Spinning particles in vacuum spacetimes of different curvature types

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We consider the motion of spinning test particles with nonzero rest mass in the "pole-dipole" approximation, as described by the Mathisson-Papapetrou-Dixon (MPD) equations, and examine its properties in dependence on the spin supplementary condition added to close the system. In order to better understand the spin-curvature interaction, the MPD equation of motion is decomposed in the orthonormal tetrad whose time vector is given by the four-velocity V^μ chosen to fix the spin condition (the "reference observer") and the first spatial vector by the corresponding spin s^μ; such projections do not contain the Weyl scalars Ψ_0 and Ψ_4 obtained in the associated Newman-Penrose (NP) null tetrad. One natural option of how to choose the remaining two spatial basis vectors is shown to follow "intrinsically" whenever V^μ has been chosen; it is realizable if the particle's four-velocity and four-momentum are not parallel. In order to see how the problem depends on the algebraic type of curvature, one first identifies the first vector of the NP tetrad k^μ with the highest-multiplicity principal null direction of the Weyl tensor, and then sets V^μ so that k^μ belong to the spin-bivector eigenplane. In spacetimes of any algebraic type but III, it is known to be possible to rotate the tetrads so as to become "transverse," namely so that Ψ_1 and Ψ_3 vanish. If the spin-bivector eigenplane could be made to coincide with the real-vector plane of any of such transverse frames, the spinning particle motion would consequently be fully determined by Ψ_2 and the cosmological constant; however, this can be managed in exceptional cases only. Besides focusing on specific Petrov types, we derive several sets of useful relations that are valid generally and check whether/how the exercise simplifies for some specific types of motion. The particular option of having four-velocity parallel to four-momentum is advocated, and a natural resolution of nonuniqueness of the corresponding reference observer V^μ is suggested.

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I. INTRODUCTION

Curvature of physical spacetime is the major prediction of general relativity, so it is of special interest to study processes in which curvature (nonhomogeneity of the gravitational field) plays a *direct* role. Being described by the Riemann tensor, the participation of curvature usually makes the problem difficult, at least in comparison with those involving just metric and/or affine connection. One such problem is the motion of extended bodies (e.g. Ref. [1]). Even if the body is treated as nonradiating, test, and small (with all the lengths connected with its multipoles much shorter than the spacetime curvature radius), the corresponding equations of motion even contain the Riemann tensor together with its derivatives. More explicit studies thus mostly restrict to the "spinning particle" limit (the "pole-dipole" approximation) when just monopole (mass) and dipole (rotational angular momentum, spin) are taken into account and the motion is described by the Mathisson-Papapetrou-Dixon (MPD) equations, supplemented by some "spin condition." The approximation is problematic in highly nonhomogeneous fields, mainly due to the disregard for the quadrupole effect, but we adhere to it here. Of important recent references in the field, let us recommend Refs. [2–6].

The curvature properties are often best revealed in a suitable orthonormal tetrad, namely as represented in terms of the "Weyl scalars"—independent projections of the Weyl tensor in the attached Newman-Penrose (NP) null tetrad, which can be given a physical interpretation. One can then understand the geometrical/physical effect of the individual curvature terms in dependence on the scalars, and especially discuss the situations when some of the scalars vanish—the *algebraically special* cases. Such studies actually began before the birth of the NP formalism (e.g. Refs. [7,8]) and since then have notably been devoted to geodesic deviation as a universal probe of gravitational field properties [9,10] or to the interpretation of spacetime perturbations [11].[1] Surprisingly, for the spinning particle problem, a similar discussion has been published in the massless case only [16]. (The gravito-electromagnetic parallel has been applied to it by Ref. [17].)

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[1] Another major curvature interpretation direction stems from the celebrated analogy between curvature tensors and the electromagnetic field tensor or electromagnetic tidal tensor—see Refs. [12–15], for example.

In the present paper, we will consider particles with nonzero mass. We keep the cosmological constant, but restrict to *vacuum* spacetimes (with zero energy-momentum tensors), since otherwise we would also have to incorporate interaction of the particle with matter and/or other physical fields, generally including torques exerted on its spin. This would certainly obscure the effects coming from curvature and its particular algebraic type. In the following Sec. II, we first recall the spinning particle problem, consider its basic properties, including the necessity to add a certain "spin supplementary condition," and write the equations down in terms of the spin vector and Riemann tensor dual. In Sec. III, the equation of motion is expressed in a suitable orthonormal as well as complex null (NP) frame, representing the Weyl-tensor dual in terms of its complex projections Ψ_0–Ψ_4 (details are shifted to the Appendix). We show that the problem itself provides a tetrad which can be used rather generally (specifically, if the particle's four-velocity and four-momentum are not parallel) and which gives very simple results, mainly in connection with Tulczyjew's supplementary condition.

In order to discuss the equation of motion in dependence on the spacetime Petrov type, the interpretation frame is then attached to the Weyl-tensor principal directions in Sec. IV. As an alternative to the above "intrinsic" tetrad (and the related null one), there arises a generic possibility (only not applicable in algebraic type III) to use "transverse frames" in which pure-gauge longitudinal wave effects vanish, but their special turn can be aligned with the spin structure only in exceptional cases. The effect of particular spin conditions is checked in Sec. V, and several special types of motion are discussed in Sec. VI. Concluding remarks close the paper by relating the topic to a wider context and providing some tips for improvement or an alternative view.

Conventions: We use the metric signature $(-+++)$ and geometrized units in which $c=1$, $G=1$. Greek indices run 0–3, and summation convention is followed. The dot denotes the absolute derivative with respect to the particle's proper time τ, and the overbar indicates complex conjugation. The Riemann tensor is defined according to $V_{\nu;\kappa\lambda}-V_{\nu;\lambda\kappa}=R^{\mu}{}_{\nu\kappa\lambda}V_{\mu}$, and the Levi-Civitá tensor as

$$\epsilon_{\mu\nu\rho\sigma}=\sqrt{-g}\,[\mu\nu\rho\sigma],\qquad \epsilon^{\mu\nu\rho\sigma}=-\frac{1}{\sqrt{-g}}\,[\mu\nu\rho\sigma],\tag{1}$$

where g is the covariant-metric determinant and $[\mu\nu\rho\sigma]$ is the permutation symbol fixed by $[0123]\coloneqq 1$.

II. MATHISSON-PAPAPETROU-DIXON (MPD) EQUATIONS

In their seminal papers [18–21], Mathisson, Papapetrou, Tulczyjew, and Dixon provided—following somewhat different approaches—a full multipole expansion for the extended-body evolution in general relativity. In greater detail, this problem has as yet been studied in the "pole-dipole" approximation, including the computation of generic trajectories in some basic spacetimes [22–25] (and of their dynamics over a corresponding phase space [26–33]), while a similar study at the quadrupole level has only commenced quite recently [34–38]. If there is no torque exerted on the particle, the pole-dipole Mathisson-Papapetrou-Dixon (MPD) system reads

$$\dot p^{\mu}=-\frac{1}{2}R^{\mu}{}_{\nu\kappa\lambda}u^{\nu}S^{\kappa\lambda},\tag{2}$$

$$\dot S^{\alpha\beta}=p^{\alpha}u^{\beta}-u^{\alpha}p^{\beta},\tag{3}$$

where u^{μ} is a tangent to the worldline which represents the particle's history (it is assumed to be timelike and normalized by $u_{\sigma}u^{\sigma}=-1$), $S^{\mu\nu}$ is the particle-spin bivector, p^{μ} denotes the total momentum (assumed to be timelike), and $m\coloneqq -u_{\sigma}p^{\sigma}(>0)$ is the particle's mass in the frame attached to the representative worldline. Let us also introduce the particle's mass in the frame given by its momentum, $\mathcal{M}\coloneqq\sqrt{-p_{\sigma}p^{\sigma}}$, and an associated four-velocity, $\mathcal{U}^{\mu}\coloneqq p^{\mu}/\mathcal{M}$.

The second MPD equation demonstrates that the tensor $\dot S^{\alpha\beta}$ is simple and timelike, with its blade spanned by p^{α} and u^{α}. It also implies that the spin-bivector dual

$${}^{*}S_{\mu\nu}\coloneqq\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}S^{\alpha\beta}$$

evolves according to

$${}^{*}\dot S_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\dot S^{\alpha\beta}=\epsilon_{\mu\nu\alpha\beta}p^{\alpha}u^{\beta},\tag{4}$$

since $\dot\epsilon_{\mu\nu\alpha\beta}=0$, and dualization thus commutes with covariant differentiation. (Hence, ${}^{*}\dot S_{\mu\nu}$ is simple and spacelike, having p^{μ} and u^{μ} as eigendirections corresponding to zero eigenvalues.) This in turn yields

$$\frac{1}{2}\dot S^{\alpha\beta}\dot S_{\alpha\beta}=-\frac{1}{2}{}^{*}\dot S^{\mu\nu}{}^{*}\dot S_{\mu\nu}=\mathcal{M}^2-m^2\le 0,\tag{5}$$

$${}^{*}\dot S^{\mu\iota}\dot S_{\iota\beta}=0.\tag{6}$$

Several useful relations can be obtained directly by projections of the second MPD equation (3): multiplication by u_{β}, p_{β}, $\dot u_{\beta}$, and $\dot p_{\beta}$ yields, respectively,

$$p^{\alpha}=mu^{\alpha}-\dot S^{\alpha\beta}u_{\beta},\tag{7}$$

$$\mathcal{M}^2u^{\alpha}=mp^{\alpha}+\dot S^{\alpha\beta}p_{\beta},\tag{8}$$

$$\dot m u^{\alpha}=\dot S^{\alpha\beta}\dot u_{\beta},\tag{9}$$

$$\mathcal{M}\dot{\mathcal{M}}u^{\alpha}=\dot S^{\alpha\beta}\dot p_{\beta},\tag{10}$$

where in the third case the basic property $u_\beta \dot{p}^\beta = 0$ of the first MPD equation (2) has been used ("four-force acts perpendicular to four-velocity"). The first two of these equations indicate a momentum-velocity relation (m can be supplied from the normalization $u_\alpha u^\alpha = -1$), while the last two lead to mass-evolution formulas:

$$\dot{m} = \dot{u}_\alpha u_\beta \dot{S}^{\alpha\beta} = \dot{u}_\alpha \frac{p_\beta}{m} \dot{S}^{\alpha\beta} = -\dot{u}_\alpha p^\alpha, \tag{11}$$

$$\mathcal{M}\dot{\mathcal{M}} = \dot{p}_\alpha u_\beta \dot{S}^{\alpha\beta} = \dot{p}_\alpha \frac{p_\beta}{m} \dot{S}^{\alpha\beta} = -\dot{p}_\alpha p^\alpha \tag{12}$$

(the last expressions follow from the very definitions of m and $\mathcal{M}$), but in both cases they provide only a partial answer since they contain the *derivative* of $S^{\alpha\beta}$. It was shown by Ref. [39] that the MPD equations can also be inverted to

$$\mathcal{M}^2 u^\alpha = m\left(\tilde{p}^\alpha + \frac{2S^{\alpha\beta} R_{\beta\iota\kappa\lambda} \tilde{p}^\iota S^{\kappa\lambda}}{4\mathcal{M}^2 + R_{\mu\nu\gamma\delta} S^{\mu\nu} S^{\gamma\delta}}\right), \tag{13}$$

where[2]

$$\tilde{p}^\alpha \coloneqq p^\alpha + \frac{1}{m}\frac{\mathrm{D}}{\mathrm{d}\tau}(S^{\alpha\beta} p_\beta), \tag{14}$$

with τ standing for proper time and m again fixed by $u_\alpha u^\alpha = -1$. This formula still contains $\dot{p}^\mu$ and $\dot{S}^{\mu\nu}$, and thus naturally depends on the solution of the MPD system (which in general cannot be given without adding a "spin supplementary condition"—see the following subsection), so it is also *not* an explicit momentum-velocity relation, similarly to equations (7) and (8). However, it at least shows clearly that such a closed relation does follow when $S^{\alpha\beta} p_\beta = 0$ (Tulczyjew's condition, see Sec. V A).

A. Spin supplementary condition

The effective nonzero size of the "particle," required by its nonzero multipole moments, implies freedom of its internal motion. On the pole-dipole level, this freedom has three degrees and corresponds to a possibility of selecting the representative worldline. A usual choice is to identify the latter with the particle's center of mass defined with respect to some physical observer. If such an observer is represented by a future-pointing timelike field V^μ, defined "within the body" (all along its history) and normalized without loss of generality as $V_\sigma V^\sigma = -1$, this means prescribing that the corresponding relative mass dipole is zero, $S^{\mu\sigma} V_\sigma = 0$, along that worldline (which is yet to be found, however). These three conditions close the MPD system, the freedom thus being translated into the choice of the reference observer V^μ.

Several specific choices of V^μ are natural and have proven advantageous, namely $V^\mu \equiv u^\mu$ (Mathisson-Pirani spin condition), $V^\mu \equiv \mathcal{U}^\mu$ (Tulczyjew's spin condition), $V^\mu = (V^t, 0, 0, 0)$ in a coordinate system adapted to given spacetime symmetries (Corinaldesi-Papapetrou spin condition), $V^\mu \equiv \mathcal{U}^\mu + N^\mu$, where N^μ is a normalized timelike direction used for $3+1$ splitting (Newton-Wigner spin condition, being employed in the Hamiltonian treatment [40,41]), and V^μ given by any parallel vector function along u^μ (which implies $u^\mu \| p^\mu$ and thus $m = \mathcal{M}$; see Ref. [24]). Different spin conditions have slightly different character and (naturally) lead to different representative worldlines, which has been stimulating discussions on how uniquely they determine the evolution and whether they actually describe "the same body"; see Ref. [6] for a recent thorough treatise on the nature and implications of the different choices.

B. Spin bivector and spin vector

If it satisfies $S^{\mu\sigma} V_\sigma = 0$ (its "electric part" vanishes), the spin bivector must be of rank 2 (must be *simple*), having just three independent components. With the reference observer V^μ selected, it is thus possible to introduce a spin vector ("magnetic part" of the bivector) by

$$s^\mu \coloneqq -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} V_\nu S_{\rho\sigma} = -{}^*S^{\mu\nu} V_\nu \tag{15}$$

$$\Leftrightarrow S_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} V^\gamma s^\delta, \qquad {}^*S^{\mu\nu} = s^\mu V^\nu - V^\mu s^\nu. \tag{16}$$

The vector s^μ is orthogonal to V^μ as well as to $S^{\mu\nu}$ by definition, from where also

$${}^*S^{\mu\iota} S_{\iota\beta} = 0. \tag{17}$$

In other words, the spin bivector $S^{\mu\nu}$ has two different (orthogonal) eigenvectors V^μ and s^μ tied to zero eigenvalues; these vectors span the blade of the dual bivector ${}^*S^{\mu\nu}$. Since V^μ is timelike by assumption, this dual blade is timelike (so ${}^*S^{\mu\nu}$ is timelike); hence the blade of $S^{\mu\nu}$, being orthogonal to the dual one, is spacelike.

It is useful to also calculate

$$S_{\alpha\beta} S^{\mu\beta} = s^2(\delta^\mu_\alpha + V^\mu V_\alpha - s^{-2} s^\mu s_\alpha), \tag{18}$$

$$-{}^*S_{\alpha\beta}{}^*S^{\mu\beta} = s^2(-V^\mu V_\alpha + s^{-2} s^\mu s_\alpha). \tag{19}$$

Therefore, $-{}^*S_{\alpha\beta}{}^*S^{\mu\beta}$ represents the s^2-multiple of the dual-blade metric, while $S_{\alpha\beta} S^{\mu\beta}$ represents the s^2-multiple of the metric of the blade (i.e. that of the surface orthogonal to both V^μ and s^μ).

The dual blade (the eigenplane of $S^{\mu\nu}$) can alternatively be spanned, instead of V^μ and s^μ, by null vectors

[2]Mind the opposite metric signature used in Ref. [39], resulting in the opposite sign of $S^{\alpha\beta} p_\beta$.

$$k^\mu := \frac{1}{\sqrt{2}}\left(V^\mu + \frac{s^\mu}{s}\right), \qquad l^\mu := \frac{1}{\sqrt{2}}\left(V^\mu - \frac{s^\mu}{s}\right),$$

where

$$s^2 := s_\mu s^\mu = \frac{1}{2} S_{\alpha\beta} S^{\alpha\beta} = -\frac{1}{2} {}^*S_{\kappa\lambda} {}^*S^{\kappa\lambda} \quad (>0)$$

stands for the spin magnitude squared. Clearly the vectors are normalized to $k_\mu l^\mu = -1$. In terms of the null vectors, the bivectors can be expressed as

$$S_{\alpha\beta} = -s\,\epsilon_{\alpha\beta\gamma\delta} k^\gamma l^\delta, \qquad {}^*S^{\mu\nu} = s(k^\mu l^\nu - l^\mu k^\nu). \tag{20}$$

This implies $S^{\mu\nu}k_\nu = 0$ and $S^{\mu\nu} l_\nu = 0$, while ${}^*S^{\mu\nu}k_\nu = -sk^\mu$ and ${}^*S^{\mu\nu} l_\nu = s l^\mu$, so k^μ and l^μ are eigenvectors of ${}^*S^{\mu\nu}$ as well (with eigenvalues $\mp s$).

The four-momentum p^α was previously extracted from (3) by multiplication by u_β or p_β, but it can now also be expressed in a different way if multiplying the equation by V_β:

$$\gamma p^\alpha = \mu u^\alpha + S^{\alpha\beta}\dot{V}_\beta, \tag{21}$$

where

$$\mu := -V_\sigma p^\sigma \quad (>0), \qquad \gamma := -V_\sigma u^\sigma \quad (>0) \tag{22}$$

are, respectively, the particle mass measured with respect to V^μ and the relative Lorentz factor between u^μ and V^μ. By multiplying this formula once more by u_α or p_α, one obtains relations between masses m, $\mathcal{M}$, and μ (and corresponding projections of $S^{\alpha\beta}$); multiplication by V_α gives just identity, while multiplication by s_α, $\dot{V}_\alpha$, and $\dot{s}_\alpha$, respectively, yields the important equalities

$$\gamma p^\alpha s_\alpha = \mu u^\alpha s_\alpha, \tag{23}$$

$$\gamma p^\alpha \dot{V}_\alpha = \mu u^\alpha \dot{V}_\alpha, \tag{24}$$

$$\gamma p^\alpha \dot{s}_\alpha = \mu u^\alpha \dot{s}_\alpha. \tag{25}$$

The last one actually follows due to the first two, because thanks to them the $\dot{s}_\alpha$-product of the last term of (21) gives zero, too:

$$\dot{s}_\alpha S^{\alpha\beta}\dot{V}_\beta = -s_\alpha \dot{S}^{\alpha\beta}\dot{V}_\beta = s_\alpha(u^\alpha p^\beta - p^\alpha u^\beta)\dot{V}_\beta = 0.$$

The above relations tell us that the vector $(\mu u^\mu - \gamma p^\mu)$ is orthogonal to s^μ, $\dot{V}^\mu$, and $\dot{s}^\mu$ (and to V^μ as well). Due to them, it is even possible to find a *quadruple* of *mutually orthogonal* vectors, (e.g.)

$$V^\mu, s^\mu, \mu u^\mu - \gamma p^\mu, (s^2\delta^\mu_\nu - s^\mu s_\nu)\dot{V}^\nu, \tag{26}$$

which can thus be used as a basis (we will indeed use it in Sec. III D). Another simple spacetime basis—"built on" u^μ instead of V^μ—will be added in Sec. II D. Several simple orthogonal triples can also be found and are useful, like

$$\{V^\mu, \dot{V}^\mu, \mu u^\mu - \gamma p^\mu\}, \qquad \{u^\mu, \mu\dot{u}^\mu - \gamma\dot{p}^\mu, \gamma s^\mu + s^\nu u_\nu V^\mu\}.$$

Note that the last of these vectors is orthogonal to p^μ besides, so it is orthogonal to both u^μ and p^μ, which means that it is an eigenvector of $\dot{S}^{\mu\nu}$ (with zero eigenvalue).

Equations (23)–(25) further imply

$$p^\alpha s_\alpha = 0 \Leftrightarrow u^\alpha s_\alpha = 0,$$
$$p^\alpha \dot{V}_\alpha = 0 \Leftrightarrow u^\alpha \dot{V}_\alpha = 0,$$
$$p^\alpha \dot{s}_\alpha = 0 \Leftrightarrow u^\alpha \dot{s}_\alpha = 0,$$

independently of the spin condition. Whenever s^μ *is* orthogonal to u^μ and p^μ, it means that

$$S^{\alpha\beta}\dot{s}_\beta = -\dot{S}^{\alpha\beta}s_\beta = 0,$$

so $\dot{s}^\mu$ then belongs to the eigenplane of $S^{\mu\nu}$—it is a combination of V^μ and s^μ. Conversely, s^μ belongs then to the eigenplane of $\dot{S}^{\mu\nu}$, the other independent eigendirection of the latter being given by $\epsilon^{\mu\iota\kappa\lambda}s_\iota u_\kappa p_\lambda$. Similarly, when $\dot{V}^\mu$ is orthogonal to u^μ and p^μ, it means that $\dot{S}^{\alpha\beta}\dot{V}_\beta = 0$, so $\dot{V}^\mu$ is the eigenvector of $\dot{S}^{\mu\nu}$, the other one being $\epsilon^{\mu\iota\kappa\lambda}\dot{V}_\iota u_\kappa p_\lambda$. And, finally, when $\dot{s}^\mu$ is orthogonal to u^μ and p^μ, it means that $\dot{S}^{\alpha\beta}\dot{s}_\beta = 0$, so $\dot{s}^\mu$ is the eigenvector of $\dot{S}^{\mu\nu}$, the other one being $\epsilon^{\mu\iota\kappa\lambda}\dot{s}_\iota u_\kappa p_\lambda$.

The above reasoning is clearly pointless if u^μ is parallel to p^μ; this circumstance will be discussed more in Sec. V C.

1. Hidden momentum

Inspired by the concept of "hidden momentum" used in electromagnetism, Ref. [42] introduced its "gravitational" counterpart analogously as the component of p^μ orthogonal to u^μ. Since we assume the particle is torque free, it is solely given by the chosen spin supplementary condition in our case (it is purely kinematical; see Ref. [6] for details),

$$p^\mu_{\rm hidden} := (\delta^\mu_\alpha + u^\mu u_\alpha)p^\alpha = p^\mu - m u^\mu = \tag{27}$$

$$= -\dot{S}^{\mu\nu}u_\nu = \frac{1}{\gamma}(\delta^\mu_\alpha + u^\mu u_\alpha)S^{\alpha\beta}\dot{V}_\beta. \tag{28}$$

We will refer to this term in Sec. V C, where the option of making $p^\mu_{\rm hidden} = 0$ will be discussed.

C. MPD equations in terms of spin vector

Writing out the left-hand side of (3) in terms of s^μ,

$$\dot{S}_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta}\dot{V}^\gamma s^\delta + \epsilon_{\alpha\beta\gamma\delta}V^\gamma \dot{s}^\delta, \tag{29}$$

and then multiplying the equation by $\epsilon^{\mu\nu\alpha\beta}V_\nu$, one has

$$(\delta^\mu_\nu + V^\mu V_\nu)\dot{s}^\nu = \epsilon^{\mu\nu\alpha\beta}V_\nu u_\alpha p_\beta, \tag{30}$$

and hence

$$\dot{s}^\mu = V^\mu \dot{V}_\nu s^\nu + \epsilon^{\mu\nu\alpha\beta}V_\nu u_\alpha p_\beta. \tag{31}$$

Therefore, the change of spin along u^μ is parallel to V^μ in two obvious cases: (i) whenever u^μ is parallel to p^μ (which implies $\dot{S}_{\alpha\beta} = 0$ and $\mathcal{M} = m$); and (ii) if V^μ lies in the plane spanned by u^μ and $\mathcal{U}^\mu$ (i.e. if one applies some combination of the Mathisson-Pirani and Tulczyjew conditions). The spin magnitude evolves according to

$$s\dot{s} \equiv s\frac{\mathrm{d}s}{\mathrm{d}\tau} = \frac{1}{2}\frac{\mathrm{d}s^2}{\mathrm{d}\tau} = \frac{1}{2}S^{\alpha\beta}\dot{S}_{\alpha\beta} = S^{\alpha\beta}p_\alpha u_\beta \tag{32}$$

$$= s_\mu \dot{s}^\mu = \epsilon^{\mu\nu\alpha\beta}s_\mu V_\nu u_\alpha p_\beta, \tag{33}$$

which in the above two cases yields conservation. Note in passing that $s_\mu(s\dot{s}^\mu - \dot{s}s^\mu) = 0$, so regarding (23) and (25), the vectors

$$s^\mu, \qquad s\dot{s}^\mu - \dot{s}s^\mu, \qquad \mu u^\mu - \gamma p^\mu$$

are orthogonal to each other.

Similarly, by multiplying the relation (29) by $\epsilon^{\mu\nu\alpha\beta}s_\nu$ and using (33), one arrives at

$$(s^2\delta^\mu_\nu - s^\mu s_\nu)\dot{V}^\nu = (\delta^\mu_\nu + V^\mu V_\nu)\epsilon^{\nu\iota\alpha\beta}s_\iota u_\alpha p_\beta, \tag{34}$$

and hence

$$s\frac{\mathrm{D}}{\mathrm{d}\tau}(sV^\mu) = -s^\mu \dot{s}_\nu V^\nu + \epsilon^{\mu\iota\alpha\beta}s_\iota u_\alpha p_\beta. \tag{35}$$

Introducing (29) and then (31) into the mass-evolution formulas (11) and (12), we obtain, after some rearrangement,

$$\gamma^2\dot{m} = \epsilon^{\alpha\beta\rho\sigma}s_\alpha \dot{V}_\beta u_\rho \dot{u}_\iota(\delta^\iota_\sigma + V^\iota V_\sigma), \tag{36}$$

$$\gamma^2\mathcal{M}\dot{\mathcal{M}} = \epsilon^{\alpha\beta\rho\sigma}s_\alpha \dot{V}_\beta u_\rho \dot{p}_\iota(\delta^\iota_\sigma + V^\iota V_\sigma). \tag{37}$$

In particular, the $V^\mu \equiv u^\mu$ choice leads to $\dot{m} = 0$, while the $V^\mu \equiv \mathcal{U}^\mu$ choice leads to $\dot{\mathcal{M}} = 0$. On the other hand, the evolution of $\mu \equiv -V_\sigma p^\sigma$ can be expressed, using (24), as

$$\gamma\dot{\mu} = -\gamma\dot{V}_\lambda p^\lambda - \gamma V_\lambda \dot{p}^\lambda = -\mu u_\kappa \dot{V}^\kappa - \gamma V_\lambda \dot{p}^\lambda. \tag{38}$$

Naturally, this reduces to the above limits too.

Now to the main point at last: written in terms of the spin vector, the first MPD equation appears as

$$\begin{aligned}\dot{p}^\mu &= -\frac{1}{2}R^\mu{}_{\nu\kappa\lambda}u^\nu S^{\kappa\lambda} = -\frac{1}{2}g^{\mu\rho}R_{\rho\nu\kappa\lambda}\epsilon^{\kappa\lambda\alpha\beta}u^\nu V_\alpha s_\beta \\ &= -g^{\mu\rho}R^*_{\rho\nu\alpha\beta}u^\nu V^\alpha s^\beta = {}^*R^\mu{}_{\nu\alpha\beta}u^\nu s^\alpha V^\beta,\end{aligned} \tag{39}$$

where $R^*_{\rho\nu\alpha\beta}$ and ${}^*R_{\rho\nu\alpha\beta}$ are the Riemann-tensor right and left duals; in the last equality, we have used the fact that they are equal in the vacuum case (even with a nonzero cosmological constant).

D. Eigenvectors of $S^{\mu\nu}$, ${}^*S^{\mu\nu}$, $\dot{S}^{\mu\nu}$, ${}^*\dot{S}^{\mu\nu}$

Already in several places we have mentioned eigenvectors of the spin bivector and of its derivative and/or dual. All the bivectors indeed have the whole 2D eigenplanes (corresponding to zero eigenvalues) which can be spanned in a number of ways, so let us just summarize one simple possibility for each:

(i) $S_{\alpha\beta} = \epsilon_{\alpha\beta\mu\nu}V^\mu s^\nu$, so it is clear that V^β and s^β are its "default" eigenvectors—they are simple and orthogonal to each other.

(ii) ${}^*S^{\mu\nu} = s^\mu V^\nu - V^\mu s^\nu$. Thanks to the property (17), the eigenvectors of ${}^*S^{\mu\nu}$ can be found simply by any nontrivial projections of $S_{\nu\beta}$. One suitable vector for such a projection is $\dot{V}^\beta$:

$$-S_{\nu\beta}\dot{V}^\beta = \dot{S}_{\nu\beta}V^\beta = (p_\nu u_\beta - u_\nu p_\beta)V^\beta = \mu u_\nu - \gamma p_\nu.$$

We would like the second eigenvector to be orthogonal to the one just found, which is, for example, true for the vector (34). The latter is in fact orthogonal to all the vectors V^μ, s^μ, p^μ, and u^μ, so it clearly has both the desired properties. Note that actually *all* the eigenvectors suggested in (i) and (ii) are mutually orthogonal: sure, they just form the basis we already know from (26).

(iii) ${}^*\dot{S}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}p^\alpha u^\beta$, so the "default" eigenvectors seem to be p^ν and u^ν. These two are, however, *never* orthogonal, yet it is easy to fix this by taking

$$u^\nu \quad \text{and} \quad (\delta^\nu_\sigma + u^\nu u_\sigma)p^\sigma = p^\nu - mu^\nu (= -\dot{S}^{\nu\beta}u_\beta).$$

The last vector is exactly the "hidden momentum," which turned out to be useful in the understanding of the pole-dipole description [6].

(iv) $\dot{S}^{\alpha\beta} = p^\alpha u^\beta - u^\alpha p^\beta$. One finds shortly that one eigenvector, moreover orthogonal to both u^β and p^β (hence also to $p^\beta - mu^\beta$), is $(\gamma s_\beta + s_\nu u^\nu V_\beta)$. The second one, orthogonal to the latter as well as to u^β and $p^\beta - mu^\beta$, reads

$$\epsilon^{\beta\iota\kappa\lambda}u_\iota p_\kappa(\gamma s_\lambda + s_\nu u^\nu V_\lambda) = \gamma s\dot{s}V^\beta + \gamma(s^2\delta^\beta_\nu - s^\beta s_\nu)\dot{V}^\nu + u^\sigma s_\sigma(\delta^\beta_\nu + V^\beta V_\nu)\dot{s}^\nu. \quad (40)$$

Clearly the eigenvector choices suggested for the (ii)–(iv) bivectors are only realizable in the generic case; in particular, they are not valid if $p^\mu = mu^\mu$ when most of them become trivial. This special "gauge" will be discussed below, along with the other two major ones (Sec. V).

III. VACUUM MPD EQUATIONS IN A SUITABLE TETRAD

In a vacuum region, yet possibly involving a nonzero cosmological constant Λ, the usual decomposition of the Riemann tensor reduces to

$$R_{\mu\nu\kappa\lambda} = C_{\mu\nu\kappa\lambda} + \frac{\Lambda}{3}(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}), \quad (41)$$

where $C_{\mu\nu\kappa\lambda}$ denotes the Weyl tensor. In some orthonormal tetrad $\{e^\mu_{\hat\alpha}\}$ (α numbers the vectors), $g_{\mu\nu}e^\mu_{\hat\alpha}e^\nu_{\hat\beta} = \eta_{\alpha\beta}$, the decomposition reads

$$R_{\hat\alpha\hat\beta\hat\gamma\hat\delta} = C_{\hat\alpha\hat\beta\hat\gamma\hat\delta} + \frac{\Lambda}{3}(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma}), \quad (42)$$

where $\eta_{\alpha\beta}$ is the Minkowski tensor. The dual tensor thus decomposes as

$$^*R^\mu{}_{\nu\alpha\beta} = {}^*C^\mu{}_{\nu\alpha\beta} + \frac{\Lambda}{3}\epsilon^\mu{}_{\nu\alpha\beta}. \quad (43)$$

A. Orthonormal tetrad

When considering the choice of a suitable tetrad, one has three logical options—u^μ, $\mathcal{U}^\mu$, and V^μ—for the timelike vector $e^\mu_{\hat 0}$. The choice $e^\mu_{\hat 0} \equiv V^\mu$ and $e^\mu_{\hat 1} \equiv s^\mu/s$ seems the most advantageous and "universal," because

(i) The spin s^μ, fixed by V^μ and introduced as orthogonal to the latter in every case, represents the most important spacelike direction of the problem, so it is natural to discuss the curvature effects with respect to it.

(ii) The reference observer V^μ is actually the only vector that one chooses freely, the other ones depending on it: V^μ fixes the meaning of p^μ and $S^{\mu\nu}$ as moments of the energy-momentum tensor by specifying the hypersurface over which they are calculated,[3] as well as that of u^μ identified as the tangent to the worldline along which $S^{\mu\sigma}V_\sigma = 0$ holds.

(iii) The freedom in V^μ makes it easily adaptable to particular spacetimes and situations, specifically to the particular Petrov types.

(iv) Special pictures arising for particular spin supplementary conditions follow simply by selecting V^μ accordingly, e.g. as $u^\mu, \mathcal{U}^\mu$, or as some vector parallel along u^μ.

Let us note that it might seem preferable to choose u^μ as *the* timelike direction, for it is (its finding is) certainly central to the problem, and because $u_\mu\dot{p}^\mu = 0$, which makes the time component of the problem settled immediately. However, such a choice does not allow one to select the spin s^μ as one of the basis directions, because s^μ is orthogonal to V^μ rather than to u^μ (the latter would require $u_\mu s^\mu = u_\mu V_\nu{}^*S^{\nu\mu} = 0$, which is not the case in general). And, obviously, u^μ cannot be *chosen*; it is given by the MPD equations and a chosen spin condition.

Having opted for $e^\mu_{\hat 0} \equiv V^\mu$, $e^\mu_{\hat 1} \equiv s^\mu/s$, we have from (39) and (43)

$$\dot{p}^\mu = {}^*R^\mu{}_{\nu\alpha\beta}u^\nu s^\alpha V^\beta = s\,{}^*R^\mu{}_{\hat\gamma\hat1\hat0}u^{\hat\gamma} = s\,{}^*C^\mu{}_{\hat\gamma\hat1\hat0}u^{\hat\gamma} + \frac{\Lambda}{3}s\,\epsilon^\mu{}_{\hat\gamma\hat1\hat0}u^{\hat\gamma}. \quad (44)$$

The projections of equation (44) on V^μ and $e^\mu_{\hat1}$ read, respectively,

$$-V_\mu\dot{p}^\mu = \dot{\mu} + p_\mu\dot{V}^\mu = \dot{\mu} + p_{\hat k}\dot{V}^{\hat k} = s\,{}^*C_{\hat\imath\hat0\hat1\hat0}u^{\hat\imath}, \quad (45)$$

$$e^{\hat1}_\mu\dot{p}^\mu = s\,{}^*C_{\hat1\hat\gamma\hat1\hat0}u^{\hat\gamma}, \quad (46)$$

where the notation $\mu := -V_\sigma p^\sigma$ has been recalled. The cosmological term has no effect in this plane. The remaining two spatial directions $e^\mu_{\hat2}$ and $e^\mu_{\hat3}$, perpendicular to both V^μ and s^μ, are left unspecified for the moment; the respective projections of (44) are written

$$e^{\hat2}_\mu\dot{p}^\mu = s\,{}^*C_{\hat2\hat\gamma\hat1\hat0}u^{\hat\gamma} - \frac{\Lambda}{3}su^{\hat3}, \quad (47)$$

$$e^{\hat3}_\mu\dot{p}^\mu = s\,{}^*C_{\hat3\hat\gamma\hat1\hat0}u^{\hat\gamma} + \frac{\Lambda}{3}su^{\hat2}. \quad (48)$$

Hence, the time component $-V_\mu\dot{p}^\mu$ is determined purely by the magnetic part of the Weyl tensor, $B_{\hat\imath\hat1}u^{\hat\imath} := {}^*C_{\hat\imath\hat0\hat1\hat0}u^{\hat\imath}$ (see Sec. 1 of the Appendix), while the remaining components are influenced by both magnetic terms (those containing $u^{\hat0} := -V_\sigma u^\sigma \equiv \gamma$) and electric terms (containing $u^{\hat\imath}$).

[3]However, it is worth noting that in the pole-dipole order and with just gravitational effects included, the p^μ is in fact same for *any* spin condition.

B. Newman-Penrose null tetrad

Let us proceed now to the standard Newman-Penrose tetrad made of two real and two complex null vectors $(k^\mu, l^\mu, m^\mu, \bar{m}^\mu)$ introduced by

$$k^\mu := \frac{1}{\sqrt{2}}(V^\mu + e^\mu_{\hat{1}}), \qquad l^\mu := \frac{1}{\sqrt{2}}(V^\mu - e^\mu_{\hat{1}}), \quad (49)$$

$$m^\mu := \frac{1}{\sqrt{2}}(e^\mu_{\hat{2}} + \mathrm{i}e^\mu_{\hat{3}}), \qquad \bar{m}^\mu := \frac{1}{\sqrt{2}}(e^\mu_{\hat{2}} - \mathrm{i}e^\mu_{\hat{3}}), \quad (50)$$

satisfying the normalizations

$$k_\mu l^\mu = -1, \qquad m_\mu \bar{m}^\mu = 1,$$
$$k_\mu m^\mu = k_\mu \bar{m}^\mu = l_\mu m^\mu = l_\mu \bar{m}^\mu = 0$$

and giving rise to the metric decomposition

$$g_{\mu\nu} = -k_\mu l_\nu - k_\nu l_\mu + m_\mu \bar{m}_\nu + m_\nu \bar{m}_\mu. \quad (51)$$

The ten independent components of the Weyl tensor are represented by five independent complex projections:

$$\Psi_0 := C_{\mu\nu\kappa\lambda} k^\mu m^\nu k^\kappa m^\lambda, \quad (52)$$

$$\Psi_1 := C_{\mu\nu\kappa\lambda} k^\mu l^\nu k^\kappa m^\lambda, \quad (53)$$

$$\Psi_2 := C_{\mu\nu\kappa\lambda} k^\mu m^\nu \bar{m}^\kappa l^\lambda \quad (54)$$

$$= \frac{1}{2} C_{\mu\nu\kappa\lambda}(k^\mu l^\nu k^\kappa l^\lambda - k^\mu l^\nu m^\kappa \bar{m}^\lambda), \quad (55)$$

$$\Psi_3 := C_{\mu\nu\kappa\lambda} l^\mu k^\nu l^\kappa \bar{m}^\lambda, \quad (56)$$

$$\Psi_4 := C_{\mu\nu\kappa\lambda} l^\mu \bar{m}^\nu l^\kappa \bar{m}^\lambda. \quad (57)$$

We need to find how the above scalars relate to their counterparts given by the *dual* Weyl tensor now. It is clear from the latter's definition that the dualization can be shifted to the respective couple of tetrad vectors in the projections, so it is sufficient to find how the bivectors made of the tetrad elements behave under dualization; actually, due to the symmetries of the (dual) Weyl tensor, it is sufficient to know this for $k^\mu m^\nu$ and $l^\mu \bar{m}^\nu$. It is easy to check that the null tetrad is "positively" oriented:

$$\epsilon_{\alpha\beta\gamma\delta} k^\alpha l^\beta m^\gamma \bar{m}^\delta = \mathrm{i}[0123] = \mathrm{i},$$

and that the Hodge star simply brings the imaginary unit,

$${}^*(k^\mu \wedge m^\nu) = \mathrm{i}(k^\mu \wedge m^\nu), \quad {}^*(l^\mu \wedge \bar{m}^\nu) = \mathrm{i}(l^\mu \wedge \bar{m}^\nu), \quad (58)$$

which implies "anti-self-duality" of the scalars:

$${}^*\Psi_0 := {}^*C_{\mu\nu\kappa\lambda} k^\mu m^\nu k^\kappa m^\lambda = \mathrm{i}\Psi_0, \quad (59)$$

$${}^*\Psi_1 := {}^*C_{\mu\nu\kappa\lambda} k^\mu l^\nu k^\kappa m^\lambda = \mathrm{i}\Psi_1, \quad (60)$$

$${}^*\Psi_2 := {}^*C_{\mu\nu\kappa\lambda} k^\mu m^\nu \bar{m}^\kappa l^\lambda = \mathrm{i}\Psi_2, \quad (61)$$

$${}^*\Psi_3 := {}^*C_{\mu\nu\kappa\lambda} l^\mu k^\nu l^\kappa \bar{m}^\lambda = \mathrm{i}\Psi_3, \quad (62)$$

$${}^*\Psi_4 := {}^*C_{\mu\nu\kappa\lambda} l^\mu \bar{m}^\nu l^\kappa \bar{m}^\lambda = \mathrm{i}\Psi_4. \quad (63)$$

Writing, conversely,

$$V^\mu = \frac{1}{\sqrt{2}}(k^\mu + l^\mu), \qquad e^\mu_{\hat{1}} = \frac{1}{\sqrt{2}}(k^\mu - l^\mu), \quad (64)$$

$$e^\mu_{\hat{2}} = \frac{1}{\sqrt{2}}(m^\mu + \bar{m}^\mu), \qquad e^\mu_{\hat{3}} = \frac{1}{\sqrt{2}\mathrm{i}}(m^\mu - \bar{m}^\mu), \quad (65)$$

and using the Appendix, we thus obtain

$$\begin{aligned} -V_\mu \dot{p}^\mu = &-2s\,\mathrm{Im}\Psi_2 u^{\hat{1}} \\ &- s(\mathrm{Im}\Psi_3 - \mathrm{Im}\Psi_1)u^{\hat{2}} \\ &- s(\mathrm{Re}\Psi_3 + \mathrm{Re}\Psi_1)u^{\hat{3}}, \end{aligned} \quad (66)$$

$$\begin{aligned} e^{\hat{1}}_\mu \dot{p}^\mu = &-2s\,\mathrm{Im}\Psi_2 u^{\hat{0}} \\ &- s(\mathrm{Im}\Psi_3 + \mathrm{Im}\Psi_1)u^{\hat{2}} \\ &- s(\mathrm{Re}\Psi_3 - \mathrm{Re}\Psi_1)u^{\hat{3}}, \end{aligned} \quad (67)$$

$$\begin{aligned} e^{\hat{2}}_\mu \dot{p}^\mu = &+s\left(2\mathrm{Re}\Psi_2 - \frac{\Lambda}{3}\right)u^{\hat{3}} \\ &- s(\mathrm{Im}\Psi_3 - \mathrm{Im}\Psi_1)u^{\hat{0}} \\ &+ s(\mathrm{Im}\Psi_3 + \mathrm{Im}\Psi_1)u^{\hat{1}}, \end{aligned} \quad (68)$$

$$\begin{aligned} e^{\hat{3}}_\mu \dot{p}^\mu = &-s\left(2\mathrm{Re}\Psi_2 - \frac{\Lambda}{3}\right)u^{\hat{2}} \\ &- s(\mathrm{Re}\Psi_3 + \mathrm{Re}\Psi_1)u^{\hat{0}} \\ &+ s(\mathrm{Re}\Psi_3 - \mathrm{Re}\Psi_1)u^{\hat{1}}. \end{aligned} \quad (69)$$

The main feature of these equations is that they do not at all contain Ψ_0 and Ψ_4. And note again that the cosmological constant only influences motion in the spatial directions perpendicular to spin.

It may seem possible to express the above equations in terms of only two components of four-velocity, because $u^{\hat{\alpha}}$ are constrained by the relation

$$0 = u_\mu \dot{p}^\mu = u^{\hat{0}} V_\mu \dot{p}^\mu + \delta_{ij} u^{\hat{i}} e^{\hat{j}}_\mu \dot{p}^\mu$$

and by normalization

$$-1 = -(u^{\hat{0}})^2 + \delta_{ij} u^{\hat{\imath}} u^{\hat{\jmath}}.$$

But $u_\mu \dot{p}^\mu = 0$ is satisfied *automatically*; it brings no information (it can actually be used as a correctness check).

C. Simple implications from tetrad components of the MPD equation of motion

It should first be stressed that $u_\mu \dot{p}^\mu = 0$, so the overall effect is *always* perpendicular to the representative worldline, irrespective of any interpretation superstructure. In the above projections, the simplest terms are the "cosmological" ones: they act within the $(e^\mu_{\hat{2}}, e^\mu_{\hat{3}})$ plane and always perpendicular to the projection of the trajectory onto this plane. The "Newton-Coulomb" tidal field, generated by mass and given by $\mathrm{Re}\Psi_2$, acts in the same manner within the same plane, its effect having the opposite/same orientation for positive/negative Λ. The "magnetic-type" tidal field, generated by angular momentum and given by $\mathrm{Im}\Psi_2$, acts in exactly the same way, but within the orthogonal $(e^\mu_{\hat{0}}, e^\mu_{\hat{1}})$ plane. The remaining force is tied to longitudinal wave effects in the k^μ and l^μ directions, represented by Ψ_1 and Ψ_3 (cf. Sec. IV C). The last scalars Ψ_0 and Ψ_4 are not at all present, which in standard understanding (cf. Ref. [43]) means that if the NP tetrad is chosen as we did, there are no transverse wave effects, neither in the k^μ nor in the l^μ direction.

D. "Intrinsic" choice of tetrad

It was notably Ernst Mach who emphasized that the system should itself provide terms in which it will be interpreted. Unfortunately, our system needs the "reference observer" V^μ in order to be unique and to make sense. However, we know from (26) that with this chosen, the spinning particle does provide a unique orthogonal (thus also orthonormal) basis which can be used in generic situations (namely, when the vectors p^μ and u^μ are independent): besides V^μ and s^μ from which we have started, it is given by $(\mu u^\mu - \gamma p^\mu)$ and by the "vector product" of these three,

$$\epsilon^{\mu\iota\kappa\lambda} V_\iota s_\kappa (\mu u_\lambda - \gamma p_\lambda) = S^{\mu\lambda}(\mu u_\lambda - \gamma p_\lambda) = -S^{\mu\lambda} S_{\lambda\nu} \dot{V}^\nu = (s^2 \delta^\mu_\nu - s^\mu s_\nu)\dot{V}^\nu; \tag{70}$$

note again that we already mentioned this basis in (26). Hence, besides the $-V_\mu u^\mu \equiv \gamma \equiv u^{\hat{0}}$ and $s_\mu u^\mu \equiv s u^{\hat{1}}$ four-velocity components, we have

$$(\mu u_\mu - \gamma p_\mu) u^\mu = \gamma m - \mu, \tag{71}$$

$$\epsilon_{\mu\iota\kappa\lambda} V^\iota s^\kappa (\mu u^\lambda - \gamma p^\lambda) u^\mu = -\gamma \epsilon_{\mu\iota\kappa\lambda} u^\mu V^\iota s^\kappa p^\lambda = \gamma s \dot{s}. \tag{72}$$

The second and the third components of $\dot{p}^\mu$ follow from

$$(\mu u_\mu - \gamma p_\mu)\dot{p}^\mu = -\gamma p_\mu \dot{p}^\mu = \gamma \mathcal{M}\dot{\mathcal{M}}, \tag{73}$$

$$\epsilon_{\mu\iota\kappa\lambda} V^\iota s^\kappa (\mu u^\lambda - \gamma p^\lambda)\dot{p}^\mu = -\dot{p}^\mu S_{\mu\lambda} S^{\lambda\nu} \dot{V}_\nu = (s^2 g_{\mu\nu} - s_\mu s_\nu)\dot{p}^\mu \dot{V}^\nu. \tag{74}$$

Before usage of the above tetrad, one should learn the norm of the two newly specified vectors,

$$(\mu u_\mu - \gamma p_\mu)(\mu u^\mu - \gamma p^\mu) = 2\gamma\mu m - \mu^2 - \gamma^2 \mathcal{M}^2, \tag{75}$$

$$\epsilon_{\mu\alpha\beta\gamma} V^\alpha s^\beta (\mu u^\gamma - \gamma p^\gamma) \epsilon^{\mu\iota\kappa\lambda} V_\iota s_\kappa (\mu u_\lambda - \gamma p_\lambda) = s^2 (2\gamma\mu m - \mu^2 - \gamma^2 \mathcal{M}^2). \tag{76}$$

Note that the expression crucial in both these relations becomes

$$2\gamma\mu m - \mu^2 - \gamma^2 \mathcal{M}^2 \to m^2 - \mathcal{M}^2 \quad (\geq 0)$$

for the Mathisson-Pirani as well as the Tulczyjew condition.

Just to summarize, in an *orthonormal* tetrad involving the above vectors, one has

$$u^{\hat{2}} \coloneqq e^{\hat{2}}_\mu u^\mu = \frac{\gamma m - \mu}{\sqrt{2\gamma\mu m - \mu^2 - \gamma^2 \mathcal{M}^2}}, \tag{77}$$

$$u^{\hat{3}} \coloneqq e^{\hat{3}}_\mu u^\mu = \frac{\gamma \dot{s}}{\sqrt{2\gamma\mu m - \mu^2 - \gamma^2 \mathcal{M}^2}}, \tag{78}$$

$$e^{\hat{2}}_\mu \dot{p}^\mu = \frac{\gamma \mathcal{M}\dot{\mathcal{M}}}{\sqrt{2\gamma\mu m - \mu^2 - \gamma^2 \mathcal{M}^2}}, \tag{79}$$

$$e^{\hat{3}}_\mu \dot{p}^\mu = \frac{(s^2 g_{\mu\nu} - s_\mu s_\nu)\dot{p}^\mu \dot{V}^\nu}{s\sqrt{2\gamma\mu m - \mu^2 - \gamma^2 \mathcal{M}^2}}. \tag{80}$$

Let us stress that the above tetrad is quite generic (it only cannot be used with the spin condition $u^\mu \| p^\mu$, which implies $\mu u^\mu - \gamma p^\mu = 0$); in particular, it is independent of the spacetime curvature structure.

IV. EQUATION OF MOTION IN SPACETIMES OF DIFFERENT ALGEBRAIC TYPES

Let us look at whether the above projections of the MPD equation of motion can be somehow linked to the algebraic type of curvature. This mainly means to turn the interpretation tetrad so as to reflect both the spin and the curvature features. First, we leave the supplementary spin condition generic, and then we will consider the mostly used special choices.

A. Interpretation in a tetrad tied to curvature structure

One of the aims of this paper is to check whether in some cases the above structure tied to the particle spin can "fit in" spacetime curvature in such a manner that the MPD equations assume an especially simple and easily interpretable form. It is known that every spacetime hosts four null eigendirections of the Weyl tensor, called its principal null directions (PNDs); their multiplicities indicate the Petrov type of spacetime. Let us denote by k^μ the PND of the highest multiplicity. Let us also define another independent real null vector l^μ and normalize it by $k_\mu l^\mu = -1$; it can in general be chosen arbitrarily (it even *has to be* if the curvature is of type N and k^μ is a four-fold PND), though it would be especially beneficial to take as l^μ some of the other PNDs. However, such a choice is typically not feasible, if one needs to also simultaneously fix the tetrad to the spin structure.

The only way to "adapt" the spin structure to some spacetime features is to choose the reference observer V^μ accordingly and thus adjust the spin eigenspace in some preferred direction. Consider what can be achieved in this respect at some point of the particle worldline. Let us have some spacetime and use its highest-multiplicity PND k^μ as the first vector of the NP tetrad. Now, imagine first that one knows the particle's spin vector s^μ. Then it is natural to define the second null vector of the NP tetrad by

$$l^\mu := \frac{s^2 k^\mu}{2(s_\iota k^\iota)^2} - \frac{s^\mu}{s_\iota k^\iota}.$$

It is easy to check that such l^μ is really null and that $k_\mu l^\mu = -1$. Clearly the definition is not possible if $s_\iota k^\iota = 0$. Supposing that this is not the case and regarding that a null vector can be normalized arbitrarily, let us choose the normalization of k^μ so that $s_\iota k^\iota = \frac{s}{\sqrt{2}}$. Then we can specify the definition of l^μ as

$$l^\mu := k^\mu - \sqrt{2}\frac{s^\mu}{s}, \tag{81}$$

which inverts to $s^\mu = \frac{s}{\sqrt{2}}(k^\mu - l^\mu)$. Subsequently, we choose the reference observer by

$$V^\mu := \frac{1}{\sqrt{2}}(k^\mu + l^\mu)$$

and introduce the spin bivector by

$$S_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} V^\gamma s^\delta = -s\,\epsilon_{\alpha\beta\gamma\delta} k^\gamma l^\delta.$$

As the second situation, imagine—as is the case when starting from the original MPD equations—that one knows the particle's spin bivector $S^{\mu\nu}$ (rather than the vector s^μ). Then it is natural to define the second null vector l^μ of the NP tetrad as *some* null eigenvector of $S^{\mu\nu}$, i.e. as a vector satisfying $S_{\mu\nu} l^\nu = 0$, and normalize it by $k_\mu l^\mu = -1$. (If l^μ happened to be proportional to k^μ, one would have to take the other independent null eigenvector of $S^{\mu\nu}$.) Having both k^μ and l^μ, one chooses

$$V^\mu := \frac{1}{\sqrt{2}}(k^\mu + l^\mu), \qquad s^\mu = \frac{s}{\sqrt{2}}(k^\mu - l^\mu)$$

as the reference observer and the corresponding spin vector. Since $S_{\mu\nu} V^\nu = 0$ according to the required spin condition, the eigenplane of $S^{\mu\nu}$ is thus turned so as to also contain k^μ, with s^μ lying in it automatically.

Finally, the plane orthogonal to both k^μ and l^μ can be spanned by the remaining two complex null vectors m^μ and $\bar{m}^\mu$ arbitrarily (turning these suitably may slightly reduce the Weyl-tensor components; see Sec. 2 of the Appendix); the corresponding orthonormal vectors are obtained by (65).

To summarize the above in an effective way, by a suitable choice of V^μ, it is possible to rotate the eigenplane of $S^{\mu\nu}$ so as to contain the given null vector k^μ (the highest-multiplicity PND in our case). The spin structure of our problem has thus been connected with the curvature structure, which is desirable if one wishes to discuss the spinning particle motion in dependence on the Petrov type of the background. Namely, the interpretation orthonormal tetrad has thus been chosen in the way described in Sec. III A, while the first vector k^μ of the associated NP tetrad has been identified with the highest-multiplicity PND, which leads to the respective simplification (vanishing) of some of the Weyl-tensor projections. Were it possible to also identify the second NP-tetrad vector l^μ with some of the other PNDs, some further Weyl scalars could be made to vanish (in particular, in type D it is very advantageous to align l^μ with the second existing double-degenerate PND), but this is typically not the case, because the (k^μ, l^μ) plane has no reason to coincide with the eigenplane of $S^{\mu\nu}$. Practically, by a suitable choice of V^μ, one can make the eigenplane of $S^{\mu\nu}$ intersect the (k^μ, l^μ) plane right along k^μ, but the other generator s^μ is thus fixed and cannot in general be rotated to fall in the (k^μ, l^μ) plane too.

Now it is clear how the equations (66)–(69) simplify in specific algebraic types. If all the Weyl-tensor PNDs are distinct, the spacetime is of Petrov type I. If the null tetrad is chosen so that one of these directions coincides with k^μ, the corresponding scalar Ψ_0 vanishes (if the PND were aligned with l^μ, the last scalar Ψ_4 would vanish instead). This, however, does not affect equations (66)–(69), since they lack Ψ_0 and Ψ_4 *a priori*. In algebraically special cases, the PND k^μ is degenerate and further Weyl scalars vanish besides Ψ_0: in type II, k^μ is double and $\Psi_0 = \Psi_1 = 0$, and in type III, it is triple and $\Psi_0 = \Psi_1 = \Psi_2 = 0$, with obvious effect on equations (66)–(69). We see, however, that even in a type-III background, where only Ψ_3 survives in the equations, all the force components are still nonzero in general; just one of the components of Ψ_3 can be transformed out in addition by a suitable rotation of m^μ and $\bar{m}^\mu$ vectors (see Sec. 2 of the Appendix).

B. Type-N and type-D fields

The remaining Petrov types N and D are usually discussed separately, since they are algebraically the most special and cover many of the important exact spacetimes known. In type N, corresponding to purely transverse plane waves, there is just 1 four-times repeated PND. Using it as k^μ and erecting the orthonormal tetrad as described in Sec. IV A, one obtains equations with $(\Psi_0 =)\Psi_1 = \Psi_2 = \Psi_3 = 0$, hence only keeping the cosmological terms,

$$-V_\mu \dot{p}^\mu = 0, \qquad e^{\hat{1}}_\mu \dot{p}^\mu = 0, \tag{82}$$

$$e^{\hat{2}}_\mu \dot{p}^\mu = -\frac{\Lambda}{3} s u^{\hat{3}}, \qquad e^{\hat{3}}_\mu \dot{p}^\mu = \frac{\Lambda}{3} s u^{\hat{2}}. \tag{83}$$

Recalling that $u_\mu \dot{p}^\mu = 0$, we see that *three* (generically independent) projections of $\dot{p}^\mu$ vanish, so apparently it is possible to rotate $e^\mu_{\hat{2}}$ and $e^\mu_{\hat{3}}$ in order to annul one of the respective projections additionally. In the "intrinsic" tetrad suggested in Sec. III D, the third of these equations is written

$$\mathcal{M}\dot{\mathcal{M}} = -\frac{\Lambda}{3} s\dot{s}, \tag{84}$$

and the last one, thanks to the second ($s_\mu \dot{p}^\mu = 0$), is written

$$\dot{V}_\mu \dot{p}^\mu = \frac{\Lambda}{3}(\gamma m - \mu). \tag{85}$$

Finally, in spacetimes of Petrov type D, there are two independent double PNDs. Identifying one of them again with the NP-tetrad vector k^μ, one arrives at the same result as for type II, namely $(\Psi_0 =)\Psi_1 = 0$. A further simplification, $\Psi_3 = \Psi_4 = 0$, only occurs if the second NP-tetrad vector l^μ can be aligned with the second double PND (this choice is known as the Kinnersley tetrad); all the tidal field is represented then by Ψ_2, so the equations reduce to

$$-V_\mu \dot{p}^\mu = -2s\,\mathrm{Im}\Psi_2 u^{\hat{1}}, \tag{86}$$

$$e^{\hat{1}}_\mu \dot{p}^\mu = -2s\,\mathrm{Im}\Psi_2 u^{\hat{0}}, \tag{87}$$

$$e^{\hat{2}}_\mu \dot{p}^\mu = +s\left(2\mathrm{Re}\Psi_2 - \frac{\Lambda}{3}\right)u^{\hat{3}}, \tag{88}$$

$$e^{\hat{3}}_\mu \dot{p}^\mu = -s\left(2\mathrm{Re}\Psi_2 - \frac{\Lambda}{3}\right)u^{\hat{2}}. \tag{89}$$

In our "intrinsic" tetrad the third of these equations is written

$$\mathcal{M}\dot{\mathcal{M}} = \left(2\mathrm{Re}\Psi_2 - \frac{\Lambda}{3}\right)s\dot{s}. \tag{90}$$

As pointed out already, the Kinnersley-like choice is, however, not feasible in general, if we need to reconcile the tetrad with the spin structure at the same time.

C. Ideally rotated NP tetrads: Transverse frames

It is known [44,45] that instead of canceling out the scalars Ψ_0 and Ψ_4, it is quite generally possible to do the same with the couple Ψ_1 and Ψ_3. This is in fact a preferable alternative, as it means elimination of the "pure-gauge" longitudinal wave effects. Notably, Ref. [46] presented a covariant procedure, applicable to any type-I spacetime and any initially chosen NP tetrad, to find a new tetrad (the *transverse frame*) in which $\Psi_1 = 0$ and $\Psi_3 = 0$, together with prescriptions for the new values of Ψ_0, Ψ_2, and Ψ_4. Such an option seems ideal for our projections (66)–(69) of the MPD equations, since they do not contain Ψ_0 and Ψ_4 from the beginning, so their values are irrelevant. Hence, if we use in our picture as k^μ and l^μ the vectors reached in Corollary 2 of Ref. [46] (and derive V^μ, $e^\mu_{\hat{\imath}}$ from them accordingly), the MPD equation projections (66)–(69) would reduce to the type-D form (86)–(89) in *any* field. The only exception is type III, where the "transverse" frame cannot be found; the existence and (non)uniqueness of such a tetrad were summarized in Ref. [44]. Note also that in type N our MPD equation projections are even simpler to write (see previous subsection), so there is no need to use the transverse frame. To sum up, the transverse tetrad could simplify our equations in type-I and type-II spacetimes, effectively turning them into the type-D form.

There is a problem, however: as found in Ref. [46], there generally exist *three* distinct "principal" transverse frames $(k^\mu, l^\mu, m^\mu, \bar{m}^\mu)$, plus a continuous set of their derivatives obtained by renormalization $e^\phi k^\mu$, $e^{-\phi} l^\mu$ in the real timelike plane and rotation $e^{-\mathrm{i}\theta} m^\mu$, $e^{\mathrm{i}\theta}\bar{m}^\mu$ in the complex spacelike plane. Hence, for any of these alternatives, the (k^μ, l^μ) plane is *fixed*, so—like with the Kinnersley tetrad choice in the type-D case—it is not in general possible to fix it to the spin direction concurrently.

In type-D spacetimes, $\mathrm{Re}\Psi_2$ and $\mathrm{Im}\Psi_2$ represent, respectively, components of the gravitoelectric and gravitomagnetic tidal fields (see Sec. 1 of the Appendix); $\mathrm{Re}\Psi_2$ stands for expansion and $\mathrm{Im}\Psi_2$ stands for the vorticity/twist of the PNDs. Clearly $\mathrm{Re}\Psi_2$ is connected with the "scalar," centrally acting field component, while $\mathrm{Im}\Psi_2$ is connected with magnetic-type effects due to mass currents (typically due to rotation). Equations (86)–(89) reveal that in case one could make the interpretation tetrad transverse (thus definitely not in spacetimes of type III), the electric part of the curvature would drive the spinning particle within the plane orthogonal to (V^μ, s^μ) (i.e. within the blade of $S^{\mu\nu}$), while the magnetic part of the curvature would drive it within the (V^μ, s^μ) plane (i.e. within the blade of ${}^*S^{\mu\nu}$).

In type-III spacetimes where the NP tetrad cannot be rotated so as to become transverse (not even at one point), the equations of motion (66)–(69) read

$$-V_\mu \dot{p}^\mu = -s\,\mathrm{Im}\Psi_3 u^{\hat{2}} - s\,\mathrm{Re}\Psi_3 u^{\hat{3}}, \tag{91}$$

$$e^{\hat{1}}_\mu \dot{p}^\mu = -s\,\mathrm{Im}\Psi_3 u^{\hat{2}} - s\,\mathrm{Re}\Psi_3 u^{\hat{3}}, \tag{92}$$

$$e^{\hat{2}}_\mu \dot{p}^\mu = -\frac{\Lambda}{3} s u^{\hat{3}} + s\,\mathrm{Im}\Psi_3 (u^{\hat{1}} - u^{\hat{0}}), \tag{93}$$

$$e^{\hat{3}}_\mu \dot{p}^\mu = +\frac{\Lambda}{3} s u^{\hat{2}} + s\,\mathrm{Re}\Psi_3 (u^{\hat{1}} - u^{\hat{0}}). \tag{94}$$

The first two projections have the same right-hand side, and the last two would be just Im and Re parts of the same expression if Λ were zero.

Note, finally, that the transverse tetrad is (or would be) an *alternative* to the "intrinsic" tetrad suggested in Sec. III D: when using the intrinsic orthonormal tetrad, given by the pole-dipole description itself (provided that the reference observer V^μ has been fixed), it is generally *not* possible to make it transverse with respect to the curvature at the same time.

V. SPECIFIC SPIN CONDITIONS

Let us check now whether some of the usually posed spin supplementary conditions bring some advantages when employed in particular Petrov types. It must be stressed, however, that choosing a certain particular V^μ *irrespectively of* the curvature type (typically as fixed to some important direction given by the particle motion) means that this zeroth tetrad vector *cannot* in general be *at the same time* aligned with the PNDs in the desirable way, so in special Petrov types the MPD equation only gets further simplified if the highest-multiplicity PND k^μ *incidentally* belongs to the (now *a priori* selected) spin plane (V^μ, s^μ).

A. Mathisson-Pirani spin condition, $V^\mu \equiv u^\mu$

With the condition $S^{\mu\sigma}u_\sigma = 0$, the first MPD equation (39) becomes

$$\dot{p}^\mu = {}^*R^\mu{}_{\nu\alpha\beta} u^\nu s^\alpha u^\beta \equiv B^\mu_\alpha s^\alpha, \tag{95}$$

where $B_{\alpha\beta}$ is the gravitomagnetic tidal field (see Sec. 1 of the Appendix). The right-hand side thus differs from that of the geodesic deviation equation only in sign and in the Hodge dualization of Riemann. The tangent-vector tetrad components degenerate to

$$u^{\hat{0}} \equiv \gamma \equiv -u_\sigma u^\sigma = 1, \qquad u^{\hat{\imath}} = 0,$$

which simplifies the (66)–(69) projections to

$$e^{\hat{1}}_\mu \dot{p}^\mu = -2s\,\mathrm{Im}\Psi_2, \tag{96}$$

$$e^{\hat{2}}_\mu \dot{p}^\mu = -s(\mathrm{Im}\Psi_3 - \mathrm{Im}\Psi_1), \tag{97}$$

$$e^{\hat{3}}_\mu \dot{p}^\mu = -s(\mathrm{Re}\Psi_3 + \mathrm{Re}\Psi_1) \tag{98}$$

(the time component is trivial, since $u_\mu \dot{p}^\mu = 0$). Here the cosmological constant drops out completely, while all three Weyl scalars remain present; Ψ_2 is only represented by its imaginary part and determines the force which acts on the particle in the direction of its spin.

For $V^\mu \equiv u^\mu$, one has $\gamma \equiv 1$ and $\mu \equiv m$, equation (36) implies $\dot{m} = -\dot{u}_\mu p^\mu = 0$, equation (31) reduces to $\dot{s}^\mu = u^\mu \dot{u}_\nu s^\nu$, equation (33) gives $\dot{s} = 0$, and equations (23) and (25) imply

$$s_\mu u^\mu = 0 \Rightarrow s_\mu p^\mu = 0\,(\Rightarrow)\, s_\mu \dot{p}^\mu = m s_\mu \dot{u}^\mu, \tag{99}$$

which allows us to rewrite the first of the above equations (for the spin-direction projection of $\dot{p}^\mu$) directly in terms of four-acceleration $\dot{u}^\mu$:

$$m s_\mu \dot{u}^\mu (= -m u_\mu \dot{s}^\mu) = -2s^2\,\mathrm{Im}\Psi_2. \tag{100}$$

A propos, in the case of the Mathisson-Pirani condition, the four-acceleration can be isolated from the MPD system [24],

$$\dot{u}^\mu = \frac{1}{s^2}\left(\frac{1}{m}\dot{p}^\iota s_\iota s^\mu - p_\kappa S^{\mu\kappa}\right), \tag{101}$$

as also follows by equation (35).

As noted above, after choosing $V^\mu \equiv u^\mu$, the tetrad zeroth and first vectors are fixed, so even if the spacetime was algebraically special, they cannot be rotated to make $k^\mu \equiv \frac{1}{\sqrt{2}}(V^\mu + s^\mu/s)$ coincide with the desired PND. Only in the special case when the particle is moving so that $k^\mu \equiv \frac{1}{\sqrt{2}}(u^\mu + s^\mu/s)$ points just in that principal direction does the Ψ_1 scalar vanish and the above equations simplify accordingly. If the principal direction were even triple degenerate (type-III field), in that case, Ψ_2 would vanish too, implying no force (and no acceleration) in the direction of the spin vector. Finally, for a particle moving in the direction given by PND of the type-N spacetime, one has $e^{\hat{\alpha}}_\mu \dot{p}^\mu = 0$, so $\dot{p}^\mu = 0$ (no force).

B. Tulczyjew spin condition, $V^\mu \equiv p^\mu/\mathcal{M}$

With $V^\mu \equiv p^\mu/\mathcal{M}$, one has $\gamma \equiv m/\mathcal{M}$ and $\mu \equiv \mathcal{M}$, equation (37) implies $\mathcal{M}\dot{\mathcal{M}} = -\dot{p}_\mu p^\mu = 0$, equation (31) reduces to $\mathcal{M}^2 \dot{s}^\mu = p^\mu \dot{p}_\nu s^\nu$ and equation (33) to $\dot{s} = 0$, and equations (23) and (25) imply, analogously as above,

$$s_\mu p^\mu = 0 \Rightarrow s_\mu u^\mu = 0\,(\Rightarrow)\, s_\mu \dot{u}^\mu = \frac{m}{\mathcal{M}^2} s_\mu \dot{p}^\mu. \tag{102}$$

However, the main advantage of this condition is that an explicit relation exists giving u^μ in terms of p^μ and $S^{\mu\nu}$ [47]:

$$u^\mu = \frac{m}{\mathcal{M}^2}\left(p^\mu + \frac{2S^{\mu\nu}R_{\nu\iota\kappa\lambda}p^\iota S^{\kappa\lambda}}{4\mathcal{M}^2 + R_{\alpha\beta\gamma\delta}S^{\alpha\beta}S^{\gamma\delta}}\right) \tag{103}$$

(plus $u_\mu u^\mu = -1$ fixes m, which is *not* constant along u^μ here). Likewise, it is also possible to find p^μ in terms of u^μ and $S^{\mu\nu}$. Actually, relation (23) implies $u^\mu s_\mu = 0$, so p^μ and u^μ are both orthogonal to $\dot{p}^\mu$ as well as to s^μ. Hence, it is natural to decompose p^μ into mu^μ and a term orthogonal to u^μ (namely, proportional to $\epsilon^{\mu\nu\rho\sigma}u_\nu \dot{p}_\rho s_\sigma$). Multiplying this decomposition by $\epsilon_{\mu\alpha\beta\gamma}u^\alpha \dot{p}^\beta s^\gamma$, using the relation

$$(s^2\delta^\mu_\nu - s^\mu s_\nu)\dot{p}^\nu = \mathcal{M}\epsilon^{\mu\iota\alpha\beta}s_\iota u_\alpha p_\beta \tag{104}$$

following from (34)), substituting the definition relation $\mathcal{M}S^{\mu\alpha} = \epsilon^{\mu\alpha\beta\iota}p_\beta s_\iota$, and finally demanding that $p_\mu p^\mu = -\mathcal{M}^2$, one derives

$$p^\mu = mu^\mu - \frac{1}{\mathcal{M}}\epsilon^{\mu\nu\rho\sigma}u_\nu \dot{p}_\rho s_\sigma, \tag{105}$$

where $m^2 = \mathcal{M}^2 + S^{\alpha\beta}\dot{p}_\alpha u_\beta$, and $\dot{p}_\mu$ is to be substituted from the first MPD equation (2). Note that a counterpart of (104) can be found as well for the decomposition of s^μ into components parallel to $\dot{p}^\mu$ and orthogonal to it (and thus proportional to $\epsilon^{\mu\nu\rho\sigma}\dot{p}_\nu u_\rho p_\sigma$).

With Tulczyjew's condition, the first MPD equation (39) reads

$$\mathcal{M}\dot{p}^\mu = {}^*R^\mu{}_{\nu\alpha\beta}u^\nu s^\alpha p^\beta, \tag{106}$$

and its "temporal" (V_μ) projection vanishes again due to $p_\mu \dot{p}^\mu = -\mathcal{M}\dot{\mathcal{M}} = 0$. The tetrad components of four-velocity include

$$u^{\hat{0}} \equiv \gamma \equiv -\frac{p_\sigma u^\sigma}{\mathcal{M}} = \frac{m}{\mathcal{M}},$$
$$u^{\hat{1}} \coloneqq \frac{s_\sigma u^\sigma}{s} = \frac{m s_\sigma p^\sigma}{\mathcal{M}^2 s} = 0,$$

so the (66)–(69) system reduces to

$$0 = -V_\mu \dot{p}^\mu = +s(\mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3)u^{\hat{2}} - s(\mathrm{Re}\Psi_1 + \mathrm{Re}\Psi_3)u^{\hat{3}}, \tag{107}$$

$$\frac{\mathcal{M}^2}{m}e^{\hat{1}}_\mu \dot{u}^\mu = e^{\hat{1}}_\mu \dot{p}^\mu = -\frac{2ms}{\mathcal{M}}\mathrm{Im}\Psi_2 - s(\mathrm{Im}\Psi_3 + \mathrm{Im}\Psi_1)u^{\hat{2}} - s(\mathrm{Re}\Psi_3 - \mathrm{Re}\Psi_1)u^{\hat{3}}, \tag{108}$$

$$e^{\hat{2}}_\mu \dot{p}^\mu = +s\left(2\mathrm{Re}\Psi_2 - \frac{\Lambda}{3}\right)u^{\hat{3}} - \frac{ms}{\mathcal{M}}(\mathrm{Im}\Psi_3 - \mathrm{Im}\Psi_1), \tag{109}$$

$$e^{\hat{3}}_\mu \dot{p}^\mu = -s\left(2\mathrm{Re}\Psi_2 - \frac{\Lambda}{3}\right)u^{\hat{2}} - \frac{ms}{\mathcal{M}}(\mathrm{Re}\Psi_3 + \mathrm{Re}\Psi_1). \tag{110}$$

The first of these equations together with the normalization

$$-1 = -(u^{\hat{0}})^2 + \sum(u^{\hat{\imath}})^2 = -\frac{m^2}{\mathcal{M}^2} + (u^{\hat{2}})^2 + (u^{\hat{3}})^2$$

gives $u^{\hat{2}}$ and $u^{\hat{3}}$ in terms of the Weyl scalars,

$$u^{\hat{2}} = \pm\frac{\mathrm{Re}\Psi_1 + \mathrm{Re}\Psi_3}{\mathcal{M}} \times \sqrt{\frac{m^2 - \mathcal{M}^2}{(\mathrm{Re}\Psi_1 + \mathrm{Re}\Psi_3)^2 + (\mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3)^2}} = \pm\frac{\mathrm{Re}\Psi_1 + \mathrm{Re}\Psi_3}{\mathcal{M}|\Psi_1 + \bar{\Psi}_3|}\sqrt{m^2 - \mathcal{M}^2}, \tag{111}$$

$$u^{\hat{3}} = \pm\frac{\mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3}{\mathcal{M}} \times \sqrt{\frac{m^2 - \mathcal{M}^2}{(\mathrm{Re}\Psi_1 + \mathrm{Re}\Psi_3)^2 + (\mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3)^2}} = \pm\frac{\mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3}{\mathcal{M}|\Psi_1 + \bar{\Psi}_3|}\sqrt{m^2 - \mathcal{M}^2}, \tag{112}$$

which can then be used in the remaining equations to express $e^{\hat{1}}_\mu \dot{p}^\mu$ in terms of Ψ_a; for instance,

$$e^{\hat{1}}_\mu \dot{p}^\mu = -\frac{2ms}{\mathcal{M}}\mathrm{Im}\Psi_2 \mp \frac{2s\mathrm{Im}\Psi_1\mathrm{Im}\Psi_3}{\mathcal{M}|\Psi_1 + \bar{\Psi}_3|}\sqrt{m^2 - \mathcal{M}^2}. \tag{113}$$

Again, if p^μ *incidentally* points in such a direction that the plane (p^μ, s^μ) contains some of the PNDs of an algebraically special spacetime, then $\Psi_1 = 0$ and

$$e^{\hat{1}}_\mu \dot{p}^\mu = -\frac{2ms}{\mathcal{M}}\mathrm{Im}\Psi_2, \tag{114}$$

$$e^{\hat{2}}_{\mu}\dot{p}^{\mu} = -\frac{s}{\mathcal{M}}\mathrm{Im}\Psi_3\left(m \pm \frac{2\mathrm{Re}\Psi_2 - \frac{\Lambda}{3}}{|\Psi_3|}\sqrt{m^2 - \mathcal{M}^2}\right), \tag{115}$$

$$e^{\hat{3}}_{\mu}\dot{p}^{\mu} = -\frac{s}{\mathcal{M}}\mathrm{Re}\Psi_3\left(m \pm \frac{2\mathrm{Re}\Psi_2 - \frac{\Lambda}{3}}{|\Psi_3|}\sqrt{m^2 - \mathcal{M}^2}\right). \tag{116}$$

(One of the last two projections can furthermore be transformed out by a suitable rotation of m^{μ} and $\bar{m}^{\mu}$ vectors.) A similar situation in the type-III case would mean no force in the spin-vector direction (plus one more transformable out), and if it were type N, the force would vanish completely again.

A short remark on equation (104): It says that the projection of $\dot{p}^{\mu}$ onto a hypersurface orthogonal to $s^{\mu}(\equiv s e^{\mu}_{\hat{1}})$ equals $\mathcal{M}\epsilon^{\mu\iota\alpha\beta}s_{\iota}u_{\alpha}p_{\beta}$, which means that the $e^{\hat{2}}_{\mu}\dot{p}^{\mu}$ and $e^{\hat{3}}_{\mu}\dot{p}^{\mu}$ components can also be obtained directly from that expression [rather than from (106), which is more complicated)]:

$$\begin{aligned} s e^{\hat{2}}_{\mu}\dot{p}^{\mu} &= \frac{\mathcal{M}}{s}\epsilon^{\mu\iota\alpha\beta}e^{\hat{2}}_{\mu}s_{\iota}u_{\alpha}p_{\beta} = \mathcal{M}\epsilon^{\hat{2}\,\hat{1}\,\hat{\alpha}\,\hat{\beta}}u_{\hat{\alpha}}p_{\hat{\beta}} \\ &= \mathcal{M}(u_{\hat{0}}p_{\hat{3}} - u_{\hat{3}}p_{\hat{0}}) = \mathcal{M}^2u^{\hat{3}} - mp^{\hat{3}}, \\ s e^{\hat{3}}_{\mu}\dot{p}^{\mu} &= \frac{\mathcal{M}}{s}\epsilon^{\mu\iota\alpha\beta}e^{\hat{3}}_{\mu}s_{\iota}u_{\alpha}p_{\beta} = \mathcal{M}\epsilon^{\hat{3}\,\hat{1}\,\hat{\alpha}\,\hat{\beta}}u_{\hat{\alpha}}p_{\hat{\beta}} \\ &= \mathcal{M}(u_{\hat{2}}p_{\hat{0}} - u_{\hat{0}}p_{\hat{2}}) = mp^{\hat{2}} - \mathcal{M}^2u^{\hat{2}}. \end{aligned}$$

However, this is nothing new; it only reproduces what we knew from the beginning, namely relation (8). Actually, with the Tulczyjew condition, the latter reads

$$\begin{aligned} \mathcal{M}^2u^{\alpha} - mp^{\alpha} &= -S^{\alpha\beta}\dot{p}_{\beta} \\ &= -\epsilon^{\alpha\beta\gamma\delta}V_{\beta}s_{\gamma}\dot{p}_{\delta} = s\,\epsilon^{\hat{0}\,\hat{1}\,\alpha\delta}\dot{p}_{\delta}, \end{aligned}$$

which exactly yields the above projections.

1. *Using the "intrinsic" tetrad with Tulczyjew's condition*

The Tulczyjew condition very well fits together with decomposing the MPD equation in the "intrinsic" tetrad suggested in Sec. III D. Actually, substituting $V^{\mu} = p^{\mu}/\mathcal{M}$, $\gamma = m/\mathcal{M}$, $\mu = \mathcal{M}$, and $\dot{s} = 0$ into (77)–(80), and assuming $m \neq \mathcal{M}$ (otherwise the tetrad is not defined), we obtain for u^{μ} and $\dot{p}^{\mu}$ the components

$$u^{\hat{2}} = \frac{\sqrt{m^2 - \mathcal{M}^2}}{\mathcal{M}}, \tag{117}$$

$$u^{\hat{3}} = \frac{m\dot{s}}{\mathcal{M}\sqrt{m^2 - \mathcal{M}^2}} = 0, \tag{118}$$

$$e^{\hat{2}}_{\mu}\dot{p}^{\mu} = \frac{m\dot{\mathcal{M}}}{\sqrt{m^2 - \mathcal{M}^2}} = 0, \tag{119}$$

$$e^{\hat{3}}_{\mu}\dot{p}^{\mu} = \frac{-p^{\mu}\dot{S}_{\mu\lambda}\dot{S}^{\lambda\nu}p_{\nu}}{\mathcal{M}s\sqrt{m^2 - \mathcal{M}^2}} = \frac{\mathcal{M}}{s}\sqrt{m^2 - \mathcal{M}^2}, \tag{120}$$

which reduces the (107)–(110) system to

$$0 = s(\mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3), \tag{121}$$

$$e^{\hat{1}}_{\mu}\dot{p}^{\mu} = -\frac{2ms}{\mathcal{M}}\mathrm{Im}\Psi_2 - s(\mathrm{Im}\Psi_1 + \mathrm{Im}\Psi_3)u^{\hat{2}}, \tag{122}$$

$$0 = s(\mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3), \tag{123}$$

$$\mathcal{M}^2 = s^2\left(\frac{\Lambda}{3} - 2\mathrm{Re}\Psi_2\right) - \frac{ms^2(\mathrm{Re}\Psi_1 + \mathrm{Re}\Psi_3)}{\sqrt{m^2 - \mathcal{M}^2}}. \tag{124}$$

The first and the third equation are equal and imply $\mathrm{Im}\Psi_1 = \mathrm{Im}\Psi_3$; the last equation represents a *constraint* between several parameters of the exercise; evolution has only remained in the $s_{\mu}\dot{p}^{\mu}$ component.

If the (p^{μ}, s^{μ}) plane contained some of the PNDs of an algebraically special field, one would have $\Psi_1 = 0$; hence also $\mathrm{Im}\Psi_3 = 0$, and the evolution equation would shorten to

$$\left(\frac{\mathcal{M}^2}{m}e^{\hat{1}}_{\mu}\dot{u}^{\mu} =\right)e^{\hat{1}}_{\mu}\dot{p}^{\mu} = -\frac{2ms}{\mathcal{M}}\mathrm{Im}\Psi_2.$$

In type III where $\Psi_2 = 0$ as well, this right-hand side would vanish, and the constraint would reduce to

$$\mathcal{M}^2 = \frac{\Lambda}{3}s^2 - \frac{ms^2\,\mathrm{Re}\Psi_3}{\sqrt{m^2 - \mathcal{M}^2}}.$$

We have not mentioned types D and N, because in these cases the "intrinsic" tetrad is not available. To prove this, let us evaluate the term which "deviates" u^{μ} from p^{μ} according to the relation (103): by substituting consecutively

$$\begin{aligned} S^{\kappa\lambda} &= -\epsilon^{\kappa\lambda}{}_{\alpha\beta}s^{\alpha}V^{\beta}, \qquad p^{\iota} = \mathcal{M}V^{\iota}, \\ S^{\mu\nu} &= \epsilon^{\mu\nu\rho\sigma}V_{\rho}s_{\sigma}, \\ R^{*}_{\iota\hat{0}\,\hat{1}\,\hat{0}} &= {}^{*}R_{\iota\hat{0}\,\hat{1}\,\hat{0}} = {}^{*}C_{\iota\hat{0}\,\hat{1}\,\hat{0}} \quad \text{(vacuum)}, \end{aligned}$$

decomposing the result in the generic $(k^{\mu}, l^{\mu}, m^{\mu}, \bar{m}^{\mu})$-tied orthonormal tetrad, and using the Appendix, one finds

$$
\begin{aligned}
S^{\mu\nu}R_{\nu\iota\kappa\lambda}p^{\iota}S^{\kappa\lambda} &= -2S^{\mu\nu}R^{*}_{\nu\iota\alpha\beta}p^{\iota}s^{\alpha}V^{\beta} \\
&= -2\mathcal{M}sS^{\mu\nu*}R_{\nu\hat{0}\hat{1}\hat{0}} = -2\mathcal{M}s^2\epsilon^{\mu\nu\hat{0}\hat{1}*}C_{\nu\hat{0}\hat{1}\hat{0}} \\
&= 2\mathcal{M}s^2({}^*C_{\hat{3}\hat{0}\hat{1}\hat{0}}e^{\mu}_{\hat{2}} - {}^*C_{\hat{2}\hat{0}\hat{1}\hat{0}}e^{\mu}_{\hat{3}}) \\
&= -2\mathcal{M}s^2[(\mathrm{Re}\Psi_1 + \mathrm{Re}\Psi_3)e^{\mu}_{\hat{2}} \\
&\qquad + (\mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3)e^{\mu}_{\hat{3}}].
\end{aligned}
$$

Hence, in type-D and type-N spacetimes where $\Psi_1 = 0 = \Psi_3$, this term vanishes and u^μ is parallel with p^μ, which is exactly the situation when the "intrinsic" tetrad does not exist.

2. Note on the momentum-velocity relation

As seen from (14), for the momentum-velocity relation (13) to "close," a condition weaker than Tulczyjew's is in fact sufficient; namely, $\frac{\mathrm{D}}{\mathrm{d}\tau}(S^{\alpha\beta}p_\beta)$ has to be expressible in terms of $S^{\alpha\beta}$ and p^α only (for example, it must be proportional to $S^{\alpha\beta}p_\beta$ or to p^α). Concerning the importance of the momentum-velocity relation, it might be interesting to examine the range of this option, but we will not go in this direction here.

C. V^μ parallel along u^μ

We suggested in Ref. [24] to take V^μ given by some vector parallel along u^μ, i.e. satisfying $\dot{V}^\mu = 0$. Relation (21) then implies $u^\mu \| p^\mu$ (such an option was already recommended by Ref. [48]); therefore $m = \mathcal{M}$, $\mu = \gamma m$ and $\dot{S}^{\mu\nu} = 0$, ${}^*\dot{S}_{\alpha\beta} = 0$, $\dot{s} = 0$. The mass $m = \mathcal{M}$ is constant along $p^\mu = mu^\mu$, and the MPD equation's left-hand side can thus be written as $\dot{p}^\mu = \mathcal{M}\dot{u}^\mu$; its "time" component, in particular, also reads $-V_\mu\dot{p}^\mu = \dot{\mu} = \dot{\gamma}m$. From $u^\mu \| p^\mu$ and $\dot{V}^\mu = 0$ it also follows that $\dot{s}^\mu = 0$. Most of the equations in Sec. II become trivial.

Not so the first MPD equation (39). Actually, in spite of these significant simplifications, the scheme (66)–(69) does *not* reduce in general. In particular, note that although the present spin condition leaves the reference vector V^μ more free, namely only restricted by $\dot{V}^\mu = 0$, one still cannot count on correlating it with the main PND k^μ besides: in such a case V^μ and the corresponding s^μ would have to be related to k^μ by $V^\mu = \sqrt{2}k^\mu - s^\mu/s$, so the requirement $\dot{V}^\mu = 0$ would only be fulfilled if $\dot{k}^\mu = 0$ (because $\dot{s}^\mu = 0$), i.e. only if k^μ were itself parallel along the particle's worldline. Of course, it is not in general. On the other hand, if restricting to purely *local* analysis, "at any single point" of the trajectory, then it is always possible to select V^μ in the desirable way, namely to take advantage of its freedom and choose it *there* in the same way as described in Sec. IV A. Therefore, at any given point, we can keep the recipe from that section and simplify equations in some of the algebraically special situations accordingly.

1. More on the $u^\mu \| p^\mu$ option

The main benefit of choosing such a V^μ whose $\dot{V}^\mu = 0$ is that u^μ and p^μ are parallel then, $p^\mu = mu^\mu$; i.e. the "hidden" component of momentum (28) vanishes. Besides obvious simplification, this circumstance also remedies one of the inherent inconveniences of the extended-body problem. Namely, even though p^μ should be timelike in reality (i.e. one supposes $\mathcal{M}^2 > 0$), the MPD equations do *not* in general guarantee that u^μ is timelike as well (the selected representative worldline may be winding through the body's convex-hull world tube in an awkward way); in particular, u^μ has been observed to easily become spacelike in highly nonhomogeneous fields where the pole-dipole approximation is most problematic. The $u^\mu \| p^\mu$ option thus eliminates the need to control the spacetime character of u^μ, this being fixed by the character of p^μ. Moreover, since $u^\mu \| p^\mu$ implies $\dot{\mathcal{M}} = \dot{m} = 0$, the normalization of $p^\mu = mu^\mu$ is *conserved*. The same, of course, applies to the reference vector function V^μ, since it is parallel transported along u^μ.

A further advantage of having $p^\mu = mu^\mu$ is that it can keep the problem linear in spin. Actually, the MPD equations (2) and (3) themselves *are* linear in $S^{\mu\nu}$, but the spin supplementary condition—which anyway has to be added—brings the nonlinearity in general. The nonlinearity can be seen as entering through the momentum-velocity relation which has to be used on the left-hand side of (2) in order to write the latter down as an equation for u^μ; the momentum-velocity relation (103) arising for the Tulczyjew spin condition is a clear example. With the option $p^\mu = mu^\mu$ (and m constant along u^μ), equations (2) and (39) simply become

$$m\dot{u}^\mu = -\frac{1}{2}R^\mu{}_{\nu\kappa\lambda}u^\nu S^{\kappa\lambda} = {}^*R^\mu{}_{\nu\alpha\beta}u^\nu s^\alpha V^\beta. \tag{125}$$

Therefore, the problem remains linear in spin (if V^μ is not spin dependent, of course).

Let us realize now that $u^\mu \| p^\mu$ can, however, be ensured by a weaker condition than $\dot{V}^\mu = 0$; namely, it is sufficient to take $\dot{V}^\mu$ proportional to s^μ. Actually, we know that $u^\mu \| p^\mu$ implies $\dot{S}^{\alpha\beta} = 0$, ${}^*\dot{S}^{\alpha\beta} = 0$, and $\dot{s} = 0$, irrespectively of the spin supplementary condition. But since *some* spin condition ($V_\mu S^{\mu\nu} = 0$) has ultimately to be employed, it is reasonable to rewrite the $\dot{S}^{\alpha\beta} = 0$ option as $\dot{V}_\mu S^{\mu\nu} = 0$, as also given by (21). Well, we have in fact only restored the statement that the hidden momentum (28) should be zero. Or, in still other words, $u^\mu \| p^\mu \Leftrightarrow \dot{V}^\mu$ belongs to the eigenplane of $S^{\mu\nu}$. The latter is spanned by V^μ and s^μ, of which V^μ is nevertheless *perpendicular* to $\dot{V}^\mu$, so $\dot{V}^\mu$ has to be proportional to s^μ. Multiplying this proportionality by s_μ, one finds

$$\dot{V}^\mu = \frac{1}{s^2}\dot{V}^\nu s_\nu s^\mu. \tag{126}$$

This result is nothing new; we knew it from equation (34)) already. Similarly, the spin-vector evolution (31) for $u^\mu \| p^\mu$ reduces to

$$\dot{s}^\mu = \dot{V}^\nu s_\nu V^\mu. \tag{127}$$

The reference vector function V^μ is clearly not fixed uniquely. Specifically, it is constrained by $V_\mu V^\mu = -1$ and (126)), of which the latter represents just two independent conditions, because its projections onto V^μ and s^μ are satisfied automatically. The remaining indeterminacy of V^μ can be interpreted as a freedom to choose the magnitude of $\dot{V}^\mu$. If $\dot{V}^\mu$ is multiplied by some scalar, $\dot{s}_\mu$ has to be multiplied by the same one, otherwise $\frac{\mathrm{D}}{\mathrm{d}\tau}(V^\mu s_\mu)$ (and ergo also $V^\mu s_\mu$ itself) would not stay zero.

Consider now how to exploit the above freedom in order to choose V^μ in a "natural" way. One possibility is to require $\frac{\mathrm{D}}{\mathrm{d}\tau}(u^\mu s_\mu) = 0$ which, according to relation (23) (but here even more so due to $u^\mu \| p^\mu$) is equivalent to

$$\frac{\mathrm{D}}{\mathrm{d}\tau}(p^\mu s_\mu) = \dot{p}^\mu s_\mu + p^\mu \dot{s}_\mu = 0.$$

Taking now $\dot{s}_\mu (= \dot{V}^\nu s_\nu V_\mu) = \frac{\alpha s^2}{\mu \mathcal{M}^2} V_\mu$, with α some dimensionless scalar and $\mu \equiv -V_\sigma p^\sigma$, $\mathcal{M}^2 \equiv -p_\sigma p^\sigma$ ($= m^2$ when $u^\mu \| p^\mu$), we find

$$\alpha = \frac{\mathcal{M}^2}{s^2}\dot{p}^\mu s_\mu = \frac{m^2}{s^2}{}^*R_{\mu\nu\alpha\beta} s^\mu u^\nu s^\alpha V^\beta \tag{128}$$

$$= m^{2*}R_{\hat{1}\hat{\gamma}\hat{1}\hat{0}} u^{\hat{\gamma}}, \tag{129}$$

which can in general (general *vacuum*) be decomposed by equation (67).[4] Therefore, if $p^\mu s_\mu = m u^\mu s_\mu = 0$ at some (initial) point and α is chosen as above, then $u^\mu s_\mu = 0$ (s^μ is "purely spatial") along the whole representative worldline.

Finally, a natural option for how to set $u^\mu s_\mu = 0$ is to simply select $V^\mu \equiv u^\mu$ ($= p^\mu/m$) initially. With α chosen by the above prescription, and with V^μ and s^μ evolving according to

$$\dot{V}^\mu = \frac{\alpha}{\mu m^2} s^\mu, \qquad \dot{s}^\mu = \frac{\alpha s^2}{\mu m^2} V^\mu, \tag{130}$$

the four-momentum p^μ will then remain tangent and the spin s^μ orthogonal to the representative worldline u^μ.

[4]Specifically in type-N spacetimes, $\alpha = 0$.

VI. SPECIAL TYPES OF MOTION

It is useful to once more realize what can actually be chosen freely in the spinning particle exercise. Tackling it as a $3+1$ problem (e.g. when integrating the MPD equations on computer), one typically first selects V^μ and the three-vectors of initial relative velocity and initial spin with respect to some observer (which may be different from V^μ); these determine the initial four-velocity u^μ and four-spin s^μ; then the initial bivector $S_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} V^\gamma s^\delta$ is calculated. The remaining point is to obtain the initial four-momentum p^μ; this is practically done in dependence on the chosen spin supplementary condition, but *in principle* p^μ follows by integrating the energy-momentum tensor over a hypersurface fixed by V^μ. Hence, apart from initial conditions, V^μ is *the only* freely selectable quantity.

It is thus natural that we have first considered the choice of V^μ, because this may be done without loss of generality. In this section, secondarily, let us check whether some "clean" cases do not follow for special types of motion, i.e. for special u^μ or/and p^μ (whether with a special choice of V^μ or not). Note that some of these have already been mentioned within the previous section on specific spin conditions.

It should be emphasized that one must distinguish between a special setting holding *at one point* and the much stronger (and by default considered) circumstance of such a setting remaining valid *along the whole representative worldline*.

A. Special u^μ

The MPD components (66)–(69) simplify when some of the four-velocity components $u^{\hat{\alpha}}$ vanish.

1. u^μ lying in the Weyl-tensor eigenplane

If u^μ lies in the plane spanned by k^μ and l^μ, it is orthogonal to m^μ and $\bar{m}^\mu$, hence $u^{\hat{2}} = 0$ and $u^{\hat{3}} = 0$, with obvious effect on the above equations. However, the plane of k^μ and l^μ is the plane of V^μ and s^μ—namely, it is the eigenplane of $S^{\mu\nu}$—hence necessarily $S^{\mu\sigma} u_\sigma = 0$. If the above holds along the trajectory (not just at one point), one is thus back at the Mathisson-Pirani condition in Sec. V A.

2. u^μ (and thus p^μ) orthogonal to s^μ

If u^μ is perpendicular to s^μ or, in other words, u^μ is tangent to a timelike hypersurface spanned by V^μ, m^μ, and $\bar{m}^\mu$, then $u^{\hat{1}} = 0$ disappears from the equations. If the transverse NP frame could be used, thus having equations in the (86)–(89) form, the time component would vanish in that case, $-V_\mu \dot{p}^\mu = 0$.

Let us check for further consequences, mainly for how the choice of V^μ is restricted by the requirement $s^\alpha u_\alpha = 0$. Firstly, relation (23) says that $s^\alpha p_\alpha = 0$ then, too. We know that such a situation can be accomplished by selecting

$V^\mu = u^\mu$ or $V^\mu = p^\mu/\mathcal{M}$, i.e. by Mathisson-Pirani or Tulczyjew choice of the spin supplementary condition (Secs. V A and V B above), and that the simultaneous orthogonality of s^μ to both u^μ and p^μ may also happen when these two vectors are parallel (Sec. V C). However, here we want to check whether there are some other alternatives, so we assume that the three time-direction vectors V^μ, u^μ, and p^μ are independent.

Now, since s^μ is orthogonal to all of them, it can be written

$$s^\mu = \frac{s}{\dot{s}} \epsilon^{\mu\iota\kappa\lambda} V_\iota u_\kappa p_\lambda, \tag{131}$$

where (33) has been used for "normalization." Consequently, equation (31) assumes the form

$$\dot{s}^\mu = V^\mu \dot{V}_\nu s^\nu + \frac{\dot{s}}{s} s^\mu. \tag{132}$$

Does anything follow for the evolution of V^μ? Projecting the equation (35) on u^μ and p^μ, one finds, respectively,

$$u_\mu \frac{\mathrm{D}}{\mathrm{d}\tau}(sV^\mu) \equiv u_\mu(s\dot{V}^\mu + \dot{s}V^\mu) = 0, \tag{133}$$

$$p_\mu \frac{\mathrm{D}}{\mathrm{d}\tau}(sV^\mu) \equiv p_\mu(s\dot{V}^\mu + \dot{s}V^\mu) = 0, \tag{134}$$

which means orthogonality to both u^μ and p^μ, because the vector $(s\dot{V}^\mu + \dot{s}V^\mu)$ cannot be trivial. Hence, this vector has to be an eigenvector of $\dot{S}^{\mu\nu}$ (with zero eigenvalue again). It is also simple to check, e.g. by multiplying relation (132)) by $\dot{V}_\mu$, that

$$\dot{s}_\mu(s\dot{V}^\mu + \dot{s}V^\mu) = 0. \tag{135}$$

Since the vectors u^μ, p^μ, and $\dot{s}^\mu$ have to be independent,[5] this means that $\frac{\mathrm{D}}{\mathrm{d}\tau}(sV^\mu)$ is orthogonal to all u^μ, p^μ, and $\dot{s}^\mu$, so it can also be represented as

$$s\dot{V}^\mu + \dot{s}V^\mu = \frac{1}{\dot{s}} \epsilon^{\mu\iota\kappa\lambda} \dot{s}_\iota u_\kappa p_\lambda = -\frac{1}{\dot{s}} {}^*\dot{S}^{\mu\iota} \dot{s}_\iota \tag{136}$$

[5]Otherwise (132)) would be a combination of u^μ and p^μ, i.e.

$$V^\mu \dot{V}_\nu s^\nu + (\dot{s}/s)s^\mu = Au^\mu + Bp^\mu.$$

Multiplying this by s_μ, we find $s\dot{s} = 0$, which however also equals $\epsilon^{\mu\nu\alpha\beta} s_\mu V_\nu u_\alpha p_\beta$ according to (33). Now, u^μ, p^μ, and V^μ are assumed to be independent in this part, so s^μ would have to be dependent, which is in contradiction with its being orthogonal to all the three.

[cf. its generic decomposition (35)]. Besides $\frac{\mathrm{D}}{\mathrm{d}\tau}(sV^\mu)$, the other eigenvector of $\dot{S}^{\mu\nu}$ is of course s^μ, its derivative $\dot{s}^\mu$ belonging to the eigenplane of $S^{\mu\nu}$ conversely.[6]

Before continuing, two points should be stressed:

(1) It might seem that $\dot{V}^\mu$ is aligned with s^μ (which would ensure $u^\mu \| p^\mu$), because both these vectors are orthogonal to the triple V^μ, $(\mu u^\mu - \gamma p^\mu)$, and $(s\dot{s}^\mu - \dot{s}s^\mu)$ (for s^μ it is *always* so, while $\dot{V}^\mu$ is only orthogonal to the first two in general); note that the second vector of the triple is orthogonal to both the remaining two. But the suspicion is not the case, because exactly in the case when s^μ is orthogonal to u^μ and p^μ, the vector $(s\dot{s}^\mu - \dot{s}s^\mu)$ is proportional to V^μ [it is clear from equation (132))], so the triple is not independent.

(2) Mind that orthogonality to both u^μ and p^μ does *not* mean lying in the plane spanned by V^μ and s^μ: as V^μ, p^μ, and u^μ are all timelike, they are never orthogonal to each other, so the planes (V^μ, s^μ) and (u^μ, p^μ) are never orthogonal, in spite of s^μ being orthogonal to both u^μ and p^μ (in this subsection).

Let us assume that the orthogonalities $s^\alpha u_\alpha = 0$ and $s^\alpha p_\alpha = 0$ remain valid all along the representative worldline, i.e. that $\frac{\mathrm{D}}{\mathrm{d}\tau}(s^\alpha u_\alpha) = 0$, $\frac{\mathrm{D}}{\mathrm{d}\tau}(s^\alpha p_\alpha) = 0$ as well. Combining these with the relation (25), i.e. with $\gamma p^\alpha \dot{s}_\alpha = \mu u^\alpha \dot{s}_\alpha$, we also see that

$$\gamma \dot{p}^\alpha s_\alpha = \mu \dot{u}^\alpha s_\alpha, \tag{137}$$

which can further be extended on account of relation (31):

$$\begin{aligned}\gamma \dot{p}^\alpha s_\alpha &= \mu \dot{u}^\alpha s_\alpha = \gamma\mu \dot{V}^\alpha s_\alpha \\ &= -\gamma p^\alpha \dot{s}_\alpha = -\mu u^\alpha \dot{s}_\alpha = -\gamma\mu V^\alpha \dot{s}_\alpha.\end{aligned} \tag{138}$$

The above means that we already know of several vectors orthogonal to s^μ: V^μ, p^μ, u^μ, and $(\gamma\dot{p}^\mu - \mu\dot{u}^\mu)$, of which the last two are also orthogonal to each other. Hence, $(\gamma\dot{p}^\mu - \mu\dot{u}^\mu)$ must be some combination of V^μ, p^μ, and u^μ, because these three are independent by assumption.

3. $\dot{u}^\mu = 0$: *Geodesic motion*

It is known that in special situations the spin-curvature interaction may have no effect on the particle's four-velocity, thus leaving the motion free. Vanishing of acceleration $\dot{u}^\mu$ implies, irrespectively of spin condition,

$$\dot{m} = -\dot{u}_\mu p^\mu = 0, \tag{139}$$

$$\dot{p}^\mu = -\ddot{S}^{\mu\sigma} u_\sigma. \tag{140}$$

[6]The second eigenvector of $\dot{S}^{\mu\nu}$ *orthogonal* to s^μ is $\epsilon^{\mu\iota\alpha\beta} s_\iota u_\alpha p_\beta$. The vector $(s\dot{V}^\mu + \dot{s}V^\mu)$ was already decomposed into this basis in equation (35).

Note that $\dot{S}^{\alpha\beta}\dot{u}_\beta = 0$ holds due to $\dot{u}_\beta = 0$ here, so one cannot argue that $\dot{u}^\mu$ is another eigenvector of $\dot{S}^{\alpha\beta}$, the latter thus being trivial, etc.

When $\dot{u}^\mu = 0$, the Mathisson-Pirani condition $V^\mu = u^\mu$ clearly coincides with the $\dot{V}^\mu = 0$ condition. This is an advantageous option, since the latter yields $p^\mu = mu^\mu$ and $\dot{p}^\mu = m\dot{u}^\mu (= 0)$, so there is no force, and the MPD exercise reduces to the constraint

$$ {}^*R^\mu{}_{\nu\alpha\beta} u^\nu s^\alpha u^\beta \equiv B^\mu_\alpha s^\alpha = 0, \tag{141} $$

or, if written out in terms of its projections (96)–(98),

$$ \mathrm{Im}\Psi_2 = 0, \qquad \mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3 = 0, $$
$$ \mathrm{Re}\Psi_1 + \mathrm{Re}\Psi_3 = 0. \tag{142} $$

Hence, the $B_{\alpha\beta}$ field has to be zero in the rest frame of the particle, or, if this is not the case, the particle's spin has to be its eigenvector (with zero eigenvalue).

On the other hand, the Tulczyjew condition $S^{\alpha\beta}p_\beta = 0$ yields different results—in particular, it allows for nonzero force $\dot{p}^\mu$ even when $\dot{u}^\mu = 0$. Regarding (102), we however see that $s_\mu \dot{p}^\mu = 0$, which in turn reduces the expression (104) to

$$ s^2 \dot{p}^\mu = \mathcal{M}\epsilon^{\mu\iota\alpha\beta} s_\iota u_\alpha p_\beta. \tag{143} $$

Also, using $V_\nu \equiv p_\nu/\mathcal{M}$ and $\dot{p}_\nu s^\nu = 0$ in (31), we have

$$ \dot{s}^\mu = 0 \qquad (\Rightarrow \dot{s} = 0). \tag{144} $$

Finally, since $e^{\hat{1}}_\mu \dot{p}^\mu = 0$, the Tulczyjew-condition form of the MPD decomposition, (107)–(110), reduces to the last two projections, plus the condition that the expression (113) has to yield zero. The latter says that the masses m and $\mathcal{M}$ are necessarily related through the scalars Ψ_1, Ψ_3, and $\mathrm{Im}\Psi_2$. Equivalently (and more simply), one can calculate $e^{\hat{2}}_\mu \dot{p}^\mu$ and $e^{\hat{3}}_\mu \dot{p}^\mu$ from (143). As the latter is clearly spacelike and orthogonal to s^μ, it is moreover possible to rotate the $e^\mu_{\hat{2}}$, $e^\mu_{\hat{3}}$ vectors so that it has just a single component.

4. Stationary situation

Consider now a situation in which relevant scalars do not change along u^μ. It is in fact sufficient to demand

$$ \dot{\mathcal{M}} = 0 \Rightarrow \dot{S}^{\alpha\beta}\dot{p}_\beta = 0, \qquad \dot{p}_\alpha p^\alpha = 0, $$

which means that $\dot{p}_\beta$ is an eigenvector of $\dot{S}^{\alpha\beta}$. Since $\dot{p}_\beta$ (supposed to be nonzero) is independent of the generic eigenvectors of $\dot{S}^{\alpha\beta}$ given in Sec. II D, the bivector $\dot{S}^{\alpha\beta}$ has to be zero rank (antisymmetry only allows for even rank, so it cannot be 1); namely, it is trivial. Hence, ${}^*\dot{S}^{\mu\nu}$ is also zero, and p^μ and u^μ are parallel, $p^\mu = mu^\mu$, which implies (cf. Sec. V C, but mind that we do *not* assume $\dot{V}^\mu = 0$ now)

$$ m = \mathcal{M}, \qquad \mu = \gamma m, \qquad \dot{s} = 0, $$
$$ \dot{s}^\mu = V^\mu \dot{V}_\nu s^\nu, \qquad s^2 \dot{V}^\mu = -s^\mu \dot{s}_\nu V^\nu. $$

In a stationary situation, one can also assume

$$ \frac{\mathrm{d}}{\mathrm{d}\tau}(p_\mu s^\mu) = m\frac{\mathrm{d}}{\mathrm{d}\tau}(u_\mu s^\mu) = 0; \tag{145} $$

in fact, products between any dot-derivatives of p^μ (thus u^μ) and s^μ should be constant as well, which brings the chain of relations

$$ \dot{p}_\mu s^\mu = -p_\mu \dot{s}^\mu, $$
$$ \ddot{p}_\mu s^\mu = -\dot{p}_\mu \dot{s}^\mu = p_\mu \ddot{s}^\mu, $$
$$ \dddot{p}_\mu s^\mu = -\ddot{p}_\mu \dot{s}^\mu = \dot{p}_\mu \ddot{s}^\mu = -p_\mu \dddot{s}^\mu, \quad \text{etc.}\ldots $$

Let us stress, on the other hand, that we do not *a priori* demand anything of scalars involving V^μ, because this vector is auxiliary and need not respect the symmetry of the physical problem necessarily. However, in Sec. V C 1, we saw that the $u^\mu \| p^\mu$ case offers one natural possibility: to prescribe $V^\mu(\tau)$ so that s^μ is—and remains—orthogonal to p^μ.

B. Other

Other special properties are of course possible, though some of them are either contained in what has already been discussed, or they do not seem to lead to a particular simplification of the exercise.

For example, if $u_\alpha \dot{V}^\alpha = 0$, then according to (24) $p_\alpha \dot{V}^\alpha = 0$ as well, hence $\dot{S}_{\alpha\beta}\dot{V}^\beta = 0$; since $\dot{V}^\beta$ is in general independent of the eigenplane of $\dot{S}_{\alpha\beta}$ (Sec. II D), $\dot{S}_{\alpha\beta}$ has to be trivial, and we are back in the situation mentioned in the previous subsection. Similarly, $u_\alpha \dot{s}^\alpha = 0$ implies $p_\alpha \dot{s}^\alpha = 0$ by (25), from which also $\dot{S}_{\alpha\beta}\dot{s}^\beta = 0$ and again $\dot{S}_{\alpha\beta} = 0$ generically.

An interesting case follows if $s_\alpha \dot{V}^\alpha = -\dot{s}_\alpha V^\alpha = 0$. Then all the vectors V^μ, s^μ, $\dot{V}^\mu$, $\mu u^\mu - \gamma p^\mu$ are orthogonal to each other, so they form an orthogonal basis in spacetime; this basis is actually the one we introduced in Sec. III A, just with the vectors $e^\mu_{\hat{2}}$ and $e^\mu_{\hat{3}}$ specially given by $\dot{V}^\mu$ and $\mu u^\mu - \gamma p^\mu$.

VII. CONCLUDING REMARKS

The spinning particle problem is known to be inherently problematic, at least in the pole-dipole approximation, but it has a considerable history and brings a nice geometry. And it is not only of theoretical interest, as seen in growing interaction with the wide effort to understand and predict the generation of gravitational waves by collisions of compact objects, either by approximations or by purely numerical approaches. In particular, an inspiral of a binary of compact bodies has become a key process in that field, and the role of spin in its outcome is a very lively topic—see e.g. Refs. [49,50] or, especially for the question of the spin supplementary condition, Ref. [51]. The effects of spin can mainly be expected to be important in the final stages of a black hole merger, because close to the horizon they are in fact stronger than radial attraction due to mass (in case of a single black hole, this is a defining property of the static limit, thus of the ergosphere), though with distance they fall much faster than the "Newtonian" component. One particular situation where spin has been found to play a crucial role is with the "gravitational kicks" which the outcomes of binary black hole mergers can get as a result of anisotropic emission of gravitational waves—see e.g. Ref. [52] or, specifically for the role of the "hidden momentum" in this effect, Ref. [53].

Let us mention, in particular, the extreme-mass-ratio limit of the binary inspiral, because it has been given special attention recently, and because the spinning particle problem represents its limit neglecting the radiation and approximating the small body by a test pole-dipole top [35]. See, for example, Ref. [54] for a review of this field, also including a discussion of different spin supplementary conditions. The extreme-mass-ratio instance of gravitational recoils has been studied e.g. by Ref. [55].

A decent history must, of course, be also attributed to the area of the algebraic structure of curvature (Petrov types). Although in the neighborhood of compact-body astrophysics it looks more academic, it is not fully so. Actually, the Kerr solution of Einstein's equations, celebrated by S. Chandrasekhar mainly as an *astrophysical* discovery, and indeed by default considered by astrophysicists when speaking of galactic nuclei or some binary x-ray sources, *is* algebraically special (type D). If the field in the core of these systems (of galactic nuclei in particular) is really close to the Kerr one, its special structure should reveal, for example, in the inspiral of some much lower-mass compact body (mentioned above) and, consequently, also in the generated waves. In the monopole–test-particle limit, the "Kerr-like" algebraic type is actually necessary for the geodesic equation (and also several other important problems) to be completely integrable [56], hence not allowing for chaotic behavior. However, even in type-D spacetimes the motion *is* in general chaotic if the particle is endowed with spin (Ref. [32] and references therein) or higher multipoles. The issues of spin, curvature structure, and orbital dynamics are thus naturally bound together within one of today's major application directions of general relativity. We have not considered any gravitational waves emitted by the orbiting particle in this paper, but the background spacetime certainly *can* contain waves. In this respect, the often treated Petrov type N is of physical relevance as a possible approximation of the far-zone radiation fields of bounded sources (see e.g. Ref. [57] and references therein).

To summarize the present paper, we have first reviewed the Mathisson-Papapetrou-Dixon (MPD) formulation and derived (or quoted) some useful relations of generic validity, while mainly focusing on the role of the spin supplementary condition. In the second part we projected the MPD equation of motion onto a suitable tetrad and expressed the spin-curvature term on its right-hand side in terms of the Weyl scalars Ψ_{0-4} obtained in the complex null (Newman-Penrose) tetrad related to the orthonormal one. Specifically, we have chosen the orthonormal tetrad tied to the "reference" observer V^μ, fixing the spin condition, taking the corresponding spin vector s^μ (or rather its unit form) as one of the spatial legs. In such a tetrad, the MPD equation appears as (66)–(69), which does not at all contain the null-tetrad scalars Ψ_0 and Ψ_4. The remaining two spatial vectors can be chosen in various ways, of which the preferable are those fixing them "intrinsically," along some directions provided by the geometry of the problem itself. We described one such possibility, applicable when $u^\mu \nparallel p^\mu$ and given by $(\mu u^\mu - \gamma p^\mu)$ and the "vector product" of the former three, which can be expressed as $(s^2\delta^\mu_\nu - s^\mu s_\nu)\dot{V}^\nu$.

Having expressed the MPD equation in terms of the Weyl scalars, it is natural to ask whether it assumes any special, simple form in spacetimes of particular Petrov types. For such a purpose, it is advantageous to choose V^μ so that the highest-multiplicity principal null direction of the Weyl tensor falls within the eigenplane of the spin bivector, and to make it the first vector of the associated null tetrad. Even more favorable would be to make the null tetrad "transverse" in the sense that the corresponding Ψ_1 and Ψ_3 projections vanish; the spinning particle motion would then be fully determined by Ψ_2 and by the cosmological constant. Unfortunately, such a turn can only be reconciled with the spin structure in exceptional cases; namely, when it is possible—by a suitable choice of V^μ—to identify the spin eigenplane with the real-vector plane of some of the transverse tetrads.

In the last part, we first treated where the exercise leads for the main spin conditions considered in the literature, and then revisited in particular the condition $\dot{V}^\mu = 0$ (ensuring the very advantageous arrangement $u^\mu \parallel p^\mu$) and generalized it, suggesting also a natural resolution of nonuniqueness of the corresponding reference observer V^μ. Then we checked how the equations are compatible with several particular types of motion.

Our next plan is to compare the analysis with that made in a different interpretation tetrad, namely the one tied to the wordline tangent u^μ, and also to consider the case of massless particles.

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APPENDIX: ORTHONORMAL-TETRAD AND NULL-TETRAD COMPONENTS OF THE WEYL TENSOR

Here we list the Weyl-tensor components in *some* orthonormal tetrad $(V^\mu, e^\mu_{\hat{i}})$ and in the related Newman-Penrose null tetrad $(k^\mu, l^\mu, m^\mu, \bar{m}^\mu)$. Note that we actually need the Weyl-tensor *dual* in the Mathisson-Papapetrou-Dixon equations, but the null-tetrad scalars $\Psi_0, \ldots, \Psi_4$ only change by an imaginary unit when dualizing $C_{\alpha\beta\gamma\delta}$, so it does not matter if we write the relations for $C_{\alpha\beta\gamma\delta}$ itself or for its dual. Note also that ${}^*C_{\alpha\beta\gamma\delta} = C^*_{\alpha\beta\gamma\delta}$ has the same symmetries as $C_{\alpha\beta\gamma\delta}$ (and hence ten independent components).

Besides the "automatic" properties following from the Riemann-tensor–type symmetries, the additional vanishing of the Weyl-tensor nontrivial trace, $g^{\alpha\gamma}C_{\alpha\beta\gamma\delta} = 0$, implies another set of useful relations (when projected onto various null dyads):

$$C_{kmk\bar{m}} = C_{lml\bar{m}} = C_{mkml} = C_{\bar{m}k\bar{m}l} = 0, \tag{A1}$$

$$C_{km\bar{m}m} = \Psi_1, \qquad C_{lm\bar{m}m} = \bar{\Psi}_3, \tag{A2}$$

$$C_{klkl} = C_{m\bar{m}m\bar{m}} = \Psi_2 + \bar{\Psi}_2 \equiv 2\mathrm{Re}\Psi_2, \tag{A3}$$

$$C_{klm\bar{m}} = -\Psi_2 + \bar{\Psi}_2 \equiv -2\mathrm{i}\mathrm{Im}\Psi_2, \tag{A4}$$

where obvious notation has been used. Direct substitution then yields the orthonormal components

$$C_{\hat{0}\hat{1}\hat{0}\hat{1}} = C_{klkl} = 2\mathrm{Re}\Psi_2,$$

$$C_{\hat{0}\hat{2}\hat{0}\hat{2}} = \mathrm{Re}C_{kml\bar{m}} + \frac{1}{2}\mathrm{Re}(C_{kmkm} + C_{lmlm}) = -\mathrm{Re}\Psi_2 + \frac{1}{2}(\mathrm{Re}\Psi_0 + \mathrm{Re}\Psi_4),$$

$$C_{\hat{0}\hat{3}\hat{0}\hat{3}} = -C_{\hat{0}\hat{1}\hat{0}\hat{1}} - C_{\hat{0}\hat{2}\hat{0}\hat{2}} = -\mathrm{Re}\Psi_2 - \frac{1}{2}(\mathrm{Re}\Psi_0 + \mathrm{Re}\Psi_4),$$

$$C_{\hat{1}\hat{2}\hat{1}\hat{2}} = -C_{\hat{0}\hat{3}\hat{0}\hat{3}},$$

$$C_{\hat{1}\hat{3}\hat{1}\hat{3}} = -C_{\hat{0}\hat{2}\hat{0}\hat{2}},$$

$$C_{\hat{2}\hat{3}\hat{2}\hat{3}} = -C_{\hat{0}\hat{1}\hat{0}\hat{1}},$$

$$C_{\hat{0}\hat{1}\hat{0}\hat{2}} = C_{\hat{3}\hat{1}\hat{3}\hat{2}} = \mathrm{Re}C_{klmk} + \mathrm{Re}C_{klml} = \mathrm{Re}\Psi_3 - \mathrm{Re}\Psi_1,$$

$$C_{\hat{0}\hat{1}\hat{0}\hat{3}} = C_{\hat{2}\hat{1}\hat{2}\hat{3}} = -\mathrm{Im}C_{klkm} + \mathrm{Im}C_{klml} = -\mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3,$$

$$C_{\hat{0}\hat{2}\hat{0}\hat{3}} = C_{\hat{1}\hat{2}\hat{1}\hat{3}} = \frac{1}{2}\mathrm{Im}(C_{kmkm} + C_{lmlm}) = \frac{1}{2}(\mathrm{Im}\Psi_0 - \mathrm{Im}\Psi_4),$$

$$C_{\hat{0}\hat{1}\hat{2}\hat{1}} = -C_{\hat{0}\hat{3}\hat{2}\hat{3}} = \mathrm{Re}C_{klkm} + \mathrm{Re}C_{klml} = \mathrm{Re}\Psi_1 + \mathrm{Re}\Psi_3,$$

$$C_{\hat{0}\hat{1}\hat{3}\hat{1}} = -C_{\hat{0}\hat{2}\hat{3}\hat{2}} = \mathrm{Im}C_{klkm} + \mathrm{Im}C_{klml} = \mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3,$$

$$C_{\hat{0}\hat{2}\hat{1}\hat{2}} = -C_{\hat{0}\hat{3}\hat{1}\hat{3}} = \frac{1}{2}\mathrm{Re}(C_{kmkm} - C_{lmlm}) = \frac{1}{2}(\mathrm{Re}\Psi_0 - \mathrm{Re}\Psi_4),$$

$$C_{\hat{0}\hat{1}\hat{2}\hat{3}} = -\mathrm{i}C_{klm\bar{m}} = -2\mathrm{Im}\Psi_2,$$

$$C_{\hat{0}\hat{2}\hat{1}\hat{3}} = \mathrm{Im}C_{kml\bar{m}} + \frac{1}{2}\mathrm{Im}(C_{kmkm} - C_{lmlm}) = -\mathrm{Im}\Psi_2 + \frac{1}{2}(\mathrm{Im}\Psi_0 + \mathrm{Im}\Psi_4),$$

$$C_{\hat{0}\hat{3}\hat{1}\hat{2}} = C_{\hat{0}\hat{2}\hat{1}\hat{3}} - C_{\hat{0}\hat{1}\hat{2}\hat{3}} = \mathrm{Im}\Psi_2 + \frac{1}{2}(\mathrm{Im}\Psi_0 + \mathrm{Im}\Psi_4).$$

(Not all these are independent, needless to say. Others can be obtained just using the $C_{[\mu\nu][\kappa\lambda]}$ antisymmetries and the $C_{[\mu\nu]\leftrightarrow[\kappa\lambda]}$ symmetry.) The respective components of the *dual* Weyl tensor are obtained according to

$$\Psi \to {}^*\Psi = \mathrm{i}\Psi\colon\ \mathrm{Re}({}^*\Psi) = -\mathrm{Im}\Psi, \qquad \mathrm{Im}({}^*\Psi) = \mathrm{Re}\Psi.$$

1. Electric and magnetic curvature

Let us look into which Weyl scalars enter the Weyl-tensor electric and magnetic parts. These are introduced, in analogy with electric and magnetic parts of the Faraday tensor, as projections of the Weyl tensor on some timelike vector field (in our case represented by V^μ),[7]

$$E_{\alpha\beta} := C_{\alpha\mu\beta\nu}V^\mu V^\nu \equiv C_{\alpha\hat{0}\beta\hat{0}}, \tag{A5}$$

$$B_{\alpha\beta} := {}^*C_{\alpha\mu\beta\nu}V^\mu V^\nu \equiv {}^*C_{\alpha\hat{0}\beta\hat{0}}. \tag{A6}$$

Orthonormal components of these (symmetric) gravito-electric and gravitomagnetic tidal fields can be seen above—namely, $E_{\hat{\imath}\hat{\jmath}}$ are given by real parts of Ψ_a, except for $E_{\hat{1}\hat{3}}$ and $E_{\hat{2}\hat{3}}$, which are given by imaginary parts of Ψ_1 and Ψ_3, or of Ψ_0 and Ψ_4, respectively; in $B_{\hat{\imath}\hat{\jmath}}$, the appearance of real and imaginary parts is reversed.

In the so-called transverse orthonormal frames (Sec. IV C), the first MPD equation can in favorable cases be expressed in terms of just Ψ_2, which underlines the importance of type-D curvature, where the latter is the only relevant Weyl scalar. If we keep only Ψ_2, we see that the above tidal fields have only diagonal components:

$$E_{\hat{1}\hat{1}} = -2E_{\hat{2}\hat{2}} = -2E_{\hat{3}\hat{3}} = 2\mathrm{Re}\Psi_2,$$
$$B_{\hat{1}\hat{1}} = -2B_{\hat{2}\hat{2}} = -2B_{\hat{3}\hat{3}} = -2\mathrm{Im}\Psi_2.$$

[7] Alternatively, one could introduce these tensors by projections onto $u^\mu u^\nu$ or $p^\mu p^\nu/\mathcal{M}^2$, so "as measured by the particle"; cf. Ref. [17], where the $u^\mu u^\nu$ projection is employed. In our case, it is more natural to use $V^\mu V^\nu$, since we are using V^μ as the time vector of the interpretation basis.

2. Rotations within the $(m^\mu, \bar{m}^\mu)$ plane

Of the well-known four basic types of the NP-frame transformations, namely null rotations preserving k^μ or l^μ, boosts in the (k^μ, l^μ) plane and spatial rotations in the $(m^\mu, \bar{m}^\mu)$ plane, the last ones are mainly of interest for us, because in our problem m^μ and $\bar{m}^\mu$ can be rotated arbitrarily within the plane orthogonal to V^μ and s^μ. Parametrizing such a rotation as

$$m'^\mu = \exp(\mathrm{i}\alpha)m^\mu, \qquad \bar{m}'^\mu = \exp(-\mathrm{i}\alpha)\bar{m}^\mu, \tag{A7}$$

the Weyl scalars transform according to $\Psi'_2 = \Psi_2$, and

$$\Psi'_0 = \exp(2\mathrm{i}\alpha)\Psi_0, \qquad \Psi'_4 = \exp(-2\mathrm{i}\alpha)\Psi_4, \tag{A8}$$

$$\Psi'_1 = \exp(\mathrm{i}\alpha)\Psi_1, \qquad \Psi'_3 = \exp(-\mathrm{i}\alpha)\Psi_3 \tag{A9}$$

(hence, those which were zero remain so). In particular, Ψ_1 and Ψ_3 obey

$$\mathrm{Re}\Psi'_1 = +\mathrm{Re}\Psi_1 \cos\alpha - \mathrm{Im}\Psi_1 \sin\alpha, \tag{A10}$$

$$\mathrm{Im}\Psi'_1 = +\mathrm{Re}\Psi_1 \sin\alpha + \mathrm{Im}\Psi_1 \cos\alpha, \tag{A11}$$

$$\mathrm{Re}\Psi'_3 = +\mathrm{Re}\Psi_3 \cos\alpha + \mathrm{Im}\Psi_3 \sin\alpha, \tag{A12}$$

$$\mathrm{Im}\Psi'_3 = -\mathrm{Re}\Psi_3 \sin\alpha + \mathrm{Im}\Psi_3 \cos\alpha. \tag{A13}$$

By such a rotation, it is generally possible to get rid of *one* component of *one* of these two scalars, but not more.

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