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The interplay between forces in the Kerr–Newman field

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Abstract. We discuss dynamical properties of generally non-Keplerian equatorial circular orbits and zero-angular-momentum spherical polar orbits around Kerr–Newman black holes. By considering charged test particles, the thrust is simply represented by the Lorentz force due to the electromagnetic field of the hole. We analyse the properties of the rotospheres in which the acceleration of the particles depends on their orbital angular velocity in a counter-intuitive manner, and we interpret the results in terms of suitably defined forces of classical type.

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1. Introduction

In strong gravitational fields around compact objects, Einstein's theory can produce effects which, from the Newtonian viewpoint, appear strange and counter-intuitive. Recently, considerable interest has been devoted to the occurrence of rotospheres which typically exist in black-hole spacetimes. The 'rotospheres' are the regions in which the 4-acceleration of a given family of test orbits may depend on the orbital angular velocity in an 'anomalous' manner. For example, in the Schwarzschild field in the region below the circular photon orbit (at $r = 3M$), an increase of the angular velocity of a test particle, forced to move on a given circular (non-Keplerian) orbit, requires an increase of the thrust in the *outward* direction. Although the effect can be seen in its 'purest' form in the Schwarzschild field, most attention has recently been paid to the Kerr spacetime in which it is modified by dragging; interestingly, the position of the photon orbits is again crucial.

It was a series of papers by Abramowicz and his collaborators (see [1, 4] and references therein) which revived an interest in the definition of various 'forces' of classical type in general relativity, in particular the discussion of the centrifugal force. In their approach, the centrifugal effect typically becomes attractive in the closest vicinity of black holes. However, there is no physically unique way to define individual forces in relativity and, indeed, de Felice [12], Semerák [20] and Barrabès *et al* [8] suggested an alternative split of the total acceleration, where the centrifugal force does not change the sign at the photon orbits, the change of the direction of a thrust being caused by an increase of a (velocity-dependent) gravitational force. The existence and the form of a rotosphere is, of course, independent of the split of the total acceleration (force) we choose.

In the present paper we consider the equatorial circular orbits (sections 2 and 4.1) and zero-angular-momentum spherical polar motions (section 3 and 4.2) outside Kerr–Newman

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centres when, together with the stationary gravitational field, the electromagnetic field is also present. Our aim is not only to obtain a generalization to the electromagnetic case of the previous works [15, 19, 13, 20, 22], which analysed the problem in the Kerr field. We also wish to demonstrate the effects when the thrust is represented by the simplest (in fact the only) fundamental macroscopic force available—the electromagnetic Lorentz force. In this sense we generalize [3], dealing with the static Reissner–Nordström field, to a stationary situation. The circular motion of charged particles in the equatorial plane of the Kerr–Newman spacetime was already considered in terms of forces in [10], where the electromagnetic thrust is split according to the powers of the conserved particle's azimuthal angular momentum. In the Schwarzschild limit this proposal goes over to the conclusion of [2], involving the centrifugal-force reversal at $r = 3M$.

In [3] it is shown that the counter-intuitive properties of circular motion of charged particles around a Reissner–Nordström black hole can be understood easily in terms of the 'centrifugal force' (defined after [1]) which changes sign at the photon orbits. The same type of definition of forces, introduced by [4] in a general case, has recently been applied to investigate the circular motion in stationary axisymmetric spacetimes [5], the gyroscopic precession [18] and the equilibrium of a charged particle [6] in the Kerr–Newman field, and the ellipticity behaviour of a slowly rotating ultracompact object [16]. In section 4 we present the interpretation based on another possible formalism, in which the forces are defined according to [21]†. This formalism is also simple, general and covariant, and, in black-hole spacetimes, it preserves some usual classical properties such as the repulsive nature of the centrifugal force. It has proved to yield satisfactory results in the case of motion in Kerr and simpler fields [20, 21], and also in the interpretation of gyroscope precession [24, 25].

We use Boyer–Lindquist coordinates (t, r, θ, ϕ), geometrized units $(c = G = 1)$ and the $(-+++)$ signature, Greek indices running 0–3 and Latin indices running 1–3.

2. Charged particle on an equatorial circular orbit

Particles orbiting uniformly in the ϕ-direction on circular (r = constant, θ = constant) orbits in the Kerr–Newman spacetime have the 4-velocity

$$u^\mu = u^t(1, 0, 0, \omega) \tag{1}$$

and the 4-acceleration $a^\mu = \mathrm{D}u^\mu/\mathrm{d}\tau$ (τ being the proper time) of the form

$$a^t = 0, \tag{2}$$

$$a^r = \Delta\Sigma^{-3}(u^t)^2\{[M(2r^2 - \Sigma) - rQ^2](1 - a\omega\sin^2\theta)^2 - r(\Sigma\omega\sin\theta)^2\}, \tag{3}$$

$$a^\theta = -\Sigma^{-3}(u^t)^2\sin\theta\cos\theta\{(2Mr - Q^2)[a - (r^2 + a^2)\omega]^2 + \Delta\Sigma^2\omega^2\}, \tag{4}$$

$$a^\phi = 0, \tag{5}$$

where

$$(u^t)^{-2} = 1 - \frac{2Mr - Q^2}{\Sigma}(1 - a\omega\sin^2\theta)^2 - (r^2 + a^2)\omega^2\sin^2\theta, \tag{6}$$

$$\Delta = r^2 - 2Mr + Q^2 + a^2, \qquad \Sigma = r^2 + a^2\cos^2\theta, \qquad \mathcal{A} = (r^2 + a^2)^2 - \Delta a^2\sin^2\theta, \tag{7}$$

† This approach was first suggested within the Schwarzschild background in [12]. A similar view was recently presented in [8].

$\omega = \mathrm{d}\phi/\mathrm{d}t\,(=\text{constant})$, and M, Q and a are mass, charge and specific angular momentum of the Kerr–Newman centre†. (See [17], p 878, for a description of the Kerr–Newman fields.)

In the equatorial plane ($\theta = 90^\circ$), $a^\theta = 0$ and the dependence of the radial component of acceleration on the azimuthal angular velocity, $a^r(\omega)$, at various fixed radii r is similar to that in the Kerr geometry (see appendix B of [19]). Close to the lower limit, ω_{min}, of permitted ω's, $a^r(\omega)$ behaves 'intuitively' above the radius of the counter-rotating (i.e. outer) photon orbit $r_{\mathrm{ph}-}$, whereas it behaves 'counter-intuitively' below $r_{\mathrm{ph}-}$: for any fixed $r > r_{\mathrm{ph}-}$, a^r goes to $-\infty$ for $\omega \to \omega_{\mathrm{min}}$, i.e. the particle needs greater and greater inward thrust as its velocity approaches the velocity of light; for $r < r_{\mathrm{ph}-}$, on the other hand, a^r goes to $+\infty$ for $\omega \to \omega_{\mathrm{min}}$, i.e. the particle needs greater and greater *outward* thrust as its velocity approaches that of light. At the upper limit, ω_{max}, of permitted ω's, we find an analogous behaviour—with co-rotating (inner) photon orbit, $r_{\mathrm{ph}+}$, now dividing the 'intuitive' and 'counter-intuitive' regions. The limiting values of ω at given r are determined by

$$\omega_{\mathrm{min,max}} = \omega_{\mathrm{KN}} \mp \frac{r^2\sqrt{\Delta}}{\mathcal{A}} = \frac{a \mp \sqrt{\Delta}}{r^2 + a^2 \mp a\sqrt{\Delta}}, \tag{8}$$

where

$$\omega_{\mathrm{KN}} = -g_{t\phi}/g_{\phi\phi} = (2Mr - Q^2)a/\mathcal{A} \tag{9}$$

is the angular velocity of the Kerr–Newman dragging. The radii of the outer/inner photon orbits, $r_{\mathrm{ph}\mp}$, are solutions of the equation

$$r^2 - 3Mr + 2Q^2 \mp 2a\sqrt{Mr - Q^2} = 0. \tag{10}$$

At each r, except for those between the photon orbits, there exists the 'extremally accelerated observer' [20, 23]. These observers are given by maxima of $a^r(\omega)$ at $r > r_{\mathrm{ph}-}$ and by minima of $a^r(\omega)$ at $r < r_{\mathrm{ph}+}$; between photon orbits there are no extremes of $a^r(\omega)$. The extremally accelerated observers move with the angular velocities

$$\begin{aligned}\omega_{0\pm} = \{&r^3(3M - r) + 2Ma^2r - 2Q^2(r^2 + a^2)\\ &\pm r^2\sqrt{r^2(r - 3M)^2 - 4Ma^2r + 4Q^2(\Delta - Mr)}\}\\ &\times\{2a[Mr(3r^2 + a^2) - Q^2(2r^2 + a^2)]\}^{-1},\end{aligned} \tag{11}$$

where the upper (lower) sign applies to $r > r_{\mathrm{ph}-}$ ($r < r_{\mathrm{ph}+}$). The discriminant in the equation above vanishes just at the photon orbits, as it is explicitly seen from its alternative form

$$(r^2 - 3Mr + 2Q^2 - 2a\sqrt{Mr - Q^2})(r^2 - 3Mr + 2Q^2 + 2a\sqrt{Mr - Q^2});$$

and it is negative between these orbits. The examples of the $a^r(\omega)$ dependences are illustrated in figures 1(*a*) and (*b*).

Although the behaviour of $a^r(\omega)$, as described above, is qualitatively similar in the Kerr and Kerr–Newman fields, certain differences do exist. They arise, in particular, when charge Q is large and the 'repulsive' effects of the electromagnetic field become important. For example, in contrast to the Kerr case when $a^r = \frac{M\Delta}{r^3(r-2M)}$ for $\omega = 0$, and hence it decreases monotonically as r increases, in the Reissner–Nordström case we obtain $a^r = r^{-1}(Mr - Q^2)$ for $\omega = 0$, and thus it does not need to decrease; it reaches its maximum at $r = 3Q^2/(2M)$ which is above the horizon for $Q > \sqrt{\frac{8}{9}}M$. Also, more than two circular photon orbits may

† We take $a \geqslant 0$, $Q \geqslant 0$, and will restrict ourselves to the region $r \geqslant 0$.

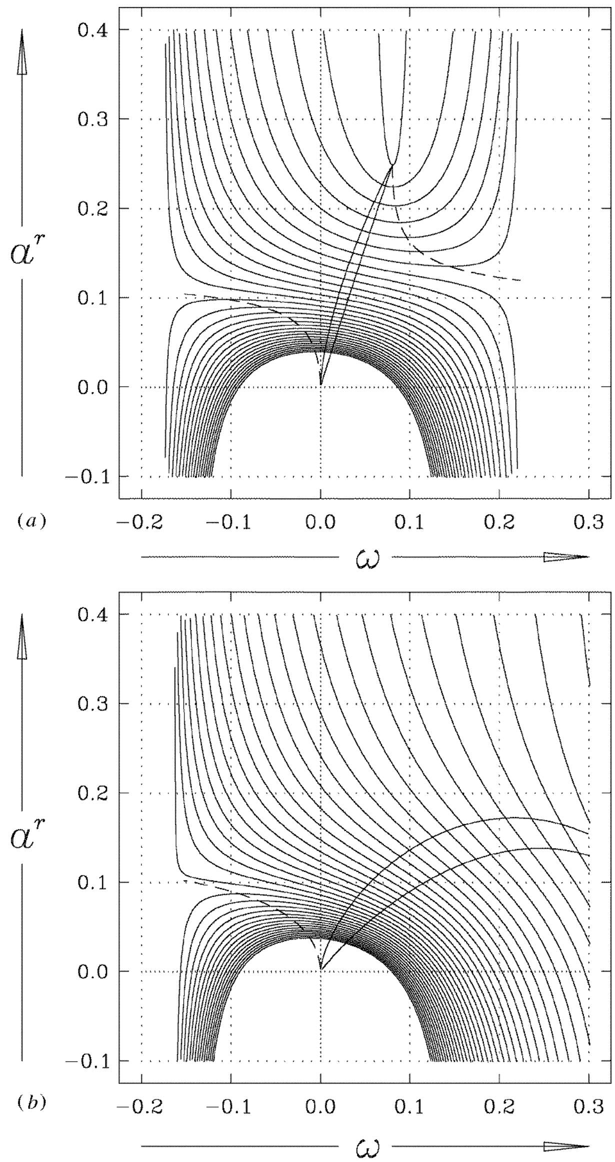

Figure 1. The dependence of the radial component of 4-acceleration, a^r (see equation (3)), on the angular velocity $\omega \in (\omega_{\rm min}, \omega_{\rm max})$ with which a particle on an equatorial circular orbit moves at various fixed radii close to the Kerr–Newman centre with (*a*) $a = Q = 0.3M$, (*b*) $a = Q = 0.72M$. The broken curves indicate the positions of the local extrema of the curves ('extremally accelerated observers'), ω_0, given by equation (11). (The fact that the lower of these—showing the maxima of $a^r(\omega)$ in the 'classical' region above the outer photon orbit—is at non-zero ω *arbitrarily far* from the source was noticed by [19, 14].) The upper and lower full curves going *across* the curves $a^r(\omega)$ represent respectively the zero-angular-momentum observers and the Carter observers. Passing from the upper right to the lower left (following for example the line of ZAMOs), one crosses the curves of $a^r(\omega)$ at $r = r_{\rm min}$, $r_{\rm min} + 0.1M$, $r_{\rm min} + 0.2M, \ldots, 5.005M$, where $r_{\rm min} = r_+ + 0.005M$ for a black hole ($a^2 + Q^2 \leqslant M^2$; case a)) and $r_{\rm min} = 0.005M$ for a naked singularity ($a^2 + Q^2 > M^2$; case b)). At the photon orbits, $r = r_{\rm ph\mp}$, the derivatives $[\partial a^r/\partial\omega](\omega \to \omega_{\rm min,max})$ change the sign. The values of both ω and a^r are in units of M^{-1}.

exist in the Kerr–Newman field [7] (though not above the coordinate radius r_+ of the outer horizon in black-hole case). However, it is not our aim here to give a detailed analysis of these extra features.

A particle is kept on its orbit purely by interaction with the Kerr–Newman source if

$$ma^\mu = eF^\mu{}_\nu u^\nu, \tag{12}$$

where m and e are the rest mass and electric charge of the particle, and the electromagnetic field tensor $F_{\mu\nu}$ in Boyer–Lindquist coordinates has non-zero components (see [17, p 878])

$$F_{tr} = Q\Sigma^{-2}(\Sigma - 2r^2), \qquad F_{r\phi} = Q\Sigma^{-2}(\Sigma - 2r^2)a\sin^2\theta, \tag{13}$$

$$F_{t\theta} = Q\Sigma^{-2}ra^2\sin 2\theta, \qquad F_{\theta\phi} = Q\Sigma^{-2}(r^2 + a^2)ra\sin 2\theta. \tag{14}$$

For the circular orbit in the equatorial plane the radial component of (12) yields

$$mu^t[(Mr - Q^2)(1 - a\omega)^2 - r^4\omega^2] = eQr(1 - a\omega). \tag{15}$$

For a given source and radius this equation determines the electric charge e which the particle has to have in order for it to circulate with angular velocity ω. By plotting the dependence $e(\omega)$ for different radii near the centre, the counter-intuitive 'rotosphere effect' is clearly demonstrated (see figure 2).

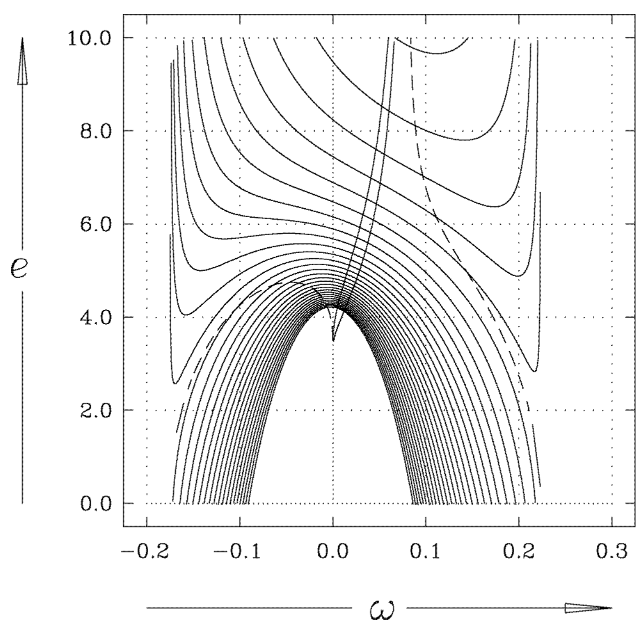

Figure 2. The dependence on ω of the electric charge e which the equatorial stationary particle orbiting at various radii close to the Kerr–Newman centre with $a = Q = 0.3M$ needs to be kept on its orbit purely by the (gravitational + electromagnetic) interaction with the source (cf equation (15)). The meaning of all the curves is otherwise the same as in figure 1. They go to $(a/(r_+^2 + a^2), +\infty)$ at the horizon and reach $(0, mM/Q)$ at asymptotic radii. The differences between figures 1(a) and 2 are discussed at the end of section 4.1. In connection with the effect discussed in [19, 14] it is interesting (though, of course, expectable) that also the maxima of the curves $e(\omega)$, such as those of $a^r(\omega)$ in figure 1, lie at $\omega \neq 0$ arbitrarily far from the source. Notice, however, that the extremally accelerated observers, indicated again by broken curves, do *not* coincide with the extrema of the curves $e(\omega)$. The values of ω are in units of M^{-1}, those of e in units of m.

The effect is not bound just to the equatorial plane—the 'ϕ-rotospheres', similarly to the ergospheres, are in fact toroidal [20]. However, we will not treat the off-equatorial circular orbits here because the charge of the particle with given ω which is needed to keep it at given $\theta \in (0^\circ, 90^\circ)$ is in general different to that needed to keep the particle at given r†. This makes the illustration of the rotosphere effect less clear. However, the effect can simply be seen also for the spherical polar orbits.

3. Charged particle on a zero-angular-momentum spherical polar orbit

Consider the particles on spherical ($r =$ constant) orbits moving uniformly in the θ-direction with the latitudinal angular velocity $\Omega = \mathrm{d}\theta/\mathrm{d}t$ ($= \pm|\Omega|$, $|\Omega| =$ constant), and in the ϕ-direction with the azimuthal angular velocity $\omega(\theta)$. They have the 4-velocity

$$u^\mu = u^t(1, 0, \Omega, \omega) \tag{16}$$

and their 4-acceleration is given by

$$\begin{aligned} a^t &= \Sigma^{-2}(u^t)^4\Omega\sin\theta\cos\theta\{\Sigma\mathcal{A}(\omega-\omega_{\mathrm{KN}})\omega_{,\theta}\tan\theta + (2Mr-Q^2)[a-(r^2+a^2)\omega]^2 \\ &\quad -2a^2(2Mr-Q^2)(1-a\omega\sin^2\theta)(u^t)^{-2} + \Sigma^2(\Delta\omega^2 - a^2\Omega^2)\}, \end{aligned} \tag{17}$$

$$a^r = \Delta\Sigma^{-3}(u^t)^2\{[M(2r^2-\Sigma) - rQ^2](1-a\omega\sin^2\theta)^2 - r\Sigma^2(\omega^2\sin^2\theta + \Omega^2)\}, \tag{18}$$

$$\begin{aligned} a^\theta &= \Delta\mathcal{A}^{-1}\Omega^{-1}a^t - \Sigma^{-3}(u^t)^2\sin\theta\cos\theta(\omega-\omega_{\mathrm{KN}}) \\ &\quad \times[\Sigma\mathcal{A}\omega_{,\theta}\tan\theta + 2(r^2+a^2)\mathcal{A}(\omega-\omega_{\mathrm{KN}}) - 2\Delta\Sigma\omega a^2\sin^2\theta], \end{aligned} \tag{19}$$

$$\begin{aligned} a^\phi &= \omega a^t + (u^t)^2\Omega(\omega-\omega_{\mathrm{KN}})_{,\theta} + 2\Sigma^{-2}(u^t)^2\Omega(\omega-\omega_{\mathrm{KN}})\cot\theta \\ &\quad \times\{\mathcal{A} - \Sigma a^2\sin^2\theta + \mathcal{A}a\omega_{\mathrm{KN}}\sin^2\theta[1-a(\omega+\omega_{\mathrm{KN}})\sin^2\theta]\}, \end{aligned} \tag{20}$$

where

$$(u^t)^{-2} = 1 - \frac{2Mr-Q^2}{\Sigma}(1-a\omega\sin^2\theta)^2 - (r^2+a^2)\omega^2\sin^2\theta - \Sigma\Omega^2. \tag{21}$$

If these particles have zero azimuthal angular momentum, then $\omega = \omega_{\mathrm{KN}}$, $(u^t)^{-2} = \Sigma(\Delta/\mathcal{A} - \Omega^2)$, and their acceleration simplifies to

$$a^t = -\Sigma^{-1}\mathcal{A}^{-1}(u^t)^2\Omega a^2\sin\theta\cos\theta[(2Mr-Q^2)(r^2+a^2) + \Delta\Sigma^2\Omega^2(u^t)^2], \tag{22}$$

$$\begin{aligned} a^r &= \Delta\Sigma^{-1}\mathcal{A}^{-2}(u^t)^2\{(r^2+a^2)^2[M(2r^2-\Sigma) - rQ^2] \\ &\quad -r(2Mr-Q^2)^2a^2\sin^2\theta - r\mathcal{A}^2\Omega^2\}, \end{aligned} \tag{23}$$

$$a^\theta = \Delta\mathcal{A}^{-1}\Omega^{-1}a^t, \tag{24}$$

$$a^\phi = \omega_{\mathrm{KN}}a^t. \tag{25}$$

Similarly to the Kerr case [22], far from the source the dependence of a^r (given by equation (23)) on $|\Omega|$ is intuitive, i.e. $\partial a^r/\partial|\Omega| < 0$, but it becomes counter-intuitive, $\partial a^r/\partial|\Omega| > 0$, below the radius of a photon spherical polar orbit. The radius of this orbit is given by the equation

$$r(\Delta + Q^2) - M(r^2 - a^2) = r^2(r-3M) + (r+M)a^2 + 2rQ^2 = 0, \tag{26}$$

the solution of which is briefly discussed in the appendix. From equation (23) we see that the sign of $\partial a^r/\partial|\Omega|$ does not depend on θ. Hence, the boundary of the region (the 'θ-rotosphere') where $\partial a^r/\partial|\Omega|$ is positive (i.e. counter-intuitive) is given by $r =$ constant. Since there is no dragging in the θ-direction, the curves of $a^r(\Omega; r)$ are symmetric about

† For the only exceptions, see [9] where specific off-equatorial circular orbits maintained by the electromagnetic interaction with the centre are found.

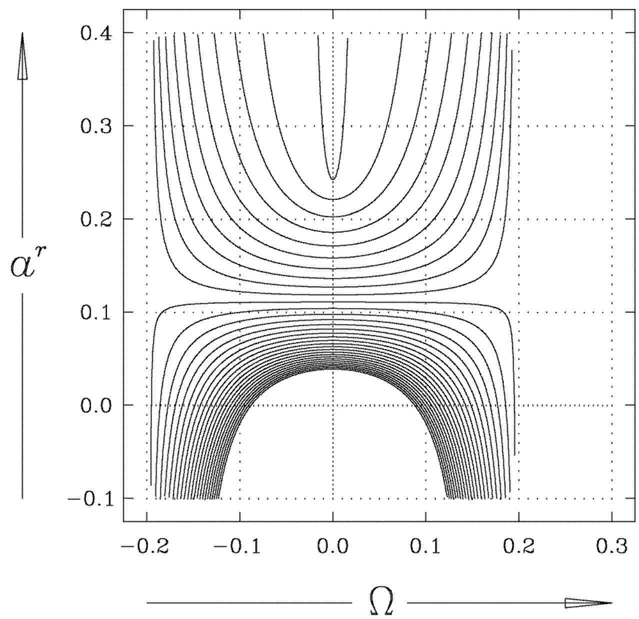

Figure 3. The dependence of $a^r(\theta = 0^\circ)$ (given by equation (23)) on the latitudinal angular velocity Ω of a particle orbiting with zero angular momentum on a spherical polar trajectory at various radii close to the Kerr–Newman centre with $a = Q = 0.3M$. The curves shown are plotted for the same radii as those in figure 1. At the photon orbit $\partial a^r/\partial|\Omega|$ changes the sign. The values of both Ω and a^r are in units of M^{-1}.

$\Omega = 0$ (figure 3). We note that the boundaries of the θ-rotospheres are *not* given by $r =$ constant in general when $\omega \neq \omega_{\rm KN}$.

If we try again to find what charge e the particle must have to remain on a given zero-angular-momentum spherical polar trajectory without any additional thrust, we discover that different components of the equation of motion (12) imply different *functions* $e(\theta)$. Therefore, in contrast to the orbits in the equatorial plane, the charge can give the prescribed acceleration only at some specific θ. Let us choose the axis of symmetry, $\theta = 0^\circ$ or 180°, where $a^t = a^\theta = a^\phi = 0$ (another simple choice, of course, would be $\theta = 90^\circ$). On the axis the required e is given unambiguously by the radial component of equation (12), i.e. by the relation

$$mu^t[M(r^2 - a^2) - rQ^2 - r(r^2 + a^2)^2\Omega^2] = eQ(r^2 - a^2). \tag{27}$$

The dependence of e on Ω at different radii illustrates the presence of the rotosphere effect below the radius of a photon spherical polar geodesic (see figure 4).

4. Interpretation in terms of forces

The results described above can be well understood in terms of quantities measured by the zero-angular-momentum observers (ZAMOs) with respect to their local orthonormal frames (locally non-rotating frames, LNRFs). The velocity of the particle with respect to the local ZAMO, $\hat{v}^\mu$, is related to its 4-velocity by

$$u^\mu = \hat{\gamma}(u^\mu_{\rm ZAMO} + \hat{v}^\mu), \tag{28}$$

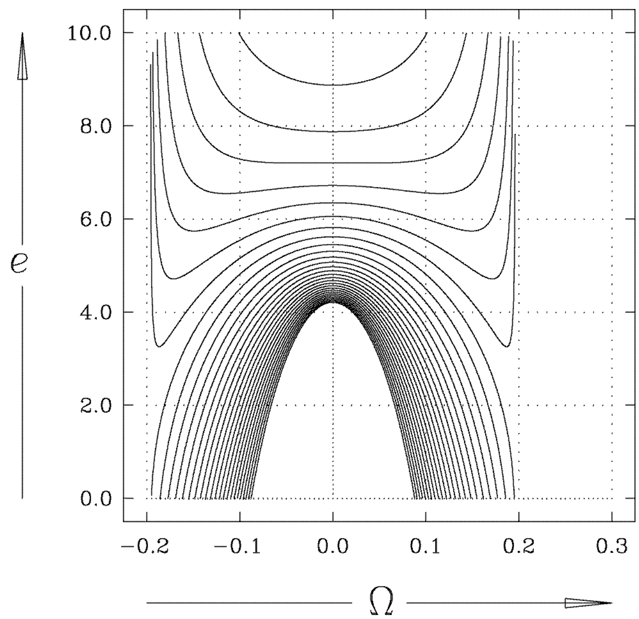

Figure 4. The dependence on $|\Omega|$ of the electric charge e which the particle orbiting on a spherical polar orbit about the Kerr–Newman centre with $a = Q = 0.3M$ needs to be kept on its orbit is drawn for the positions on the axis at the same radii as in figure 3 (e is given by equation (27)). The curves go to $(0,+\infty)$ at the horizon and reach $(0, mM/Q)$ at asymptotic radii. The values of Ω are in units of M^{-1}, those of e in units of m.

where

$$u^{\mu}_{\rm ZAMO} = \sqrt{\Delta^{-1}\Sigma^{-1}\mathcal{A}}(1, 0, 0, \omega_{\rm KN}), \tag{29}$$

$$\hat{\gamma} = (1-\hat{v}^2)^{-1/2} = -u_{\sigma}u^{\sigma}_{\rm ZAMO} = u^t/u^t_{\rm ZAMO}, \tag{30}$$

$\hat{v}^2 = \hat{v}_{\sigma}\hat{v}^{\sigma} = \hat{v}_k\hat{v}^k$ and $\hat{v}_{\sigma}u^{\sigma}_{\rm ZAMO} = 0$ (the hat means 'as measured in LNRF').

According to the definition of forces given for a motion in a general spacetime in [21], we can rewrite equation (12) in the 3-vector 'classical' form as a balance between the gravitational, the dragging, the Coriolis, the centrifugal, the 'tangent-inertial-resistance', and the Lorentz forces,

$$m(\vec{a}_{\rm g} + \vec{a}_{\rm d} + \vec{a}_{\rm C} + \vec{a}_{\rm cf} + \vec{a}_{\rm ti}) = \vec{F}_{\rm L}, \tag{31}$$

where

$$a^{\mu}_{\rm g} = \hat{\gamma}^2 a^{\mu}_{\rm ZAMO}, \tag{32}$$

$$a^{\mu}_{\rm d} = \hat{\gamma}^2\hat{v}^{\nu}\nabla_{\nu}u^{\mu}_{\rm ZAMO}, \tag{33}$$

$$a^{\mu}_{\rm C} = \hat{\gamma}^2[u^{\mu}_{\rm ZAMO}\hat{v}_{\nu}a^{\nu}_{\rm ZAMO} + (\vec{\Omega}_{\rm LNRF}\times\vec{\hat{v}})^{\mu}], \tag{34}$$

$$a^{\mu}_{\rm cf} = \hat{\gamma}\hat{v}{\rm D}\hat{v}^{\mu}_0/{\rm d}\tau - a^{\mu}_{\rm C}, \tag{35}$$

$$a^{\mu}_{\rm ti} = \hat{\gamma}^3(\hat{v}^{\mu}_0 + \hat{v}u^{\mu}_{\rm ZAMO}){\rm D}\hat{v}/{\rm d}\tau, \tag{36}$$

and the arrows denote the spatial parts of the corresponding contravariant 4-vectors. In the above formulae the ZAMO's 4-acceleration is given by

$$a^{\mu}_{\rm ZAMO} = \Sigma^{-2}\mathcal{A}^{-1}(0, (r^2+a^2)^2[M(2r^2-\Sigma) - rQ^2] - r(2Mr-Q^2)^2a^2\sin^2\theta, \\ (r^2+a^2)(Q^2-2Mr)a^2\sin\theta\cos\theta, 0), \tag{37}$$

the angular velocity of the LNRF relative to the (Fermi–Walker transported) gyroscopes carried by the ZAMO reads

$$\Omega^{\mu}_{\rm LNRF} = \Sigma^{-2}\mathcal{A}^{-1}a\sin\theta(0, \Delta(Q^2-2Mr)a^2\sin\theta\cos\theta,\\ r(r^2+a^2+\Sigma)(Q^2-2Mr)+M\Sigma(r^2+a^2), 0), \tag{38}$$

and the vector product is defined by

$$(\vec{\Omega}_{\rm LNRF}\times\vec{\hat{v}})^{\mu} = u_{{\rm ZAMO}\nu}\epsilon^{\nu\mu}{}_{\rho\sigma}\Omega^{\rho}_{\rm LNRF}\hat{v}^{\sigma}, \tag{39}$$

with $\epsilon^{\nu\mu}{}_{\rho\sigma}$ being the Levi-Civita tensor and $\hat{v}^{\mu}_0 = \hat{v}^{\mu}/\hat{v}$. The Lorentz force on the right-hand side of (31) has the usual form

$$\vec{F}_{\rm L} = \hat{\gamma}e(\vec{\hat{E}}+\vec{\hat{v}}\times\vec{\hat{B}}), \tag{40}$$

where the electric and the magnetic fields felt by ZAMO have the components

$$\hat{E}^i = Q\Sigma^{-3}(u^t_{\rm ZAMO})^{-1}((r^2+a^2)(2r^2-\Sigma), -a^2r\sin 2\theta, 0), \tag{41}$$

$$\hat{B}^i = aQ\Sigma^{-3}(u^t_{\rm ZAMO})^{-1}(2r(r^2+a^2)\cos\theta, (2r^2-\Sigma)\sin\theta, 0) \tag{42}$$

in Boyer–Lindquist coordinates.

4.1. Equatorial circular orbit

The particles on circular orbits move with the velocities

$$\hat{v}^i = u^t_{\rm ZAMO}(0, 0, \omega-\omega_{\rm KN}), \tag{43}$$

and with $\omega =$ constant equations (32)–(36) can be written in the form

$$\vec{a}_{\rm g} = \hat{\gamma}^2\vec{a}_{\rm ZAMO}, \tag{44}$$

$$\vec{a}_{\rm d} = \vec{a}_{\rm C} = \hat{\gamma}^2\vec{\Omega}_{\rm LNRF}\times\vec{\hat{v}}, \tag{45}$$

$$\vec{a}_{\rm cf} = \hat{\gamma}^2\hat{v}^2\vec{n}, \tag{46}$$

$$\vec{a}_{\rm ti} = \vec{0}, \tag{47}$$

where the (interior) normal $\vec{n}$ to the projection of the particle's trajectory into the local ZAMO's 3-space is given by

$$n^i = \Gamma^i_{\phi\phi}/g_{\phi\phi} = -\Sigma^{-2}\mathcal{A}^{-1}(\Delta\{r\Sigma^2+[rQ^2-M(2r^2-\Sigma)]a^2\sin^2\theta\},\\ [\Delta\Sigma^2+(2Mr-Q^2)(r^2+a^2)^2]\cot\theta, 0). \tag{48}$$

It is seen that as $\omega\to\omega_{\rm min,max}$ (i.e. $\hat{v}\to 1$), all the components (44)–(46) individually diverge as $\hat{\gamma}^2\to\infty$. Whether their sum then diverges to minus or plus infinity, i.e. whether we are in an 'intuitive' region or inside the ϕ-rotosphere, is determined by the interplay between $\vec{a}_{\rm ZAMO}$, $2\vec{\Omega}_{\rm LNRF}\times\vec{\hat{v}}$, and $\hat{v}^2\vec{n}$.

In the equatorial plane the latitudinal components of all the forces vanish and for the radial components we obtain

$$a^r_{\rm ZAMO} = \frac{1}{r^3}\frac{(Mr-Q^2)(r^2+a^2)^2-(2Mr-Q^2)^2a^2}{r^2(r^2+a^2)+(2Mr-Q^2)a^2}, \tag{49}$$

$$2(\vec{\Omega}_{\rm LNRF}\times\vec{\hat{v}})^r = -2a\frac{\sqrt{\Delta}}{r^3}\frac{(Mr-Q^2)(3r^2+a^2)+Q^2r^2}{r^2(r^2+a^2)+(2Mr-Q^2)a^2}\hat{v}^{\hat{\phi}}\\ = -2ar^{-5}(\omega-\omega_{\rm KN})[(Mr-Q^2)(3r^2+a^2)+Q^2r^2], \tag{50}$$

$$\hat{v}^2 n^r = -\frac{\Delta}{r^3}\frac{r^4+(Q^2-Mr)a^2}{r^2(r^2+a^2)+(2Mr-Q^2)a^2}(\hat{v}^{\hat{\phi}})^2$$
$$= -r^{-7}(\omega-\omega_{\rm KN})^2[r^4+(Q^2-Mr)a^2][r^2(r^2+a^2)+(2Mr-Q^2)a^2], \tag{51}$$

where $\hat{v}^{\hat{\phi}}$ is the azimuthal component of the particle's velocity in the LNRF.

As discussed in [20], outside *Kerr* black holes a particle on the circular orbit in the equatorial plane is always attracted to the centre by the gravitational force and repelled by the centrifugal force, while the 'dragging + Coriolis' force ($\vec{a}_{\rm d}+\vec{a}_{\rm C}$) attracts the counter-rotating particles (having $\omega < \omega_{\rm K}$) and repels the co-rotating ones (with $\omega > \omega_{\rm K}$), so that the counter-rotating particles feel a stronger gravitational field than the co-rotating ones. According to equations (49)–(51) all this remains true outside Kerr–Newman black holes—figures 1(a) and (b) can be given the same interpretation as figures 4(a)–(c) in [19], describing the Kerr situation. The reversal in the dependence of the particle's acceleration on the angular velocity arises from the fact that, when approaching the horizon (at $r = r_+$), i.e. $\Delta \to 0$, or, $\omega \to \omega_{\rm KN} \to a/(r_+^2+a^2)$, then the dragging+Coriolis and centrifugal forces become weaker (and vanish in the limit $r \to r_+$) as compared with the gravitational force. Inside a certain radius (the boundary of the rotosphere), which is larger for counter-rotating than co-rotating particles due to the effect of the dragging+Coriolis force, the gravitational force fully predominates and due to its positive sign also the total radial acceleration a^r goes, in contrarst to intuition, to *plus* infinity at the limiting value of ω. In the limit of an *extreme* black hole ($a^2 + Q^2 = M^2$), even the gravitational term vanishes at the horizon at $r = M$, and so does the total a^r.

The radial component of the Lorentz force (40) in the equatorial plane reads

$$F_{\rm L}^r = \hat{\gamma} e Q r^{-3}\sqrt{\Delta/\mathcal{A}}(r^2+a^2-a\sqrt{\Delta}\hat{v}^{\hat{\phi}}). \tag{52}$$

To understand the plots $e(\omega; a, Q, r, \theta = 90^\circ)$ showing the dependence of the particle's charge on ω (see figure 2), one must consider the interplay between forces discussed above and the Lorentz force given by (52). As compared with figures 1(a) and (b), showing just $a^r(\omega)$, an extra feature arises in figure 2—the curves of $e(\omega)$ are 'more concave' than their counterparts in the plots of $a^r(\omega)$. This is best seen on the curves between the photon orbits: a^r monotonously decreases there with ω increasing from $\omega_{\rm min}$ to $\omega_{\rm max}$, whereas some of the plots of e have local minima near their (counter-intuitive) divergences at $\omega_{\rm min}$.

This additional effect is purely due to the velocity dependence of the Lorentz force and can be best understood in the Reissner–Nordström field (see figure 5): with $a = 0$, we have $\vec{B} = \vec{0}$ and $\vec{\Omega}_{\rm LNRF} = \vec{0}$, and the radial component of equation (31) thus has the form

$$m\hat{\gamma}(a^r_{\rm ZAMO} + \hat{v}^2 n^r) = e\hat{E}^r. \tag{53}$$

Since then $a^r_{\rm ZAMO} = r^{-2}(M - Q^2/r)$, $n^r = -\Delta/r^3$ and $\hat{E}^r = \sqrt{\Delta}Q/r^3$, one finds that

$$\frac{\partial e}{\partial \hat{v}^2} = -\frac{m\hat{\gamma}^3}{2Q\sqrt{\Delta}}[(r-r_{\rm ph1})(r-r_{\rm ph2}) + \Delta(1-\hat{v}^2)], \tag{54}$$

where the radii of the photon orbits $r_{\rm ph1} > r_{\rm ph2}$ (only the first being above the horizon) are solutions of $r^2 - 3Mr + 2Q^2 = 0$. Clearly, at $r > r_{\rm ph1}$ the derivative $\partial e/\partial \hat{v}^2$ is always negative, i.e. intuitive. At $r < r_{\rm ph1}$, it is positive: (i) close to the horizon (where Δ is negligible) and (ii) for ultrarelativistic particles for which $1 - \hat{v}^2$ is negligible. The extra feature not present in the behaviour of a^r (which is qualitatively the same as that in the Schwarzschild case) is that below the photon orbit there still exist the curves whose 'central'

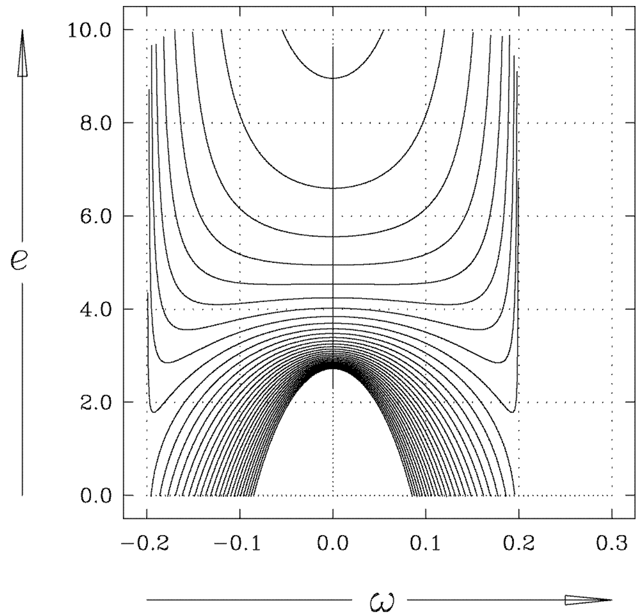

Figure 5. A similar pattern—of $e(\omega)$—to that in figure 2 (or 4), for the Reissner–Nordström black hole with $Q = 0.45M$. See the end of section 4.1 for a discussion.

parts, corresponding to non-ultrarelativistic velocities, are concave. This occurs near $r_{\rm ph1}$, where $(r - r_{\rm ph1})(r - r_{\rm ph2}) < 0$ is small and $\Delta(1 - \hat{v}^2) > 0$ makes expression (54) negative. Note that in the bracket in (54) the term $(r - r_{\rm ph1})(r - r_{\rm ph2}) - \Delta = Q^2 - Mr < 0$ is the gravitational part, and $\Delta(2 - \hat{v}^2) > 0$ is the centrifugal part. Thus, the counter-intuitive features in the particle dynamics near the Reissner–Nordström black hole appear because

$$\frac{\partial a^r_{\rm g}/\partial \hat{v}^2}{\partial a^r_{\rm cf}/\partial \hat{v}^2} \to -\infty \qquad \text{for } r \to r_+.$$

An alternative interpretation of the rotosphere effect in the Reissner–Nordström field, based on a different definition of the gravitational and centrifugal forces [1], was suggested in [3].

4.2. Zero-angular-momentum spherical polar orbit

Such an orbit is purely latitudinal in the LNRF,

$$\hat{v}^i = u^t_{\rm ZAMO}(0, \Omega, 0), \tag{55}$$

and equations (32)–(36) with $|\Omega| = \text{constant}$ produce the form

$$\vec{a}_{\rm g} = \hat{\gamma}^2 \vec{a}_{\rm ZAMO}, \tag{56}$$

$$\vec{a}_{\rm d} = -\vec{a}_{\rm C} = -(u^t)^2 \vec{\Omega}_{\rm LNRF} \times \vec{\hat{v}}, \tag{57}$$

$$\vec{a}_{\rm cf} = \hat{\gamma}^2 \hat{v}^2 \vec{n}, \tag{58}$$

$$\vec{a}_{\rm ti} = \hat{\gamma}^3 (\vec{\hat{v}}_0 + \hat{v}\vec{u}_{\rm ZAMO}) {\rm d}\hat{v}/{\rm d}\tau, \tag{59}$$

where the normal to the projected trajectory reads

$$n^i = \Gamma^i_{\theta\theta}/g_{\theta\theta} = -\Sigma^{-2}(r\Delta, a^2 \sin\theta \cos\theta, 0). \tag{60}$$

It is thus seen that for these orbits the effect of dragging is cancelled out by the Coriolis force.

From the explicit form of (59),

$$a^i_{\rm ti} = -\frac{\Delta\Omega^2 a^2 \sin 2\theta}{2\Sigma(\Delta - \mathcal{A}\Omega^2)^2}(0, \Delta, \mathcal{A}\omega_{\rm KN}\Omega), \tag{61}$$

we notice that $\vec{a}_{\rm ti} = \vec{0}$ on the axis of symmetry. Here $a^\theta_{\rm g}$ and $a^\theta_{\rm cf}$ also vanish and so the motion is governed by the interplay between

$$a^r_{\rm ZAMO} = \frac{M(r^2 - a^2) - rQ^2}{(r^2 + a^2)^2} \tag{62}$$

and

$$\hat{v}^2 n^r = -\frac{r\Delta}{(r^2 + a^2)^2}(\hat{v}^{\hat{\theta}})^2 = -r\Omega^2, \tag{63}$$

where $\hat{v}^{\hat{\theta}}$ is the latitudinal LNRF component of the velocity.

Thus, again we obtain the attractive gravitational force and repulsive centrifugal force. As in section 4.1, the reversal in the dependence of a^r on $|\Omega|$ is due to the fact that the gravitational force [$\sim \mathcal{O}(1)$] prevails over the centrifugal force [$\sim \mathcal{O}(\Delta)$] near the horizon.

The radial component of the Lorentz force $\vec{F}_{\rm L}$ on the axis has only electric part non-vanishing,

$$F^r_{\rm L} = \frac{\hat{\gamma} e Q}{(r^2 + a^2)^2}\sqrt{\frac{\Delta}{r^2 + a^2}}(r^2 - a^2), \tag{64}$$

and one arrives at the equation of motion of the same form as equation (53) which is valid to circular orbits in the equatorial plane. Then

$$\frac{\partial e}{\partial \hat{v}^2} = -\frac{m\hat{\gamma}^3}{2Q\sqrt{\Delta}}\frac{\sqrt{r^2 + a^2}}{r^2 - a^2}[r(\Delta + Q^2) - M(r^2 - a^2) + r\Delta(1 - \hat{v}^2)], \tag{65}$$

which has a similar structure as expression (54)—the first part in the brackets has its roots just at the photon spherical polar geodesics (cf equation (26)). The discussion of the Reissner–Nordström limit of the above results would correspond to that given at the end of the previous section, so it will not be repeated here.

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Appendix. The θ-rotosphere for $\omega = \omega_{\rm KN}$

According to equation (26), there is only one photon spherical polar geodesic outside black holes. Its radius, given by (see [11])

$$r_{\rm ph>} = M + 2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\arccos\frac{-q/2}{(-p/3)^{3/2}}\right], \tag{66}$$

where

$$p = -(3M^2 - 2Q^2 - a^2)(< 0), \qquad q = -2M(M^2 - Q^2 - a^2)(\leqslant 0), \tag{67}$$

decreases from $r_{\text{ph}>} = 3M$ for $Q = a = 0$ to $r_{\text{ph}>} = M + \sqrt{M^2 + a^2}$ for the extreme case with $Q^2 + a^2 = M^2$ $(q = 0)$. The θ-rotosphere is the region between the outer horizon, $r_+ = M + \sqrt{M^2 - Q^2 - a^2}$, and $r_{\text{ph}>}(> r_+)$. When $Q^2 + a^2$ is greater than M^2 (the source is thus naked), the other root of equation (26) becomes the lower limit of the θ-rotosphere. It reads

$$r_{\text{ph}<} = M + 2\sqrt{-\frac{p}{3}}\cos\left[120^\circ - \frac{1}{3}\arccos\frac{-q/2}{(-p/3)^{3/2}}\right], \tag{68}$$

and starts from the horizon's last position, $r = M$, to meet the still-decreasing outer root (66) at

$$r_{\text{ph}0} = M(1 + \sqrt[3]{Q^2/M^2 + a^2/M^2 - 1}) \tag{69}$$

when $-(p/3)^3 - (q/2)^2$ falls to zero. For still greater $Q^2 + a^2$ (giving negative $-(p/3)^3 - (q/2)^2$), there is no region of $\partial a^r/\partial|\Omega| > 0$ because then equation (26) has no positive roots. For example, in the Reissner–Nordström case the θ-rotosphere disappears at $r_{\text{ph}0} = \frac{3}{2}M$ when $Q^2 = \frac{9}{8}M^2$, in the case of $a^2 = Q^2$ it disappears at $r_{\text{ph}0} = M[1 + (\sqrt{5} - 2)^{1/3}] \doteq 1.62M$ when $Q^2 + a^2 = (\sqrt{5} - 1)M^2$, and in the Kerr case it disappears at $r_{\text{ph}0} = \sqrt{3}M$ when $a^2 = 3M^2(2\sqrt{3} - 3)$ (see [22]).

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