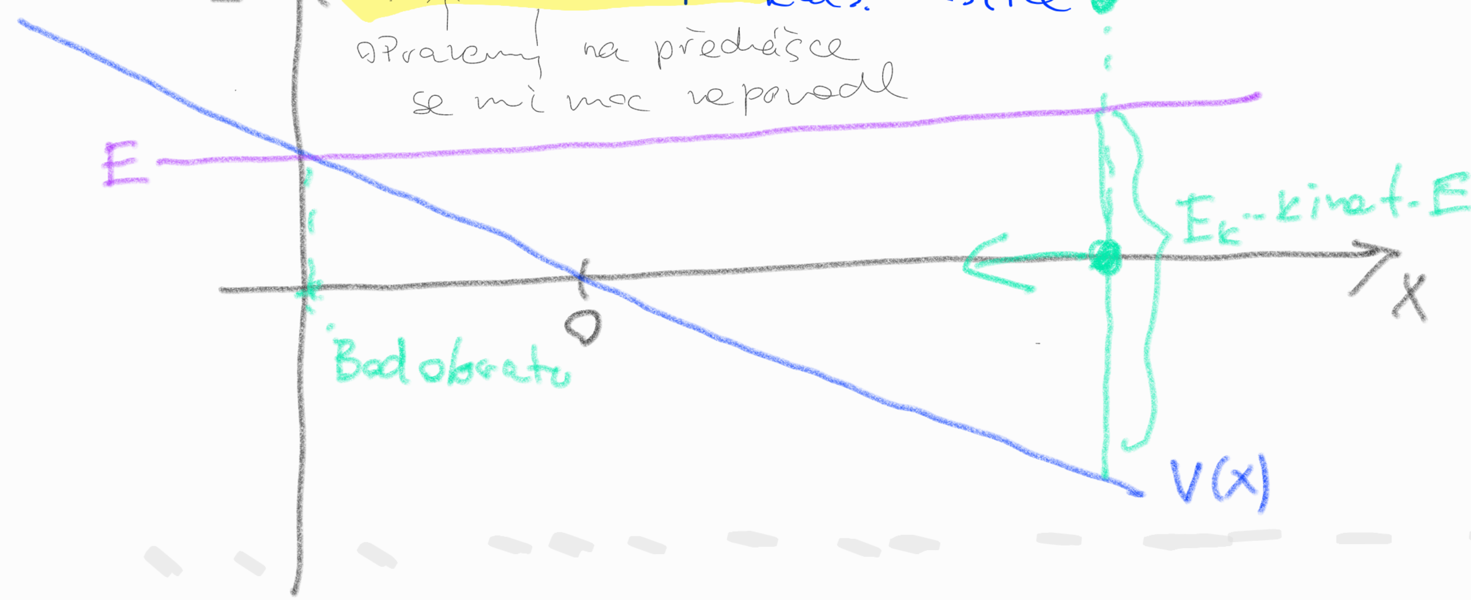
Obvykle se zavádí spec **Airiho fce** – spec. př. Bessel. fcí

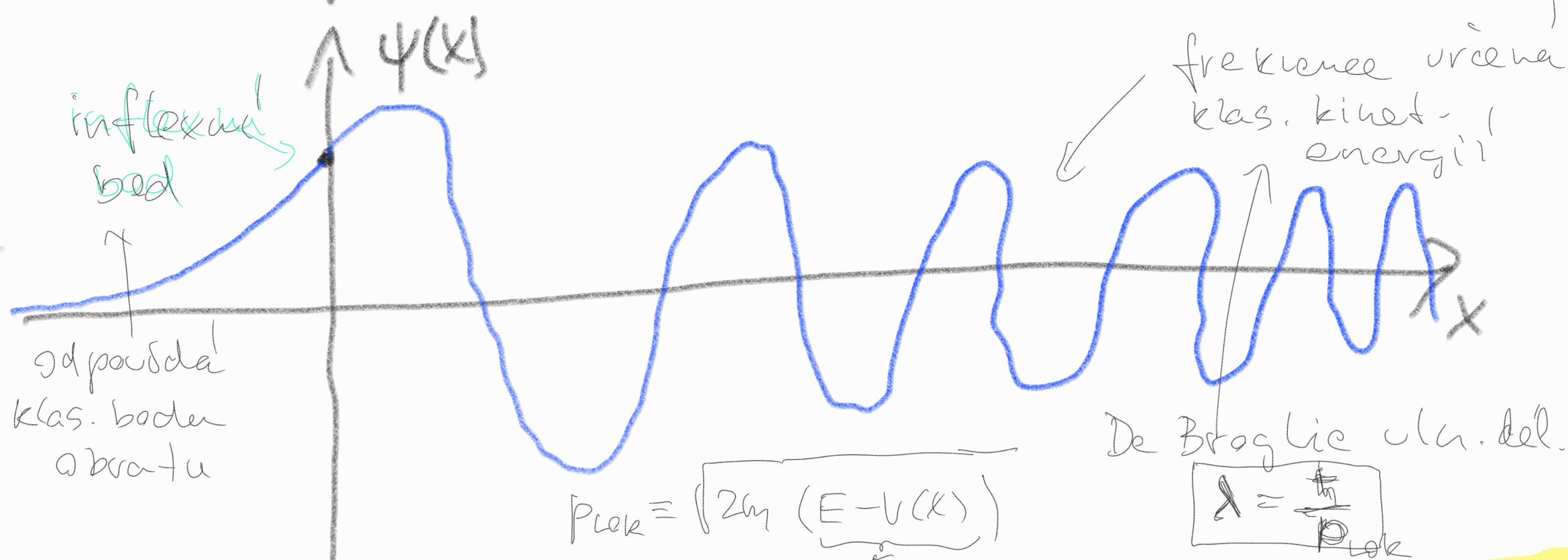
$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{1}{3}t^3\right)dt$$

souvisí s $\boxed{\sqrt{z}\, J_{\pm\frac{1}{3}}\left[\frac{2}{3}z^{3/2}\right]}$

$$\psi(x) = C\cdot Ai\left[\sqrt[3]{\frac{2mF}{\hbar^2}}\left(x+\frac{E}{F}\right)\right]$$

OBRÁZEK $\psi(x)$ a jeho souvislost s pohybem klas. částice





$\boxed{\text{• ÚSKO } \hat{H}_0, \hat{P}}$

$$\hat{H}_0 = T = \frac{p^2}{2m} \in (0,\infty)$$

$\phi_{p_0}(p)$ $\qquad E_p = \frac{p^2}{2m}$

$p \in \mathbb{R}$ $\qquad p_E = \pm\sqrt{2mE}$

$\beta|p_E\rangle + \alpha|p_{-E}\rangle$

parita, prostorová inverze

$$\hat{P}\psi(x) = \psi(-x)$$

$\sigma(\hat{P}) = \{1, -1\}$ sudé a liché

$\hat{P}^2\psi = \hat{P}(\psi(-x)) = \psi(--x) = \psi(x)$

$\lambda^2 = 1$

$\boxed{\lambda=1}$ $P\psi(x) = \underbrace{\overset{\lambda\psi}{\psi(x)} = \psi}(-x)$ -- sudé ψ -- $\mathcal{H}_+$

$\boxed{\lambda=-1}$ $P\psi(x) = \underset{\lambda\ \psi}{-\psi(x)} = \psi(-x)$ -- liché ψ -- $\mathcal{H}_-$

$\mathcal{H} = \underbrace{\mathcal{H}_+ \oplus \mathcal{H}_-}$ -- $\psi(x) = \psi_+(x) + \psi_-(x)$

$$\psi(x) = \underbrace{\frac{\psi(x)}{2} + \frac{\psi(-x)}{2}}_{\psi_+(x)} + \underbrace{\frac{\psi(x)}{2} - \frac{\psi(-x)}{2}}_{\psi_-(x)}$$

$$= P_{\lambda=1}\psi(x) + P_{\lambda=-1}\psi(x)$$

$\hat{I} = \hat{P}_1 + \hat{P}_{-1}$ $\qquad P_{\pm 1} = \frac{1}{2}(\psi(x) \pm \psi(-x)) = \frac{1}{2}(\hat{I} \pm \hat{P})$

$H_0 P = P H_0$ komutují

$$[\hat{H}_0, \hat{P}]\psi(x) = -\frac{\hbar^2}{2m}\left\{\frac{d^2}{dx^2}\psi(-x) - \frac{d^2\psi}{dx^2}\Big|_{x\to -x}\right\}$$

$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$ $\qquad \underset{1}{(-1)^2}\psi''(-x) - \psi''(-x) = 0$

úplný syst? spol. vl. v.?

H_0 --- E --- $p_E = \pm\sqrt{2mE}$

$|E\rangle$ -- $\psi_E(x) = \alpha\frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}p_E x} + \frac{\beta}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}p_E x}$

$\hat{P}$ $\qquad \underbrace{\lambda = 1}_{\text{soda}} \qquad \underbrace{\lambda = -1}_{\text{licha}}$

$|E\lambda\rangle$

$|E+\rangle = N \cos \frac{px}{\hbar}$ $\qquad N \, ? \, ?$

$|E-\rangle = N \sin \frac{px}{\hbar}$