GEODESIC CHAOS AROUND BLACK HOLES WITH DISCS

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In the fields of isolated stationary black holes, the geodesic dynamics is regular. However, due to the presence of unstable periodic orbits, it easily becomes chaotic under various perturbations. Here we study the chaos induced by the presence of an additional source in the Schwarzschild space-time. Following the astrophysical motivation, we consider thin discs or rings lying symmetrically around the hole and describe the total fields in terms of exact static and axially symmetric solutions of Einstein's equations. The character of geodesic dynamics can be revealed by Poincaré sections, Fourier spectra of the "vertical"-position time series and evolution of the test-particle "latitudinal action".

Keywords: Black holes; exact solutions; geodesics; chaos.

"Almost all" is known about isolated and stationary black holes in asymptotically flat space-times. But "real", astrophysical black holes, assumed to play a central role in most galactic nuclei, some X-ray binaries and gamma-ray bursts, are neither isolated nor stationary, and probably not living in an asymptotically flat universe. Apropos, all the observational data about them come from their *interaction* with matter and fields around. In order to describe such processes in a consistent manner, the gravity of this matter/fields should be incorporated. Concerning the gravitational potential and its gradient ("field"), the "external" contribution is almost certainly negligible, but higher derivatives ("curvature") may be affected much more. Moreover, some features of motion of the external matter (mainly its stability) are highly sensitive to the details of the field, and thus to its own effect. Self-gravitating matter may thus settle down to a different accretion configuration than the test one.

One of indicators of the influence of additional matter should be geodesic dynamics. Namely, (uncharged) isolated stationary black holes in asymptotically flat universes are necessarily described by the Kerr metric whose geodesics are completely integrable (bound by 4 isolating integrals) [1], whereas the metric perturbed by the additional matter no longer yields "the 4th, Carter's" integral and the test motion thus may become chaotic. This is even true when the matter leaves the space-time stationary and axially symmetric, like in the case of an equatorial disc or ring. We point out such most symmetric configurations, since only they lead to a manageable form of Einstein equations. Fortunately, they are also astrophysically relevant as they may approximate a steady accretion flow onto a black hole.

This note summarises some of our results on the response of geodesic dynamics to the presence of a disc or ring, obtained within exact static and axisymmetric space-times. Restricting to the case without any EM field and with zero cosmological constant, the corresponding metric can be put into the Weyl form

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu}\mathrm{d}t^2 + \rho^2\mathrm{e}^{-2\nu}\mathrm{d}\phi^2 + \mathrm{e}^{2\lambda-2\nu}(\mathrm{d}\rho^2 + \mathrm{d}z^2),$$

where t, ϕ are tied to the symmetries and ρ, z cover isotropically the meridional plane; the functions ν and λ depend only on ρ, z. [Nevertheless, regarding the presence of the horizon, it is natural to put the results in Schwarzschild coordinates r, θ, given by $\rho = \sqrt{r(r-2M)}\,\sin\theta$, $z = (r-M)\cos\theta$.]

We specifically considered thin annular discs from the inverted counter-rotating Morgan–Morgan family and also the ("more smooth") discs whose density is power-law in Weyl radius. For such thin sources, the space-time is vacuum everywhere, with the energy-momentum tensor related to the jump of the normal derivatives of metric across the source. Einstein equations then reduce to the Laplace equation for ν and to the quadrature for λ (which in general has to be computed numerically at each point). The fields of multiple sources can thus be obtained by adding the respective component potentials ν and then integrating out λ from this "total" ν. Superpositions of both families of discs with Schwarzschild were shown to contain broad ranges of parameters where the results have reasonable physical properties.

Starting from a pure Schwarzschild field, we examined how the phase space of free test motion evolves with parameters (mass and radius) of the external source and with energy of the (massive) test particles. The results were presented in [2], together with a thorough introduction and a number of figures. We tracked the geodesic phase space of the hole+disc field by several hundreds of particles launched from a chain of relevant radii in the equatorial plane, with given energy $(-u_t)$ and angular momentum (u_ϕ) and with several different values of u^r. Using the Runge–Kutta (actually the Huťa) 6th-order method to integrate numerically the geodesic equations, the particles were followed for about $150000M$–$250000M$ of proper time (usually some 200–1000 orbital periods) and their passages across the equatorial plane $\theta = \pi/2$ (it is the plane of symmetry of our sources) were recorded in terms of r and u^r. The sequences of Poincaré sections obtained by changing the parameters revealed typical features of a "nearly integrable" ("weakly perturbed") Hamiltonian dynamical system that behaves according to the KAM theory. As expected, the degree of its stochasticity grows with compactness and relative mass of the external source and with energy of the particles. However, for large values of these parameters, the dynamics rather inclines back towards regular regime.

Besides the overall tendencies in dependence of phase portrait on parameters, it is also interesting to study particular trajectories, trying to distinguish different degrees and types of their irregularity. Various methods can be followed of which we sampled time series of phase variables and their spectra so far. For example, we plotted the time series of vertical position (z) and their Fourier image, and also the evolution of the particle's "latitudinal action" $J_\theta \equiv \frac{1}{\Delta\tau}\oint \sqrt{g_{\theta\theta}u^\theta u^\theta}\,\mathrm{d}\tau$, obtained by averaging the latitudinal four-velocity over an orbital period ($\Delta\tau$). (For these purposes, we followed the geodesics up to a coordinate time of about $10^6 M$ which typically corresponds to several thousand orbital periods.) Along regular orbits, the quantities undergo regular oscillations around certain constant mean value, so the respective Fourier spectra only contain several distinctive peaks and fall off to a flat profile at low frequencies. Chaotic orbits, on the other hand, correspond

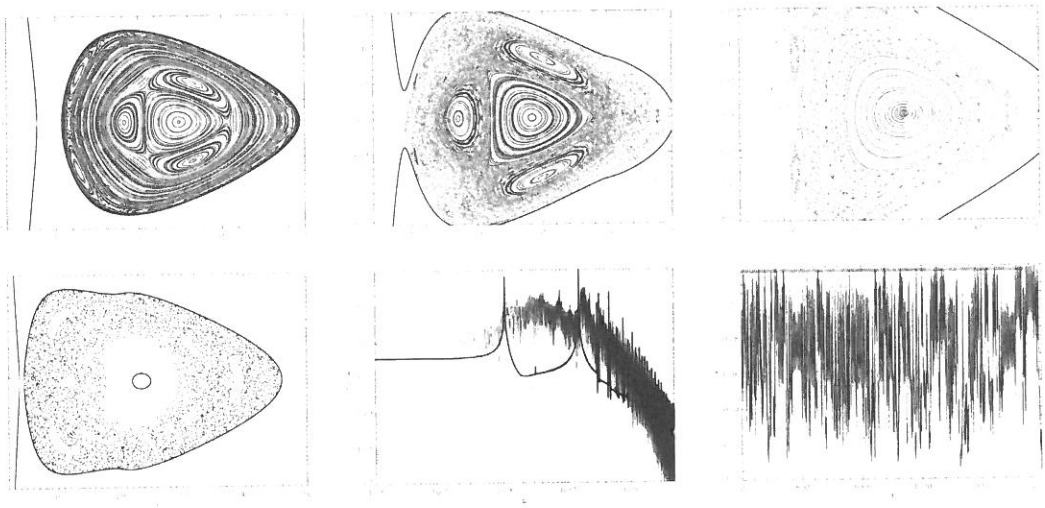

Fig. 1. Geodesic dynamics in the field of a Schwarzschild black hole (of mass M) surrounded by the inverted 1st Morgan-Morgan disc with mass $0.5M$ and inner radius r=$20M$. *Top row:* the dynamics changes with energy $-u_t$ from almost regular (left, $-u_t$=0.952) to quite chaotic (middle, $-u_t$=0.956) and back to more regular (right, $-u_t$=0.975); angular momentum is set to u_ϕ=$3.75M$. *Bottom row:* distinguishing between regular and chaotic orbit (from left to right:) on Poincaré section (regular orbit makes a closed curve in the middle, chaotic one is filling the "sea"), on power spectrum of the z-evolution (the regular is flat at low frequencies, it is plotted thicker) and on evolution of the latitudinal action J_θ (for the regular orbit it oscillates about 0.117 evenly).

to complicated, inharmonic time series without any obvious mean value and with fuzzy Fourier spectra spanning over "all" frequencies including the low ones; the irregularity of their evolution can be detected at various scales.

Within the chaotic motions, one can further recognise those which adhere to regular orbits for most of the time and those which fill the "chaotic sea" rather uniformly. The "weakly chaotic" orbits typically produce "1/frequency" power spectrum at low frequencies (it ecreases with frequency), whereas "strongly chaotic" orbits produce "white-noise" there (fine jagged curve at rather low values). These two types of spectra, identified before on spin particles in a Schwarzschild field [3], appear in our system, too. We also confirmed that the "spectral type" is correlated with the behaviour of the latitudinal action J_θ, as suggested by [4].

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