QMI-8 Bodová částice v magnetickém poli

1) klasická mechanika

- Newton. pohyb. rov.: $m\frac{d\vec{v}}{dt} = \vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ Lorentz

$\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}$

$\vec{B} = \vec{\nabla}\times\vec{A}$

kalibrační inv.

$\vec{A} \to \vec{A}' = \vec{A} + \nabla\chi$

$\phi \to \phi' = \phi - \partial_t \chi$

- Lagrange: $\mathcal{L}(\vec{x}, \vec{v}, t) = \frac{1}{2}m\vec{v}^2 - q\phi(\vec{x},t) + q\vec{v}\cdot\vec{A}(\vec{x},t)$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v_i}\right) = \frac{\partial \mathcal{L}}{\partial x_i}$

- Hamiltonův formalismus: $\boxed{\vec{p} = \frac{\partial \mathcal{L}}{\partial \vec{v}} = m\vec{v} + q\vec{A}}$

$H(\vec{x}, \vec{p}) = \vec{v}\cdot\vec{p} - \mathcal{L}(x, v) = \frac{1}{2m}\left[\vec{p} - q\vec{A}(\vec{x},t)\right]^2 + q\phi(x,t)$

$\frac{d}{dt}(p_i) = \frac{\partial \mathcal{L}}{\partial x_i}$

2) kvantový popis

$\hat{H} = \frac{(\hat{\vec{p}} - q\hat{\vec{A}})^2}{2m} + q\hat{\phi}$

$\hat{p}_\alpha = -i\hbar\frac{\partial}{\partial x_\alpha}$ $\quad q\hat{\phi} \equiv V(\hat{\vec{x}})$ $\quad q\vec{A}(\hat{\vec{x}})$

$\to \hat{p}_\alpha, \hat{x}_\beta \ldots$ $\quad \boxed{[\hat{x}_\alpha, \hat{p}_\beta] = i\hbar\delta_{\alpha\beta}}$

- operátor rychlosti:

OBECNÁ POZN: veličina $\hat{X}$ .. operátor čas změny $\boxed{\dot{\hat{X}} = \frac{i}{\hbar}[\hat{H}, \hat{X}]}$

$-\frac{d}{dt}\langle\hat{X}\rangle = \langle\dot{\hat{X}}\rangle = \frac{1}{i\hbar}\langle\psi|\hat{X}\hat{H} - \hat{H}\hat{X}|\psi\rangle$ $\quad \partial_t\psi = \frac{1}{i\hbar}H\psi$

$\frac{d}{dt}\langle\psi(t)|\hat{X}|\psi(t)\rangle$ $\quad \frac{1}{i\hbar}[X, H]$

$\leftarrow$ Heisenberg .. $|\psi_H\rangle$ .. $X_H(t) = U^\dagger(t) X_S U(t)$

$$\hat V_\alpha = \frac{i}{\hbar}[\hat H, \hat x_\alpha] = \frac{i}{\hbar}\left[\frac{(\vec p - q\vec A)^2}{2m} + q\hat\phi, x_\alpha\right]$$

$$= \frac{i}{\hbar}\frac{1}{2m}\underbrace{\left[(p_\beta - qA_\beta)(p_\beta - qA_\beta), x_\alpha\right]}_{(p-qA)\,[p_\beta - \cancel{qA_\beta}, x_\alpha] + [p_\beta - \cancel{qA_\beta}, x_\beta]\, p_\beta - qA_\beta} = \frac{1}{m}(p_\alpha - qA_\alpha)$$

$$[p_\beta, x_\alpha] = i\hbar\delta_{\alpha\beta}$$

$$\boxed{\hat V_\alpha = \frac{1}{m}(\hat p_\alpha - q\hat A_\alpha)} \qquad \hat H = \frac{1}{2}m\hat V_\alpha \hat V_\alpha + q\hat\phi$$

platí

1) $[\hat x_\alpha, \hat V_\beta] = \dfrac{i\hbar\,\delta_{\alpha\beta}}{m}$

2) $[\hat V_\alpha, \hat V_\beta] = \dfrac{i\hbar}{m^2}\, q\, \varepsilon_{\alpha\beta\gamma}\hat B_\gamma$

Důkaz 2):

$$[\hat V_x, \hat V_y] = \frac{1}{m^2}[p_x - qA_x, p_y - qA_y] = -\frac{q}{m^2}\left([p_x, A_y] + [A_x, p_y]\right)$$

obecně $[p_\alpha, f(x_\beta)] = -i\hbar f'(x)\delta_{\alpha\beta}$

$$= -\frac{qi\hbar}{m^2}\underbrace{\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)}_{-(\mathrm{rot}\,A)_z} = \frac{i\hbar q}{m^2}B_z$$

• operátor Lorentzovy síly $\qquad m\dfrac{dV}{dt} = F$

$$\hat F_\alpha = m\dot V_\alpha = m\frac{i}{\hbar}[\hat H, \hat V_\alpha] = \frac{mi}{\hbar}\frac{m}{2}\underbrace{[V_\beta V_\beta, V_\alpha]}_{V_\beta \underbrace{[V_\beta, V_\alpha]}_{\sim B} + \underbrace{[V_\beta, V_\alpha]}_{\sim B} V_\beta} + \frac{mi}{\hbar}q\underbrace{[\phi, V_\alpha]}_{x,\,V}$$

$$\boxed{\frac{i}{\hbar} q \frac{\partial\phi}{\partial x_\alpha}\, i\hbar}$$

$$= m\frac{i}{\hbar}\frac{1}{2}m\,\varepsilon_{\beta\alpha\gamma}(V_\beta B_\gamma + B_\gamma V_\beta) - q\frac{\partial\phi}{\partial x_\alpha} \qquad E_\alpha(\hat x) = \hat E_\alpha$$

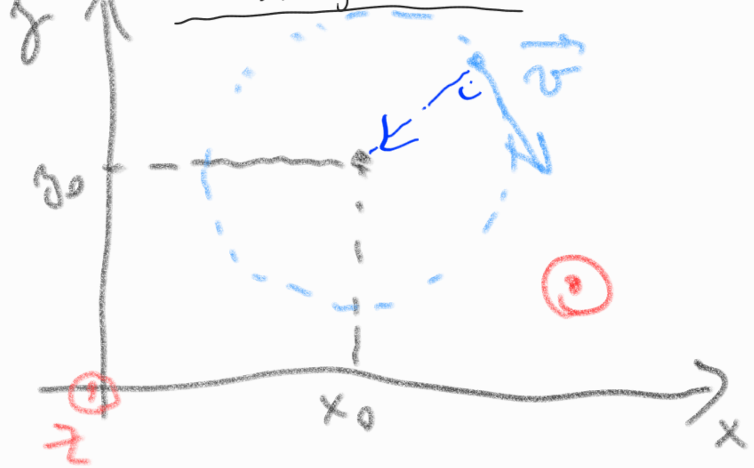
$$\hat{\vec F} = m\dot{\vec V} = q\hat{\vec E} + \frac{q}{2}\left(\hat{\vec V}\times\hat{\vec B} - \hat{\vec B}\times\hat{\vec V}\right) \qquad \vec B = \mathrm{rot}\,A(\vec x)$$

pro komutující by dalo $q\vec v\times\vec B$, ale

nekomutují, záleží na pořadí operátorů

3) Pohyb částice v homogenním mg. poli

• klasický obrázek:



$\vec B = B\vec e_z$

$\vec F = q v B = m\frac{v^2}{r}$ ←

→ $\omega_c = \frac{v}{r} = \boxed{q\frac{B}{m}}$

cyklotronová frekv.

klasické řešení:

$z = z_0 + v_z t$ volná část.

$x = x_0 + r\cos(\omega_c t + \delta)$

$y = y_0 - r\sin(\omega_c t + \delta)$

$v_x = -\omega_c r\sin(\omega_c t + \delta)$

$v_y = -\omega_c r\cos(\omega_c t + \delta)$

$\vec A = (-yB, \overset{y}{0}, \overset{z}{0})$ $\quad \nabla\cdot\vec A = 0$, $\nabla\times\vec A = \vec B$

• algebraické řešení: $\hat H = \frac{m}{2}(\hat v_x^2 + \hat v_y^2) + \frac{m}{2}\hat v_z^2 = \hat H_{xy} + \hat H_z$

$[v_\alpha, v_\beta] \sim \varepsilon_{\alpha\beta\gamma} B_\gamma \qquad \vec B = (0, 0, B)$

$[v_y, v_z] \sim B_x = 0$

$[v_x, v_z] \sim B_y = 0$ $\Rightarrow [\underset{E_{xy}}{\hat H_{xy}}, \underset{E_z}{\hat H_z}] = 0$ $\qquad E = E_{xy} + E_z$

$v = (p - qA)/m$

1) $\hat H_z = \frac{1}{2} m v_z^2 = \frac{p_z^2}{2m} = -\frac{\hbar^2}{2m}\frac{d^2}{dz^2}$ -- volná částice $\qquad v_z = p/m$

$e^{izk_z} \to \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} z p_z}$ ... $E_z = \frac{p_z^2}{2m}$

2) $\hat H_{xy} = \frac{1}{2} m(\hat v_x^2 + \hat v_y^2) = \underbrace{\frac{1}{2}\omega_c(\hat Q^2 + \hat P^2)}_{\text{jako LHO .. stejné řešení}}$

$\boxed{\omega_c = m\frac{qB}{m^2} = \frac{qB}{m}}$

$[v_x, v_y] = \frac{i\hbar q B}{m^2}$ def $\hat Q = \frac{m}{\sqrt{|q|B}} v_x$ $\qquad \hat P = \frac{m}{\sqrt{|q|B}} v_y$

$[\hat Q, \hat P] = (\pm) i\hbar \qquad \frac{q}{|q|}$

$\boxed{q > 0}$ BÚNO

$q < 0$ .. $Q \sim v_y$, $P \sim v_x$

$\hat a = \frac{1}{\sqrt 2}(\hat Q + i\hat P)$ --- $[\hat a, \hat a^+] = \hbar$ ... $\hat N = \hat a^+ \hat a$

$[\hat N, \hat a^+] = \hbar\hat a^+ \Rightarrow \{\sigma_N = \{n\hbar\}$

$\dots\ E_{xy} = \hbar\omega_c\left(n+\frac{1}{2}\right) \qquad n = 0, 1, 2, \dots$

<u>závěr:</u> $E_n(v_z, \alpha) = \hbar\omega_c\left(n+\frac{1}{2}\right) + \frac{1}{2}m v_z^2$ ← spoj. spektrum pro z-dimenzi

↑ Landauovy hladiny

- <u>řešení v souřad. reprez.:</u>

$$\hat{H} = \frac{1}{2m}\left[(\hat{p}_x + \hat{y}qB)^2 + \hat{p}_y^2 + \hat{p}_z^2\right] \qquad p_\alpha = -i\hbar\frac{\partial}{\partial x_\alpha}$$

$$[\hat{H}, \hat{p}_x] = [\hat{H}, \hat{p}_z] = 0 \qquad \text{ÚSKO}: [\hat{p}_x, \hat{p}_z, \hat{H}] \dots$$

$$\rightarrow \psi(x,y,z) = e^{i(k_x x + k_z z)}\phi(y) \sim$$

$$\hat{H}\psi = E\psi$$

$$\rightarrow -\frac{\hbar^2}{2m}\nabla^2\psi - \frac{i\hbar q}{m}By\,\partial_x\psi + \frac{q^2B^2y^2}{2m}\psi - E\psi = 0$$

$$\rightarrow -\frac{\hbar^2}{2m}\phi''(y) + \frac{\hbar q B k_x}{m}y\,\phi(y) + \left[\frac{q^2B^2}{2m}y^2 + \frac{\hbar^2}{2m}(k_x^2 + k_z^2) - E\right]\phi = 0$$

$y \rightarrow y - y_0 \quad \leftarrow \quad y^2 + \lambda y \quad \dots$ <u>na čtverec</u>

$$(y-y_0)^2 \qquad y_0^2$$

$$y_0 = -\frac{\hbar k_x}{qB} \qquad \epsilon = E - \frac{\hbar^2 k_z^2}{2m}$$

$$\rightarrow -\frac{\hbar^2}{2m}\phi''(y) + \frac{m}{2}\omega_c^2(y-y_0)^2\phi = \epsilon\phi$$

<u>LHO</u> ... řešení známe

$$\psi(x,y,z) = N e^{ik_x x + ik_z z} H_n(\alpha(y-y_0))\, e^{-\frac{1}{2}(\alpha(y-y_0))^2}$$

$$\alpha = \sqrt{\frac{m|\omega_c|}{\hbar}} = \sqrt{\frac{|q|B}{\hbar}}$$

$$E_n(k_x, k_z) = \hbar\omega_c\left(n+\frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m} \qquad n = 0, 1, 2, \dots \qquad k_x, k_z \in \mathbb{R}$$

↑ energie nezávisí na k_x ... degenerační index