

Spinning particles in general relativity: Momentum-velocity relation for the Mathisson-Pirani spin condition

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(Received 8 January 2018; published 16 April 2018)

The Mathisson-Papapetrou-Dixon (MPD) equations, providing the “pole-dipole” description of spinning test particles in general relativity, have to be supplemented by a condition specifying the worldline that will represent the history of the studied body. It has long been thought that the Mathisson-Pirani (MP) spin condition—unlike other major choices made in the literature—does *not* yield an explicit momentum-velocity relation. We derive here the desired (and very simple) relation and show that it is in fact *equivalent* to the MP condition. We clarify the apparent paradox between the existence of such a definite relation and the known fact that the MP condition is degenerate (does not specify a unique worldline), thus shedding light on some conflicting statements made in the literature. We then show how, for a given body, this spin condition yields infinitely many possible representative worldlines, and derive a detailed method how to switch between them in a curved spacetime. The MP condition is a convenient choice in situations when it is easy to recognize its “nonhelical” solution, as exemplified here by bodies in circular orbits and in radial fall in the Schwarzschild spacetime.

DOI: [10.1103/PhysRevD.97.084023](https://doi.org/10.1103/PhysRevD.97.084023)

I. INTRODUCTION

The problem of motion of a “small body” in general relativity has been widely studied in the “pole-dipole” test-particle approximation when the body is not itself contributing to the gravitational field and when it is only characterized by mass and spin (proper angular momentum), with all the higher multipoles neglected. If the particle interacts solely gravitationally, the only force it is subjected to comes from the spin-curvature interaction and the pole-dipole problem is described by the Mathisson–Papapetrou–Dixon (MPD) equations

$$\frac{DP^\mu}{d\tau} = -\frac{1}{2}R^\mu{}_{\nu\kappa\lambda}U^\nu S^{\kappa\lambda} \equiv F^\mu, \quad (1)$$

$$\frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha}U^{\beta]} \equiv P^\alpha U^\beta - U^\alpha P^\beta, \quad (2)$$

where P^μ and $S^{\mu\nu}$ denote, respectively, the body’s 4-momentum and spin tensor (spin bivector), $U^\mu \equiv dz^\mu/d\tau$ the 4-velocity of the body’s representative worldline $z^\mu(\tau)$, and

$$\frac{D}{d\tau} \equiv \nabla_U \equiv ;_\mu U^\mu$$

denotes the covariant derivative along U^μ . Both P^μ and U^μ are assumed to be timelike, with U^μ normalized to $U_\mu U^\mu = -1$, which implies that τ is the proper time. Contractions of P^μ and U^μ provide the masses \mathcal{M} and m ,

$$-P_\mu P^\mu \equiv \mathcal{M}^2 > 0; \quad -P_\mu U^\mu \equiv m > 0,$$

respectively, the mass as measured in the zero 3-momentum and in the zero 3-velocity frames. The timelike character of both P^μ and U^μ is however not guaranteed automatically by the MPD equations, with possible breakdown of this requirement indicating ultimate limits of the pole-dipole description. The spin bivector is assumed to be spacelike, so

$$\frac{1}{2}S_{\mu\nu}S^{\mu\nu} \equiv S^2 > 0.$$

Since the MPD set (1)–(2) possesses 13 unknowns¹ for only 10 equations, in order to be closed, it has to be

¹Four independent components of P^α , 3 independent components of U^α , and 6 independent components of $S^{\alpha\beta}$.

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supplemented by 3 auxiliary constraints. These are provided by the so-called spin supplementary condition (SSC), standardly written as

$$S^{\mu\nu}V_\nu = 0,$$

where, in case of a particle with nonzero rest mass, V^μ is some (freely chosen) timelike vector field [defined at least along $z^\mu(\tau)$] which is supposed to normalize as $V_\mu V^\mu = -1$. This condition is a choice of a representative worldline $z^\mu(\tau)$; more precisely, it demands $z^\mu(\tau)$ to be, at each instant, the body's center of mass (or ‘‘centroid’’) as measured by an observer with instantaneous 4-velocity V^μ . Four choices of V^μ have proven particularly convenient:

- (i) $V^\mu \equiv U^\mu$ (Mathisson-Pirani (MP) condition [1,2], originally due to Frenkel [3]), which states that the reference worldline $z^\mu(\tau)$ is the centroid as measured in its own rest frame (the zero 3-velocity frame);
- (ii) $V^\mu \equiv P^\mu/\mathcal{M}$ (Tulczyjew-Dixon (TD) condition, [4,5]), which states that $z^\mu(\tau)$ is the centroid as measured in the zero 3-momentum frame;
- (iii) $V^\mu \propto u_{\text{lab}}^\mu + P^\mu/\mathcal{M}$ (Newton-Wigner (NW) condition, [6,7]), where $u_{\text{lab}}^\mu \propto \partial_t^\mu$ is the 4-velocity of the congruence of ‘‘laboratory’’ observers, at rest in the given coordinate system (typically somehow privileged by symmetries of the host spacetime);
- (iv) $P^\mu = mU^\mu$ ($P^\mu \parallel U^\mu$ condition, known also as Ohashi-Kyrian-Semerák (OKS) condition [8,9]), which demands V^μ to be such that $DV^\mu/d\tau$ belongs to the eigenplane of $S^{\mu\nu}$ [10], for instance when V^μ parallel transports along $z^\mu(\tau)$, $DV^\mu/d\tau = 0$.

A fifth, less popular choice, is $V^\mu \equiv u_{\text{lab}}^\mu$ (Corinaldesi-Papapetrou (CP) condition, [11]), which states that $z^\mu(\tau)$ is the centroid as measured in the ‘‘laboratory’’ frame [12].

The TD choice has been used most frequently, mainly because it leads to an explicit expression of the tangent U^μ in terms of P^μ , $S^{\mu\nu}$, and z^μ , the so-called momentum-velocity relation [13],

$$U^\mu = \frac{m}{\mathcal{M}^2} \left(P^\mu + \frac{2S^{\mu\nu}R_{\nu\kappa\lambda}P^\kappa S^{\lambda\sigma}}{4\mathcal{M}^2 + R_{\alpha\beta\gamma\delta}S^{\alpha\beta}S^{\gamma\delta}} \right). \quad (3)$$

Such relation is important, mainly in numerical treatment, where the integration of the MPD system is done recurrently using the instantaneous tangent U^μ (see [14] for details). The $P^\mu = mU^\mu$ option in itself represents the momentum-velocity relation and it turned out to simplify the spinning-particle problem considerably [10]. For the CP and NW conditions a momentum-velocity relation is obtainable, but complicated [12], and no explicit expression has yet been put forth. Finally, the MP SSC has also been used many times, but it has been thought that it does not lead to an explicit momentum-velocity relation (it has only been shown to provide such an expression for the

four-acceleration $DU^\mu/d\tau$, [8]). For recent discussions of the subject, see e.g. [10,12].

Units and notation: Geometric units are used throughout the article, $G = c = 1$. Greek letters denote the indices corresponding to spacetime, while Latin letters denote indices corresponding only to space. We use the Riemann tensor convention $R^\alpha_{\beta\gamma\delta} = \partial_\gamma\Gamma^\alpha_{\beta\delta} - \partial_\delta\Gamma^\alpha_{\beta\gamma} + \dots$, with metric signature $(-, +, +, +)$. We use abstract index notation for tensors $T^{\alpha\beta\gamma\dots}$ and 4-vectors V^α ; arrow notation \vec{V} denotes space components of a vector in a given frame. The Levi-Civita tensor is $\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\tilde{\epsilon}_{\mu\nu\rho\sigma}$, with the Levi-Civita symbol $\tilde{\epsilon}_{0123} = 1$.

II. EQUATIONS OF MOTION UNDER A SPIN SUPPLEMENTARY CONDITION (SSC)

First, just for self-completeness, let us repeat several simple formulas from [10,15]. Writing, for a *general* vector V^μ , the spin bivector in terms of the corresponding spin vector $S^\mu \equiv -\epsilon^\mu_{\nu\alpha\beta}V^\nu S^{\alpha\beta}/2$,

$$S_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta}V^\gamma S^\delta, \quad (4)$$

its evolution along U^μ is just

$$\frac{DS_{\alpha\beta}}{d\tau} = \epsilon_{\alpha\beta\gamma\delta} \frac{DV^\gamma}{d\tau} S^\delta + \epsilon_{\alpha\beta\gamma\delta} V^\gamma \frac{DS^\delta}{d\tau}. \quad (5)$$

In order to ‘‘extract’’ the evolution of S^μ , one substitutes Eq. (2) in Eq. (5) and multiplies this with $\epsilon^{\mu\nu\alpha\beta}V_\nu$, i.e.

$$(\delta_\nu^\mu + V^\mu V_\nu) \frac{DS^\nu}{d\tau} = \epsilon^{\mu\nu\alpha\beta} V_\nu U_\alpha P_\beta, \quad (6)$$

and hence

$$\frac{DS^\mu}{d\tau} = V^\mu \frac{DV_\nu}{d\tau} S^\nu + \epsilon^{\mu\nu\alpha\beta} V_\nu U_\alpha P_\beta. \quad (7)$$

This yields

$$S \frac{dS}{d\tau} = \frac{1}{2} \frac{dS^2}{d\tau} = S_\mu \frac{DS^\mu}{d\tau} = \epsilon^{\mu\nu\alpha\beta} V_\mu S_\nu P_\alpha U_\beta \quad (8)$$

for evolution of the spin magnitude $S = \sqrt{S_\mu S^\mu} = \sqrt{S_{\alpha\beta}S^{\alpha\beta}/2}$.

In order to express the evolution of V^μ instead, one multiplies the relation (5) by $\epsilon^{\mu\nu\alpha\beta}S_\nu$ and uses Eq. (8), to arrive at

$$(S^2 \delta_\nu^\mu - S^\mu S_\nu) \frac{DV^\nu}{d\tau} = (\delta_\nu^\mu + V^\mu V_\nu) \epsilon^{\nu\alpha\beta} S_\nu U_\alpha P_\beta, \quad (9)$$

and hence

$$S \frac{D(SV^\mu)}{d\tau} = -S^\mu \frac{DS_\nu}{d\tau} V^\nu + \epsilon^{\mu\nu\alpha\beta} S_\nu U_\alpha P_\beta. \quad (10)$$

Substituting Eq. (4) into (1), we can also express the force in terms of the spin vector S^μ ,

$$F^\alpha \equiv \frac{DP^\alpha}{d\tau} = \star R^{\sigma\tau\alpha\mu} S_\sigma V_\tau U_\mu, \quad (11)$$

where $\star R_{\alpha\beta\gamma\delta} \equiv \epsilon_{\alpha\beta}{}^{\mu\nu} R_{\mu\nu\gamma\delta}/2$.

Let us stress again that, up to now, everything has been valid for a *generic* time-like vector V^μ . For this generic vector one can obtain a general $P-U$ relation by contracting the spin evolution equation (2) with V_β , and noticing that, by virtue of $S^{\alpha\beta}V_\beta = 0$, $V_\beta DS^{\alpha\beta}/d\tau = -S^{\alpha\beta}DV_\beta/d\tau$, leading to [16–19]

$$P^\alpha = \frac{1}{\gamma(V, U)} \left(\mu U^\alpha + S^{\alpha\beta} \frac{DV_\beta}{d\tau} \right), \quad (12)$$

where $\mu \equiv -P^\alpha V_\alpha$ is the mass as measured by an observer of 4-velocity V^α , and $\gamma(V, U) \equiv -U^\alpha V_\alpha$ is the Lorentz factor between U^α and V^α .

A. MPD system under the MP SSC

Consider now the MP SSC, i.e., let $V^\mu \equiv U^\mu$. The force equation (11) becomes [15]²

$$F^\alpha \equiv \frac{DP^\alpha}{d\tau} = \mathbb{H}^{\beta\alpha} S_\beta, \quad (13)$$

where $\mathbb{H}^\alpha{}_\beta \equiv \star R^\alpha{}_{\mu\nu} U^\mu U^\nu = \epsilon^\alpha{}_{\mu\sigma\tau} R^{\sigma\tau}{}_{\beta\nu} U^\mu U^\nu/2$ is the “gravitomagnetic tidal tensor” (or “magnetic part of the Riemann tensor”) as measured by an observer of 4-velocity U^α . The spin evolution equation becomes the Fermi-Walker transport law (e.g. [20]),

$$\frac{DS^\mu}{d\tau} = U^\mu \frac{DU^\nu}{d\tau} S_\nu. \quad (14)$$

These expressions are a unique feature of the MP SSC. Equation (14) tells us that S^μ has fixed components in the locally nonrotating frame comoving with the centroid. A locally nonrotating frame is mathematically defined as Fermi-Walker transported frame, and is physically realized as a frame where the Coriolis forces vanish. This means that S^α follows the “compass of inertia” [21], which is the most natural spin behavior (in the absence of torques), since gyroscopes are well known for opposing to changes in direction of their rotation axis. (The spin vectors of other spin conditions, by contrast, are not fixed, in general, relative to the comoving nonrotating frame).

²There is a sign difference compared to the expression in [15], due to the different sign convention for the Levi-Civita tensor.

B. The momentum-acceleration relation for the MP SSC

For $V^\mu = U^\mu$, Eq. (8) implies that S is a constant, thus Eq. (10) is rewritten as

$$\begin{aligned} S^2 \frac{DU^\mu}{d\tau} &= -S^\mu \frac{DS_\nu}{d\tau} U^\nu + \epsilon^{\mu\nu\alpha\beta} S_\nu U_\alpha P_\beta \\ &= -S^\mu \frac{DS_\nu}{d\tau} U^\nu - S^{\mu\beta} P_\beta, \end{aligned} \quad (15)$$

since now

$$S^{\mu\beta} = \epsilon^{\mu\beta\alpha\nu} U_\alpha S_\nu; \quad S^\alpha = -\frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} U_\beta S_\gamma. \quad (16)$$

Note that the first term on the right-hand side of Eq. (15) can also be rewritten as $-\frac{1}{m} S^\mu P^\nu DS_\nu/d\tau$ thanks to the generally valid relation

$$\gamma(V, U) P^\nu \frac{DS_\nu}{d\tau} = \mu U^\nu \frac{DS_\nu}{d\tau}$$

(which specifically for $V^\mu \equiv U^\mu$ means $P^\nu DS_\nu/d\tau = m U^\nu DS_\nu/d\tau$). One, thus, obtains the momentum-acceleration relation reached in [8],

$$a^\alpha \equiv \frac{DU^\alpha}{d\tau} = \frac{1}{S^2} \left(\frac{1}{m} F^\mu S_\mu S^\alpha - P_\gamma S^{\alpha\gamma} \right), \quad (17)$$

where a^α is the acceleration.

C. The momentum-velocity relation for the MP SSC

For $V^\mu = U^\mu$, Eq. (12) yields

$$P^\mu = m U^\mu + S^{\mu\nu} a_\nu, \quad (18)$$

where $a^\mu = DU^\mu/d\tau$. The desired momentum-velocity relation³ follows simply by substituting in Eq. (18) the acceleration a_ν from Eq. (15),

$$m U^\mu = P^\mu + \frac{1}{S^2} S^{\mu\nu} S_{\nu\beta} P^\beta. \quad (19)$$

(One only employs the fact that $S^{\mu\nu} S_\nu = 0$ by definition.) The relation (19) contains two scalars, m and S , which are constant in case of the MP SSC. Therefore, they are fixed by the initial conditions. Contracting Eq. (19) with P^α , we get

$$m^2 = \mathcal{M}^2 - \frac{1}{S^2} S^{\alpha\mu} S_{\mu\beta} P^\beta P_\alpha; \quad (20)$$

³Equation (19) has recently been obtained, for the *special case* of flat spacetime, in [12]. Herein we show it to be valid in general.

substituting back into Eq. (19) leads to an explicit equation for U^α in terms of P^μ and $S^{\mu\nu}$ only, $U^\alpha = U^\alpha(P^\mu, S^{\mu\nu})$.

The existence of the relation (19) might seem strange, mainly due to the long history of assertions that no such relation is available for the MP condition. Such assertions were actually followed by a debate on the freedom in choosing initial conditions and on the subsequent option for “helical” motion. These issues shall be discussed in detail in Sec. III.

1. Simple checks of the relation

As a first check, we note that, since $S^{\mu\nu}S_{\nu\beta} = S^\mu S^\beta - h^{\mu\beta}S^2$, where

$$h^\mu_\beta \equiv \delta^\mu_\beta + U^\mu U_\beta \quad (21)$$

is the space projector orthogonal to U^μ , substituting into (19) yields the trivial relation $mU^\mu = P^\mu - h^\mu_\beta P^\beta$ ($\Leftrightarrow mU^\mu = mU^\mu$), stating that mU^μ is the component of P^μ parallel to U^μ . That (19) verifies the 4-velocity normalization also follows trivially from this relation.

Let us imagine now that the relation (19) is considered in a generic case, *without specifying any spin condition*. It is (in any case) useful to express

$$\begin{aligned} S^{\mu\nu}S_{\nu\beta} &= \epsilon^{\mu\nu\kappa\lambda}V_\kappa S_\lambda \epsilon_{\nu\beta\rho\sigma}V^\rho S^\sigma \\ &= S^2(-\delta^\mu_\beta - V^\mu V_\beta + S^{-2}S^\mu S_\beta), \end{aligned} \quad (22)$$

and thus to rewrite Eq. (19) as

$$mU^\mu = (-V^\mu V_\beta + S^{-2}S^\mu S_\beta)P^\beta, \quad (23)$$

which reveals that geometrically it means projection of P^μ on the eigenplane of $S^{\mu\nu}$ (or, equivalently, on the blade of its dual bivector). Note that since $S_\beta U^\beta = 0 \Leftrightarrow S_\beta P^\beta = 0$ [this is generally valid, see Eq. (12)] and the former is true if the MP condition holds, the relation reduces to trivial $mU^\mu = mU^\mu$ in that case.

An important property is evident now: if multiplied by $S_{\alpha\mu}$, relation (23) gives

$$mS_{\alpha\mu}U^\mu = S_{\alpha\mu}(-V^\mu V_\beta + S^{-2}S^\mu S_\beta)P^\beta = 0$$

immediately, because $S_{\alpha\mu}V^\mu = 0$ as well as $S_{\alpha\mu}S^\mu = 0$ by definition. Therefore, relation (19) implies the MP SSC, and so (since the MP SSC likewise implies (19)) it is *equivalent* to the latter.

2. “Hidden momentum”

The component of P^μ orthogonal to U^μ , $h^\mu_\nu P^\nu \equiv P^\mu_{\text{hid}}$, has been dubbed in some literature “hidden momentum.” The reason for the denomination is seen taking the perspective of an observer comoving with the centroid (the zero 3-velocity frame). In such frame the spatial

momentum is precisely $h^\mu_\nu P^\nu$, and is in general nonzero. However, by definition, the body is “at rest” in this frame (since this is the rest frame of the center of mass, or centroid, chosen to represent it); hence such momentum must be hidden somehow. It may be cast as analogous (albeit with a very different nature [15,22]) to the hidden momentum first found in electromagnetic systems [23] (namely in magnetic dipoles subjected to electric fields [15,22–25]). The concept proved useful in simplifying the interpretation of some exotic motions of the centroid in Refs. [12,18,22] (amongst them the Mathisson helical motions [18], discussed below). It reads, for the MP condition,

$$P^\mu_{\text{hid}} \equiv h^\mu_\alpha P^\alpha = P^\mu - mU^\mu \quad (24)$$

$$= S^{\mu\nu}a_\nu = -\epsilon^{\mu\beta\gamma\delta}S^\beta a^\gamma U^\delta, \quad (25)$$

the last two equalities holding for the MP SSC, where we used Eqs. (18), (16). Relation (19) yields an alternative expression for the hidden momentum, in terms of P^μ and $S^{\mu\nu}$:

$$P^\mu_{\text{hid}} = -\frac{1}{S^2}S^{\mu\nu}S_{\nu\beta}P^\beta. \quad (26)$$

III. THE DUALITY BETWEEN THE DEGENERACY OF CENTROID AND THE DETERMINACY OF THE EQUATIONS

Equations (19)–(20) yield a momentum-velocity relation of the form $U^\alpha = U^\alpha(P^\mu, S^{\mu\nu})$. This means that the equations of motion can be written as the *explicit* functions

$$\begin{aligned} \frac{dz^\alpha}{d\tau} &= U^\alpha(z^\mu, P^\mu, S^{\mu\nu}); \\ \frac{dP^\alpha}{d\tau} &= f^\alpha(z^\mu, P^\mu, S^{\mu\nu}); \\ \frac{dS^{\alpha\beta}}{d\tau} &= \psi^{\alpha\beta}(z^\mu, P^\mu, S^{\mu\nu}) \end{aligned} \quad (27)$$

which, given the initial values $\{z^\alpha, P^\alpha, S^{\alpha\beta}\}|_{\text{in}}$, form a determinate system. That is, the solution is unique given this type of initial data and hence, from this point of view, the MP SSC works like the other SSC’s in the literature. This fact might be surprising at first, since, contrary to other SSCs like the TD, the MP SSC is known for not specifying a unique worldline through the body [4,5,8,12,17,22,26], being infinitely degenerated. This led to (apparent) contradictions in the literature, between authors noticing that it does not uniquely specify a worldline, and those arguing [27,28] that it does, given certain initial conditions (see in particular the comments made in [27]).

The conflict between the two perspectives is only seeming. Given a test body, with matter distribution

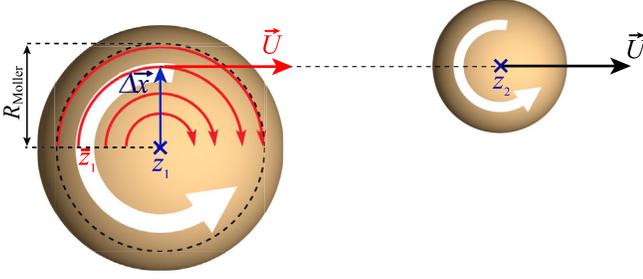


FIG. 1. Two centroids, \bar{z}_1^α and z_2^α , of two different bodies for which the initial data $\{z^\alpha, m, U^\alpha, S^{\alpha\beta}\}|_{\text{in}}$ is the same. The centroid \bar{z}_1^α is a *helical* motion of body 1, whose bulk (i.e., its nonhelical centroid z_1^α) is at rest; z_2^α is a *nonhelical* motion of body 2, which moves uniformly with velocity \vec{v} . The figure corresponds to flat spacetime, where the nonhelical centroids coincide with the centroids as measured in the zero 3-momentum frames (defined by the TD condition $S^{\alpha\beta}P_\beta = 0$). Body 2 has a smaller “intrinsic” spin, but is more massive: $S_{\star 2} = S_{\star 1}/\gamma$, $\mathcal{M}_2 = \gamma\mathcal{M}_1$, $\gamma = 1/\sqrt{1-v^2}$. By giving the initial shift relative to the nonhelical centroid, $\Delta x^\alpha = S^{\alpha\beta}P_\beta/\mathcal{M}^2$, or, equivalently, the initial acceleration $a^\alpha = -\Delta x^\alpha\mathcal{M}^2/S^2$, the degeneracy is removed. This is why the initial data $\{z^\alpha, P^\alpha, S^{\alpha\beta}\}|_{\text{in}}$ (equivalent to $\{z^\alpha, m, U^\alpha, S^{\alpha\beta}, a^\alpha\}|_{\text{in}}$) uniquely fixes the solution, and a definite velocity-momentum relation $U^\alpha = U^\alpha(P^\mu, S^{\mu\nu})$ exists.

described by some energy-momentum tensor $T^{\alpha\beta}$, the condition $S^{\alpha\beta}U_\beta = 0$ does not indeed specify a unique centroid; the MP SSC is obeyed by an infinite set of worldlines. In the simplest case of flat spacetime, as shown in [18,26], every point within the so-called “disk of centroids,”⁴ *counterrotating* relative to the body with a certain fixed angular velocity ($\Omega = \mathcal{M}/S_\star$, where S_\star is the angular momentum about the nonhelical centroid), yields a worldline obeying the condition $S^{\alpha\beta}U_\beta = 0$; this is depicted in body 1 of Fig. 1 (red semicircles therein).

The impact of this degeneracy in the initial value problem for the equations of motion is not trivial though. This is why we devote the rest of this section to explain how this difference between the MP and other SSCs is reflected in the initial data set needed to determine the solution. For all the SSCs apart from the MP one, one can apply the initial data set $\{z^\alpha, P^\alpha, S^{\alpha\beta}\}|_{\text{in}}$ equally well as the set $\{z^\alpha, m, U^\alpha, S^{\alpha\beta}\}|_{\text{in}}$, since both fix the solution uniquely. In the case of the MP SSC, however, only the former data set provides a unique solution, whereas the latter has to be supplemented by the initial acceleration, i.e., one needs the data set $\{z^\alpha, m, U^\alpha, S^{\alpha\beta}, a^\alpha\}|_{\text{in}}$. The reason for this can be seen from the generic $P - U$ relation (12). Namely, for all

⁴In the zero 3-momentum frame (i.e., in the rest space orthogonal to P^α), the set of all possible positions of the center of mass as measured by the different observers spans a disk, orthogonal to S_\star^α , of radius (56), centered at the centroid as measured in that frame (TD centroid), see Fig. 1. Such disk is dubbed the “disk of centroids.” For more details see e.g. [12,26].

the usual SSCs except from the MP one, it can be shown that $DV_\beta/d\tau$ is a function of $(z^\mu, U^\mu, S^{\mu\nu})$, which allows to obtain P^α as a function of $(z^\mu, U^\mu, S^{\mu\nu}, m)$, thereby rendering the set $\{z^\alpha, P^\alpha, S^{\alpha\beta}\}|_{\text{in}}$ equivalent to $\{z^\alpha, m, U^\alpha, S^{\alpha\beta}\}|_{\text{in}}$. Below we provide the proof of the above statement.

In the case of the TD SSC, $V_\beta = P_\beta/\mathcal{M}$, we have $\mu = \mathcal{M}$ and $\gamma(V, U) = m/\mathcal{M}$, therefore Eq. (1) gives that

$$\frac{DV_\beta}{d\tau} = \frac{1}{\mathcal{M}} \frac{DP_\beta}{d\tau} = -\frac{1}{2\mathcal{M}} R_{\beta\gamma\mu\nu} S^{\mu\nu} U^\gamma.$$

By contracting Eq. (12) with U^α one obtains

$$\mathcal{M}^2 = m^2 - \frac{1}{2} S^{\alpha\beta} U_\alpha R_{\beta\gamma\mu\nu} S^{\mu\nu} U^\gamma,$$

and by substituting the above \mathcal{M} in (12) one obtains the momentum P^α in terms of $(z^\mu, U^\mu, S^{\mu\nu}, m)$.

In the case of the Corinaldesi-Papapetrou SSC [11], $V^\beta = u_{\text{lab}}^\beta$ is the congruence of “laboratory” observers [12] (at rest in the given coordinate system). Thus, contracting Eq. (12) with U^α leads to

$$\mu_{\text{lab}} = m\gamma(u_{\text{lab}}, U) + S_{\alpha\beta} u_{\text{lab}}^{\beta;\sigma} U_\sigma U^\alpha,$$

where $\mu_{\text{lab}} \equiv -P_\alpha u_{\text{lab}}^\alpha$, $\gamma(u_{\text{lab}}, U) = -u_{\text{lab}}^\alpha U_\alpha$, and $u_{\text{lab}}^{\alpha;\beta}$ is determined by the kinematics of the observer congruence [12]. Substituting into (12) one obtains P^α in terms of $(z^\mu, U^\mu, S^{\mu\nu}, m)$.

For the OKS condition, simply $P^\alpha = mU^\alpha$.

The case of the MP condition is different, as (12) yields

$$P^\alpha = mU^\alpha + S^{\alpha\beta} \frac{DU_\beta}{d\tau};$$

so clearly the initial values $\{z^\alpha, m, U^\alpha, S^{\alpha\beta}\}|_{\text{in}}$ are not sufficient, since one cannot from them determine the acceleration $a^\alpha = DU^\alpha/d\tau$, which is needed in order to obtain P^α . Physically, this is because the same data $\{z^\alpha, m, U^\alpha, S^{\alpha\beta}\}|_{\text{in}}$ might correspond to a nonhelical solution of a given physical body, as well as to helical solutions of an indiscriminate number of physical bodies. This is exemplified, for the case of flat spacetime, in Fig. 1: a given tensor $S^{\alpha\beta}$ and a 4-velocity U^α might correspond to a *helical* solution of body 1, whose bulk (i.e., its nonhelical centroid z_1^α) is at rest, or to a *nonhelical* solution of body 2, which is uniformly moving with velocity $\vec{v} = \vec{U}/\gamma$. This is so when their “intrinsic” spins (i.e., their angular momentum about the nonhelical centroids z_1^α and z_2^α) and masses $\mathcal{M} = \sqrt{-P^\alpha P_\alpha}$ obey specific relations that we shall now derive.

First notice, from Eq. (12) applied to the Tulczyjew-Dixon SSC ($V^\alpha = P^\alpha/\mathcal{M}$), that, in the absence of forces ($DP^\alpha/d\tau = 0$), one has $P^\alpha = mU^\alpha$; this implies that the TD centroid coincides with a centroid of the MP SSC, more

precisely the nonhelical one, since $DP^\alpha/d\tau = 0 \Rightarrow DU^\alpha/d\tau = 0$ for such worldline. Therefore, the nonhelical centroids z_1^α and z_2^α are TD centroids. Now, recall the well known flat spacetime expression (e.g. [20]; see also Sec. IV) relating the angular momentum tensors ($S^{\alpha\beta}$ and $\bar{S}^{\alpha\beta}$) of a given body about two different points z^α and $\bar{z}^\alpha = z^\alpha + \Delta x^\alpha$,

$$\bar{S}^{\alpha\beta} = S^{\alpha\beta} + 2P^{[\alpha}\Delta x^{\beta]}. \quad (28)$$

Let us moreover denote, as in [12,18], by $S_\star^{\alpha\beta}$ the angular momentum of a given body taken about its TD centroid (so that $S_\star^{\alpha\beta}P_\beta = 0$), and by $S_\star^\alpha = -\epsilon^\alpha{}_{\beta\gamma\delta}S_\star^{\gamma\delta}P^\beta/(2\mathcal{M})$ the corresponding spin vector. The condition that $S^{\alpha\beta}$ be simultaneously the angular momentum of body 1 about its *helical* centroid \bar{z}_1^α , and the angular momentum of body 2 about its *nonhelical* centroid z_2^α (the TD centroid of body 2), implies, for body 2, $S_{\star 2}^{\alpha\beta} = S^{\alpha\beta} \Rightarrow S_{\star 2}^\alpha = S^\alpha$, and, for body 1, cf. Eq. (28),

$$S^{\alpha\beta} = S_{\star 1}^{\alpha\beta} + 2P_1^{[\alpha}\Delta x^{\beta]}; \quad (29)$$

$$\Delta x^\alpha = -S_{\star 1}^{\alpha\beta} \frac{U_\beta}{m}, \quad (30)$$

where Eq. (30) follows from contracting (29) with U_β , and making $\Delta x^\beta U_\beta = 0$. The vector $\Delta x^\alpha = \bar{z}_1^\alpha - z_1^\alpha$ is the ‘‘shift’’ of the centroid \bar{z}_1^α relative to z_1^α ; it is a vector orthogonal to the worldlines of both centroids, that yields their instantaneous spatial displacement (as measured in the rest frames of either of them). It is the analogue of the Newtonian displacement vector, as illustrated in Fig. 1. It follows that

$$S^\alpha \equiv -\frac{1}{2}\epsilon^\alpha{}_{\beta\gamma\delta}U^\delta S^{\beta\gamma} = \gamma S_{\star 1}^\alpha - \epsilon^\alpha{}_{\beta\gamma\delta}U^\delta P_1^\beta \Delta x^\gamma = \frac{S_{\star 1}^\alpha}{\gamma}, \quad (31)$$

where $\gamma \equiv -U_\alpha P_1^\alpha/\mathcal{M}_1 = m/\mathcal{M}_1$ satisfies⁵ $\gamma > 1$. In both the second and third equalities of (31) we notice that $U_\delta S_{\star 1}^\delta = 0$. To obtain this relation, one first notes that substituting (29) into $S_{\star 1}^\alpha = -\epsilon^\alpha{}_{\beta\gamma\delta}S_{\star 1}^{\gamma\delta}P_1^\beta/(2\mathcal{M}_1)$, and contracting with U_α , yields $S_{\star 1}^\alpha U_\alpha = -S_\alpha P_1^\alpha/\mathcal{M}_1$; then one just has to note, from Eq. (19), that $S_\alpha P_1^\alpha = 0$. We thus see that the data $\{z^\alpha, m, U^\alpha, S^{\alpha\beta}\}_{\text{in}}$ is the same for bodies 1 and 2 provided that

$$S_{\star 1}^\alpha = \gamma S_{\star 2}^\alpha = \gamma S^\alpha$$

(so body 1 has a larger intrinsic spin than body 2, $S_{\star 1} = \gamma S_{\star 2} > S_{\star 2}$), and

⁵The factor γ can also be written as $\gamma = 1/\sqrt{1-v^2}$, where \vec{v} is the velocity of the centroid \bar{z}_1^α relative to the zero 3-momentum frame (the reference frame depicted in Fig. 1, where $\vec{v} = \vec{U}/\gamma$), or, equivalently, the velocity of \bar{z}_1^α with respect to z_1^α .

$$m = \gamma\mathcal{M}_1 = \mathcal{M}_2$$

(so body 2 is more massive than body 1: $\mathcal{M}_1 < \mathcal{M}_2$).

Such degeneracy is removed by additionally fixing the initial acceleration $a^\alpha|_{\text{in}}$. In fact, the initial data $\{z^\alpha, m, U^\alpha, S^{\alpha\beta}, a^\alpha\}_{\text{in}}$ and $\{z^\alpha, P^\alpha, S^{\alpha\beta}\}_{\text{in}}$ are equivalent under this spin condition, since from the latter one immediately obtains $U^\alpha|_{\text{in}}$ via (19), and also $a^\alpha|_{\text{in}}$ via the explicit expression for the acceleration (17).

The way these things play out is especially intuitive again in the flat spacetime case in Fig. 1: as shown in detail in [18,26], for a given body (body 1 in Fig. 1), the MPD system (1)–(2) supplemented by the MP SSC is satisfied by an infinite set of worldlines which, as viewed from the perspective of the body’s zero 3-momentum frame (the frame represented in Fig. 1), consist of a set of circular motions (red semicircles), of radius $R = \|\Delta x^\alpha\|$, centered at the nonhelical centroid (z_1^α in Fig. 1). Since, as explained above, the latter coincides with the body’s TD centroid, let us denote it henceforth by $z^\alpha(P)$. In other frames, the solutions consist of a combination of such circular motion with a boost parallel to \vec{P} . If one is given just the initial data $\{z^\alpha, m, U^\alpha, S^{\alpha\beta}\}_{\text{in}}$, as explained above, one has no way of knowing to which kind of solution (helical or nonhelical) of which kind of body it corresponds to (i.e., which are the defining moments P^α and $S_\star^{\alpha\beta}$, whether its bulk at rest or moving, etc). This is exemplified by bodies 1 and 2 of Fig. 1, for which such data is the same. For the initial data $\{z^\alpha, P^\alpha, S^{\alpha\beta}\}_{\text{in}}$ the situation is very different: the momentum P^α tells us immediately the 4-velocity of the nonhelical centroid: $dz^\alpha(P)/d\tau = P^\alpha/\mathcal{M}$; P^α and $S^{\alpha\beta}$ combined give us the shift $\Delta x^\alpha = z^\alpha - z^\alpha(P)$ via the expression

$$\Delta x^\alpha = \frac{S_{\star 1}^{\alpha\beta}P^\beta}{\mathcal{M}^2} \quad (32)$$

which follows from contracting (29) with P_β (identifying $P_1^\alpha \rightarrow P^\alpha$, $S_{\star 1}^{\alpha\beta} \rightarrow S_\star^{\alpha\beta}$ therein), and noting that $\Delta x^\beta P_\beta = 0$. From this one gets the coordinates of the TD centroid $z^\alpha(P)$. In other words, as depicted in Fig. 1, the vector Δx^α tells us whether the motion is helical or not, and *which one of the helices*. Alternatively, the same information is given by the initial acceleration, since, from Eqs. (17) and (32), $a^\alpha = -\Delta x^\alpha \mathcal{M}^2/S^2$. Moreover, the angular momentum $S_\star^{\alpha\beta}$ about the nonhelical centroid $z^\alpha(P)$ can be obtained from Δx^α and $S^{\alpha\beta}$ using, again, (29). The motion is then totally determined, because we know the center [$z^\alpha(P)$, that is, z_1^α in Fig. 1] and the radius (Δx^α) of the circular motion described by z^α around $z^\alpha(P)$; and we know moreover its angular velocity, which, as shown in [18,26], is the same for all helices and equal to $\vec{\Omega} = -\mathcal{M}\vec{S}_\star/S_\star^2$. In this way we get an intuitive picture of why the motion (and hence U^α) is completely and uniquely determined given the initial data

$\{z^\alpha, P^\alpha, S^{\alpha\beta}\}|_{\text{in}}$, making natural the existence of the momentum-velocity relation (19).

IV. DIFFERENT SOLUTIONS CORRESPONDING TO THE SAME PHYSICAL BODY

In this section we discuss the degeneracy of the MP SSC, and the description of a given physical body through the different representative worldlines obeying this spin condition. First of all one needs to establish what, in the framework of a pole-dipole approximation, defines a physical body. In a multipole expansion, the energy-momentum tensor $T^{\alpha\beta}$ and the charge current density 4-vector (j^α) of an extended body are represented by its multipole moments (see e.g. [29,30]). To pole-dipole order, and in the absence of an electromagnetic field, the momentum P^α and the spin tensor $S^{\alpha\beta}$ are the only of such moments entering the equations of motion. Such moments are taken with respect to a reference worldline $z^\alpha(\tau)$ and defined as integrals over a certain space-like hypersurface. Different methods have been proposed for precisely defining the moments in a curved spacetime. Some of them are based on bitensors [5,22,30], while others employ an exponential map [29]. In the latter case the moments take the form [12,29]

$$P^{\hat{\alpha}} \equiv \int_{\Sigma(z,V)} T^{\hat{\alpha}\hat{\beta}} d\Sigma_{\hat{\beta}}, \quad (33)$$

$$S^{\hat{\alpha}\hat{\beta}} \equiv 2 \int_{\Sigma(z,V)} x^{[\hat{\alpha}} T^{\hat{\beta}]\hat{\gamma}} d\Sigma_{\hat{\gamma}}, \quad (34)$$

in a system of Riemann *normal* coordinates $\{x^{\hat{\alpha}}\}$ originating at z^α . Here $\Sigma(z, V)$ is the spacelike hypersurface generated by all geodesics orthogonal to the timelike vector V^α at the point z^α , $d\Sigma$ is the 3-volume element on $\Sigma(z, V)$, and $d\Sigma_\gamma \equiv -n_\gamma d\Sigma$, where n^α is the unit vector normal to $\Sigma(z, V)$ (at z^α , $n^\alpha = V^\alpha$).

For a free particle in flat spacetime, the conservation equations $T^{\alpha\beta}{}_{;\beta} = 0$, along with the existence of a maximal number of Killing vectors, imply that both P^α and $S^{\alpha\beta}$ are independent of $\Sigma(z, V)$ (see, e.g., [20]). Thus, $S^{\alpha\beta}$ is just a function of the reference point z^α , and P^α is a constant vector independent of the point. Hence, given P^α and $S^{\alpha\beta}$ about a reference worldline $z^\alpha(\tau)$, the moments of the same body relative to another reference worldline $\bar{z}^\alpha(\bar{\tau})$ are such that, in a *global rectangular coordinate system*, the components of P^α remain the same, and $S^{\alpha\beta}$ is transformed by the well-known expression (28).

In curved spacetime the situation is more complicated because the moments depend on the hypersurface of integration Σ (which in turn are not simply hyperplanes, as in flat spacetime), and a simple, *exact* relation between the moments $\{\bar{P}^\alpha, \bar{S}^{\alpha\beta}\}$ taken with respect to $\bar{z}^\alpha, \Sigma(\bar{z}, \bar{V})$, and the moments $\{P^\alpha, S^{\alpha\beta}\}$ evaluated with respect to z^α ,

$\Sigma(z, V)$, does not exist. However, it is still possible to devise a simple set of transformation rules that, to a very good approximation, allows us to obtain the moments taken about \bar{z}^α from the knowledge of the moments about z^α , if the size of the test body is small compared to the scale of the curvature. More precisely, the latter assumption holds when $\lambda = \|\mathbf{R}\|\rho^2 \ll 1$, where $\|\mathbf{R}\|$ is the magnitude of the Riemann tensor and ρ is the radius of the body.

To obtain these transformation rules one starts by noticing that when $\lambda \ll 1$, then for any point z^α within the convex hull of the body's worldtube, P^α and $S^{\alpha\beta}$ are independent of the argument V^α of $\Sigma(z, V)$. This is explicitly shown in the Appendix of [12]. Now, let $\{x^{\hat{\alpha}}\}$ be a system of normal coordinates originating from the point \bar{z}^α . These coordinates can be chosen such that

$$x^{\hat{\alpha}} = x^{\hat{\alpha}} - \bar{z}^{\hat{\alpha}} + \mathcal{O}(\|\mathbf{R}\| \|x^{\hat{\alpha}} - \bar{z}^{\hat{\alpha}}\|^2 \|\bar{z}^{\hat{\alpha}} - z^{\hat{\alpha}}\|),$$

cf. Eq. (11.12) of [31]. Therefore, $x^{\hat{\alpha}} \simeq x^{\hat{\alpha}} - \bar{z}^{\hat{\alpha}}$, provided that z^α and \bar{z}^α are two points within the body's convex hull⁶ (as is the case for two centroids) *and* that the condition $\lambda \ll 1$ holds. Aligning the time axis of the coordinate system $\{x^{\hat{\alpha}}\}$ with V^α , $\partial_0^\alpha|_z = V^\alpha$, we can thus take $\bar{P}^{\hat{\alpha}}, \bar{P}^{\hat{\alpha}}, S^{\hat{\alpha}\hat{\beta}}, \bar{S}^{\hat{\alpha}\hat{\beta}}$ as integrals over the *same hypersurface* $x^{\hat{0}} = 0$, which, using (33)–(34), leads to

$$\bar{P}^{\hat{\alpha}} = P^{\hat{\alpha}}; \quad \bar{S}^{\hat{\alpha}\hat{\beta}} = \bar{S}^{\hat{\alpha}\hat{\beta}} = S^{\hat{\alpha}\hat{\beta}} + 2P^{[\hat{\alpha}}\Delta x^{\hat{\beta}]}, \quad (35)$$

where $\Delta x^{\hat{\alpha}} = \bar{z}^{\hat{\alpha}} - z^{\hat{\alpha}} \equiv \bar{z}^{\hat{\alpha}}$. This yields a rule for transition between different representations of the same body: they are such that, in a normal coordinate system originating at z^α , the components $P^{\hat{\alpha}}$ of the momentum are the same at both points, and the components of the angular momentum obey relation (35). The setting of normal coordinates is however laborious in practical situations.

A practical covariant approach to implement these rules can be devised as follows. First one notes that, since the system $\{x^{\hat{\alpha}}\}$ is constructed from geodesics radiating out of z^α , the components $\Delta x^{\hat{\alpha}} = \bar{z}^{\hat{\alpha}}$ are identified with the vector Δx^μ at z^α , tangent to the geodesic $c^\alpha(s)$ connecting z^α and \bar{z}^α , and whose length equals that of the geodesic segment. And the point \bar{z}^α is, thus, the image by the exponential map of Δx^μ [12] (see Fig. 2):

$$\begin{aligned} \bar{z}^\alpha &= \exp_z^\alpha(\Delta x) = e^{\Delta s \partial/\partial s} c^\alpha(s)|_{s=s_z} \\ &= z^\alpha + \dot{c}^\alpha(s_z)\Delta s + \frac{1}{2}\ddot{c}^\alpha(s_z)\Delta s^2 + \dots, \end{aligned} \quad (36)$$

where $\Delta s \equiv s_{\bar{z}} - s_z$. Choosing s as the proper length of c^α , we have $\Delta s = \|\Delta x^\mu\|$ and

⁶More precisely, within the intersection of the body's worldtube with any spacelike hypersurface $\Sigma(z, V)$, that can be interpreted as the rest space of some observer of 4-velocity V^α .

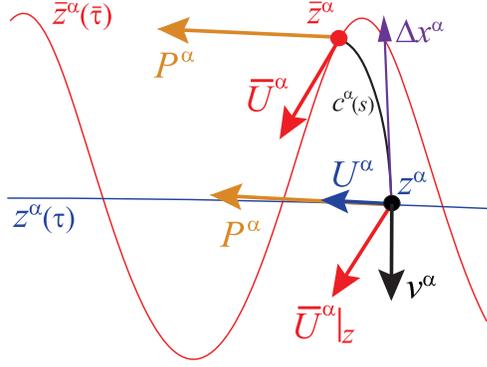


FIG. 2. Two different centroids [of worldlines $\bar{z}^\alpha(\bar{\tau})$ and $z^\alpha(\tau)$] corresponding to the same physical body. The point $\bar{z}^\alpha = \exp_{z^\alpha}(\Delta x^\alpha)$ is the image produced by the exponential map of the shift vector $\Delta x^\alpha = -S^{\alpha\beta}\bar{U}_\beta|_z/\bar{m}$ at z^α , where $\bar{m} = -P^\gamma\bar{U}_\gamma|_z$. $\bar{U}_\beta|_z$ is the vector resulting from parallel transporting \bar{U}^α from \bar{z}^α to z^α along the geodesic $c^\alpha(s)$ connecting these two points. $v^\alpha = \bar{U}^\alpha|_z/\gamma - U^\alpha$ is the “kinematical” relative velocity [33] of \bar{z}^α with respect to z^α (i.e., of \bar{U}^α with respect to U^α). Given a worldline $z^\alpha(\tau)$ and its corresponding moments (P^α and $S^{\alpha\beta}$), a new solution $\bar{z}^\alpha(\bar{\tau})$ of the MP SSC is completely set up by prescribing, at some point z^α , the vector v^α obeying the constraint (50), and then by using Eqs. (36), (41)–(43) for $\bar{V}^\alpha|_z = \bar{U}^\alpha|_z$.

$$\dot{c}^\alpha(s_z) = \frac{\Delta x^\alpha}{\|\Delta x^\mu\|}; \quad (37)$$

reading \ddot{c}^α from the geodesic equation $\ddot{c}^\alpha + \Gamma_{\beta\gamma}^\alpha \dot{c}^\beta \dot{c}^\gamma = 0$, we obtain

$$\bar{z}^\alpha = z^\alpha + \Delta x^\alpha - \frac{1}{2}\Gamma_{\beta\gamma}^\alpha|_z \Delta x^\beta \Delta x^\gamma + \dots \quad (38)$$

Now let $\bar{g}^\alpha_\alpha \equiv \bar{g}^\alpha_\alpha(\bar{z}, z)$ denote the bitensor that parallel propagates tensors $\mathcal{A}^{\alpha_1 \dots \alpha_n}$ from z^α to \bar{z}^α along $c^\alpha(s)$ [30,32]:

$$\mathcal{A}^{\alpha_1 \dots \alpha_n}|_{\bar{z}} = \bar{g}^{\alpha_1}_{\beta_1} \dots \bar{g}^{\alpha_n}_{\beta_n} \mathcal{A}^{\beta_1 \dots \beta_n}|_z. \quad (39)$$

Using the parallel transport equation $d\mathcal{A}^{\alpha_1 \dots \alpha_n}/ds = -\Gamma_{\beta\gamma}^{\alpha_1} \mathcal{A}^{\beta\alpha_2 \dots \alpha_n} \dot{c}^\gamma - \dots - \Gamma_{\beta\gamma}^{\alpha_n} \mathcal{A}^{\alpha_1 \dots \alpha_{n-1}\beta} \dot{c}^\gamma$, this is

$$\begin{aligned} \mathcal{A}^{\alpha_1 \dots \alpha_n}|_{\bar{z}} &= \mathcal{A}^{\alpha_1 \dots \alpha_n}|_z - \int_z^{\bar{z}} \Gamma_{\beta\gamma}^{\alpha_1}(x) \mathcal{A}^{\beta\alpha_2 \dots \alpha_n} dx^\gamma \\ &\quad - \dots - \int_z^{\bar{z}} \Gamma_{\beta\gamma}^{\alpha_n}(x) \mathcal{A}^{\alpha_1 \dots \alpha_{n-1}\beta} dx^\gamma. \end{aligned} \quad (40)$$

Noting that, in the normal coordinate system $\{x^{\hat{\alpha}}\}$, $\|\Gamma_{\beta\gamma}^{\hat{\alpha}}(x)\| \sim \|\mathbf{R}\| \|x\|$, it follows that $\mathcal{A}^{\hat{\alpha}_1 \dots \hat{\alpha}_n}|_{\bar{z}} = \mathcal{A}^{\hat{\alpha}_1 \dots \hat{\alpha}_n}|_z + \mathcal{O}(\|\mathbf{A}\| \|\mathbf{R}\| \|\Delta x^\mu\|^2)$. Therefore, under the assumption $\lambda \ll 1$, $\mathcal{A}^{\hat{\alpha}_1 \dots \hat{\alpha}_n}|_{\bar{z}} \simeq \mathcal{A}^{\hat{\alpha}_1 \dots \hat{\alpha}_n}|_z$. In other words, the condition that, under the assumption $\lambda \ll 1$, the components $\mathcal{A}^{\hat{\alpha}_1 \dots \hat{\alpha}_n}|_{\bar{z}}$ of a tensor at \bar{z}^α equal those of a tensor $\mathcal{A}^{\hat{\alpha}_1 \dots \hat{\alpha}_n}|_z$ at z^α in

normal coordinates originating from z^α , is equivalent to saying that $\mathcal{A}^{\alpha_1 \dots \alpha_n}|_{\bar{z}}$ is obtained by parallel transporting the tensor $\mathcal{A}^{\alpha_1 \dots \alpha_n}|_z$ from z^α to \bar{z}^α along $c^\alpha(s)$.

Thus, given two points z^α and \bar{z}^α , or a point z^α and a shift vector Δx^α , we have a covariant method for “transforming” and then “transferring” the moments from z^α to \bar{z}^α . Namely, first one has to transform the spin tensor using Eq. (28), in which Δx^α is a vector at the point z^α [defined by Eq. (37)]. Note that this is a well-defined operation for tensors at z^α : it yields a tensor $\bar{S}^{\alpha\beta}|_z$ at z^α , one whose components in the normal coordinates $\{x^{\hat{\alpha}}\}$ of z^α happen to equal the components $\bar{S}^{\hat{\alpha}\hat{\beta}}$ of the spin tensor about \bar{z}^α in the normal coordinates $\{x^{\hat{\alpha}}\}$ of \bar{z}^α [see Eq. (35)]. Then, one parallel transports P^α and $\bar{S}^{\alpha\beta}|_z$ to \bar{z}^α , i.e.

$$\begin{aligned} \bar{S}^{\alpha\beta}|_{\bar{z}} &= \bar{g}^\alpha_\gamma \bar{g}^\beta_\delta \bar{S}^{\gamma\delta}|_z \\ &= \bar{g}^\alpha_\gamma \bar{g}^\beta_\delta (S^{\gamma\delta} + 2P^{[\gamma} \Delta x^{\delta]})|_z, \end{aligned} \quad (41)$$

$$P^\alpha|_{\bar{z}} = \bar{g}^\alpha_\beta P^\beta|_z. \quad (42)$$

The above procedure can be used to shift between different representative (centroid) worldlines of a given body. Usually one has a solution $z^\alpha(\tau)$ corresponding to some spin condition $S^{\alpha\beta}V_\beta = 0$, and wishes to know how to shift to a worldline $\bar{z}^\alpha(\bar{\tau})$ specified by another SSC $\bar{S}^{\alpha\beta}\bar{V}_\beta = 0$. That can be done as follows. Starting from a point z^α along the worldline $z^\alpha(\tau)$, a point \bar{z}^α of the new worldline, such that the method above holds, is reached via Eq. (36) by an appropriate shift vector Δx^α . The vector Δx^α is obtained in turn as follows. One prescribes a vector $\bar{V}^\alpha|_z$ at z^α (understood to result from the parallel transport of the actual $\bar{V}^\alpha \equiv \bar{V}^\alpha|_{\bar{z}}$, at the yet to be determined \bar{z}^α , to z^α , i.e. $\bar{V}^\alpha = \bar{g}^\alpha_\beta \bar{V}^\beta|_z$); then

$$\Delta x^\alpha = -\frac{S^{\alpha\beta}\bar{V}_\beta|_z}{\bar{\mu}}, \quad (43)$$

where $\bar{\mu} \equiv -P^\alpha \bar{V}_\alpha$. In order to derive Eq. (43), one must recall, from [32], some properties of the parallel propagator $\bar{g}_{\alpha\beta}$ in Eq. (39). Namely, this tensor is not symmetric: its second slot parallel transports vectors from z^α to \bar{z}^α , as indicated in Eq. (39), whereas the first slot does the inverse path. That is, let $\bar{g}_{\alpha\beta}(z, \bar{z})$ be the bitensor whose second slot parallel transports tensors from \bar{z}^α to z^α [i.e., the reciprocal of the tensor $\bar{g}_{\alpha\beta}(\bar{z}, z) \equiv \bar{g}_{\alpha\beta}$ in Eq. (39)]; we have

$$\bar{g}_{\alpha\beta}(z, \bar{z}) = \bar{g}_{\beta\alpha}(\bar{z}, z) \equiv \bar{g}_{\beta\alpha}, \quad (44)$$

cf. Eq. (1.36) of [32]. Now, contracting Eq. (41) with \bar{V}_β , noting, from relation (44), that $\bar{V}_\beta \bar{g}^\beta_\delta = \bar{V}_\delta|_z$, and that, by definition, $\bar{S}^{\alpha\beta}|_{\bar{z}} \bar{V}_\beta = 0$ and $\Delta x^\delta \bar{V}_\delta|_z = 0$, one obtains Eq. (43). The vector Δx^α is orthogonal to both V^α and $\bar{V}^\alpha|_z$; it yields, in the sense of the exponential map, the

instantaneous *spatial* (with respect to either V^α or $\bar{V}^\alpha|_z$) displacement⁷ of the centroid \bar{z}^α measured by an observer of 4-velocity $\bar{V}^\alpha|_z$, relative to the centroid z^α . Equation (43), together with Eq. (36) and Eqs. (41)–(42), provide all the initial data needed to evolve the equations of motion (1)–(2), *provided* that they are coupled to a velocity-momentum relation $U^\alpha = U^\alpha(z^\mu, P^\mu, S^{\mu\nu})$, thereby uniquely determining the new worldline $\bar{z}^\alpha(\bar{\tau})$.

A. Transition between different Mathisson-Pirani centroids

According to the procedure above, given a solution $z^\alpha(\tau)$, in order to change to a different worldline corresponding to a different centroid of the same body, all one needs is prescribing the initial vector $\bar{V}^\alpha|_z$ (i.e., the 4-velocity of the observer with respect to which the new centroid is to be measured). All the other quantities follow from Eqs. (36), (41), (42), (43), that is: the new initial position, spin vector, momentum, and shift vector, respectively.

The MP SSC demands \bar{V}^α to be tangent to the centroid worldline, i.e. $\bar{V}^\alpha = \bar{U}^\alpha \equiv d\bar{z}^\alpha/d\bar{\tau}$. This demand does not specify a unique worldline, as already discussed in Sec. III; but still it restricts the choice of the eligible $\bar{V}^\alpha|_z = \bar{U}^\alpha|_z$, as we shall now see. The conditions that \bar{U}^α must obey can be found from the velocity-momentum relation (19) re-written in terms of barred quantities, that is

$$\bar{m}\bar{U}^\alpha = P^\alpha + \frac{1}{\bar{S}^2}\bar{S}^{\alpha\mu}\bar{S}_{\mu\beta}P^\beta. \quad (45)$$

First note that $\bar{S}^{\alpha\mu}\bar{S}_{\mu}{}^\beta = \bar{S}^\alpha\bar{S}^\beta - \bar{h}^{\alpha\beta}\bar{S}^2$, where $\bar{h}_{\alpha\beta} \equiv g_{\alpha\beta} + \bar{U}_\alpha\bar{U}_\beta$ is the space projector orthogonal to \bar{U}^α and

$$\bar{S}^\alpha = -\epsilon^{\alpha\beta\gamma\delta}\bar{S}_{\gamma\delta}\bar{U}_\beta/2 \quad (46)$$

is Eq. (16) in barred quantities, it follows that

$$\bar{m}\bar{U}^\alpha = P^\alpha - \bar{h}^\alpha{}_\beta P^\beta + \frac{1}{\bar{S}^2}\bar{S}^\alpha\bar{S}^\beta P_\beta \Leftrightarrow \bar{S}^\beta P_\beta = 0,$$

since $P^\alpha - \bar{h}^\alpha{}_\beta P^\beta = \bar{m}\bar{U}^\alpha$. Thus, Eq. (45) is reduced to the orthogonality between \bar{S}^β and P^β , confirming the condition suggested in [8] (p. 1928) through a different route.

To see what this orthogonality implies for \bar{U}^α , we note that by contracting Eq. (46) with P_α one gets

⁷This can readily be seen by aligning the time axis of the coordinate system $\{x^{\hat{\alpha}}\}$ in Eq. (34) with $\bar{V}^\alpha|_z$, i.e. $\partial_{\hat{0}}^\alpha|_z = \bar{V}^\alpha|_z$, leading to

$$-S^{\hat{\alpha}\hat{\beta}}\bar{V}_{\hat{\alpha}}|_z = S^{\hat{0}\hat{0}} = \int_{\Sigma(z,\bar{V})} x^{\hat{i}}T^{\hat{0}\hat{\gamma}}d\Sigma_{\hat{\gamma}} \equiv \bar{\mu}\bar{z}^{\hat{i}}.$$

$$\bar{S}^\alpha P_\alpha = -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\bar{S}_{\gamma\delta}\bar{U}_\beta P_\alpha = 0. \quad (47)$$

If $S_{\star}^{\gamma\delta}|_{\bar{z}}$ is the angular momentum about the centroid $z^\alpha(P)$ measured in the zero 3-momentum frame (the TD centroid), parallel transported to \bar{z}^α , using $\bar{S}^{\gamma\delta} = S_{\star}^{\gamma\delta}|_{\bar{z}} + 2P^{[\gamma}\zeta^{\delta]}$, where $\zeta^{\hat{\alpha}} = \bar{z}^{\hat{\alpha}} - z^{\hat{\alpha}}(P)$ is the shift vector from $z^\alpha(P)$ to \bar{z}^α , Eq. (47) gives

$$\frac{1}{2}\epsilon^{\alpha}{}_{\beta\gamma\delta}S_{\star}^{\gamma\delta}|_{\bar{z}}\bar{U}^\beta P_\alpha = 0 \Leftrightarrow S_{\star}^\beta|_{\bar{z}}\bar{U}_\beta = 0. \quad (48)$$

Thus, the restriction imposed on \bar{U}^α is that it has to be orthogonal to the spin vector $S_{\star}^\beta|_{\bar{z}}$ of the TD solution. Now, using properties (44) and $\bar{g}_{\alpha}{}^\beta\bar{g}^\gamma{}_\beta = \delta_\alpha{}^\gamma$ (cf. Eq. (1.35) of [32]), we can write $S_{\star}^\alpha|_{\bar{z}}\bar{U}_\alpha \equiv S_{\star}^\alpha|_{\bar{z}}\bar{g}_{\alpha}{}^\beta\bar{g}^\gamma{}_\beta\bar{U}_\gamma = S_{\star}^\alpha|_{\bar{z}}\bar{U}_\alpha|_z$, where $\bar{U}^\alpha|_z = \bar{g}_{\beta}{}^\alpha\bar{U}^\beta$ is the vector obtained by parallel transporting \bar{U}^α from \bar{z}^α to z^α . Therefore, the condition $S_{\star}^\beta|_{\bar{z}}\bar{U}_\beta = 0$ is equivalent to $S_{\star}^\beta|_z\bar{U}_\beta|_z = 0$ (this is just the statement that parallel transport preserves angles).

Consider now the spatial vector v^α defined by (see Fig. 2)

$$\bar{U}^\alpha|_z = \gamma(U^\alpha + v^\alpha); \quad \gamma = -U^\alpha\bar{U}_\alpha|_z = 1/\sqrt{1 - v_\alpha v^\alpha}. \quad (49)$$

The vector v^α is the *kinematical relative velocity* of the centroid \bar{z}^α with respect to z^α —a natural generalization of the concept of relative velocity for the case of objects located at different points [33]. Since $S_{\star}^\beta|_z U_\beta = 0$ (as condition (48) must hold for *any* solution \bar{z}^α), that, together with $S_{\star}^\beta|_z\bar{U}_\beta|_z = 0$, implies via (49) that

$$S_{\star}^\beta|_z v_\beta = 0. \quad (50)$$

In other words, compatibility of the initial data with the MP SSC amounts to the requirement that \bar{z}^α moves relative to z^α in a direction orthogonal to $S_{\star}^\beta|_z$. For a free particle in flat spacetime, as depicted in Fig. 1, this amounts to moving in a direction orthogonal to the body's axis of rotation.

Algorithm for transition between MP centroids

- (1) choose the “kinematical relative velocity” v^α of the new centroid \bar{z}^α with respect to z^α , such that it obeys (50);
- (2) determine $\bar{U}^\alpha|_z \equiv \bar{V}^\alpha|_z$ and the shift vector Δx^α through Eqs. (49) and (43).
- (3) Determine \bar{z}^α from Eq. (38).
- (4) Parallel transport P^α to \bar{z}^α using Eq. (42); transform the spin tensor and parallel transport it to \bar{z}^α using Eq. (41).
- (5) Use the obtained $\{\bar{z}^\alpha, P^\alpha, \bar{S}^{\alpha\beta}\}$ as initial data for the system (27), uniquely determining the solution.

V. EXAMPLES

In this section we will employ the Mathisson-Pirani condition in physical systems where it is easy to setup the nonhelical solution, and this spin condition is especially suitable in that it leads to the simplest description of the physical motion. In each case we will also exemplify the helical descriptions of the same (within the realm of the pole-dipole approximation) physical motion.

A. Radial fall in Schwarzschild spacetime

We wish to study the setup shown in Fig. 3, corresponding to the motion of a physical body whose bulk has initial radial velocity in the Schwarzschild spacetime. We start by setting an initial 4-velocity $U^\alpha = U^0 \partial_t^\alpha + U^r \partial_r^\alpha$. For such U^α , $\mathbb{H}_{\alpha\beta} = 0$ (cf. Eq. (50) of [15]), and so the spin-curvature force (13) is zero (regardless of the orientation of S^α). Taking this into account, and using Eqs. (18), (16), leads to the equation of motion for the centroid

$$\frac{DP^\alpha}{d\tau} = 0 \Leftrightarrow ma^\alpha + \epsilon^\alpha{}_{\beta\gamma\delta} U^\delta \frac{D(S^\beta a^\gamma)}{d\tau} = 0. \quad (51)$$

This equation admits the trivial solution $a^\alpha = 0$, which corresponds to a radial geodesic trajectory. This solution,

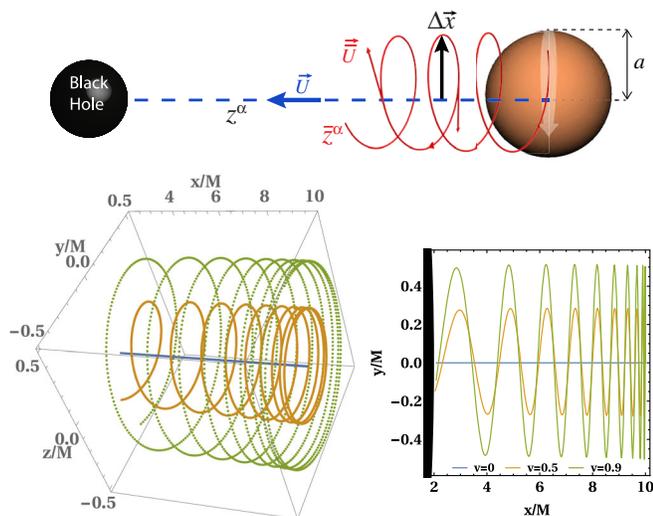


FIG. 3. Left bottom panel: three different solutions—a nonhelical centroid (blue straight line), plus two helical ones—of the Mathisson-Pirani SSC, all representing, initially, *the same physical situation*: a spinning body with radial spin $S^\alpha = S^r \partial_r^\alpha$, falling radially into the black hole (top panel). The nonhelical centroid $z^\alpha(\tau)$ (hence the physical body) starts from rest at $r = 10M$: $U_{\text{in}}^\alpha = U^0 \partial_0^\alpha$; S^α (taken about $z^\alpha(\tau)$), has magnitude $S = 0.5mM$. The helical motions (which *counterrotate* with the body) are prescribed as having initial azimuthal velocity $v^\alpha = v^\phi \partial_\phi^\alpha$ relative to $z^\alpha(\tau)$, of magnitudes $v = 0.5$ and $v = 0.9$. Their initial position is shifted from z_{in}^α by $\Delta \vec{x}|_{\text{in}} = (1/M)(\vec{S} \times \vec{v})^\theta \partial_\theta$. Right bottom panel: the corresponding 2D x-y plot (black region represents the event horizon). The coordinates $\{x, y, z\}$ relate to Schwarzschild coordinates by $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$.

call it $z^\alpha(\tau)$, is (obviously) the nonhelical MP centroid of this physical system, and it is the same for any spinning body regardless the orientation of its spin. For this special, geodesic case, Eq. (18) yields $P^\alpha = mU^\alpha$, which in turn implies that $z^\alpha(\tau)$ coincides with the (unique) centroid given by the TD SSC, i.e., it holds that $S^{\alpha\beta} P_\beta \equiv S_{\star}^{\alpha\beta} P_\beta = 0$.

Let us briefly discuss the description of the same physical motion through other spin conditions. First notice that it is only under the MP condition that the spin-curvature force takes the tidal tensor form (13), which depends only on the centroid’s 4-velocity U^α and on the spin vector S^α ; for other SSCs the force (1) depends also on V^α , as manifest in Eq. (11). Starting with the TD condition, $V^\alpha = P^\alpha/M$, the motion cannot be set up by prescribing a radial U^α , for it is not possible to obtain P^α from either the $U - P$ relation (3), nor the $P - U$ relation (12). The problem is solved instead by prescribing a radial momentum $P^\alpha = P^0 \partial_0^\alpha + P^r \partial_r^\alpha$. Then, by noticing that the numerator of the second term of Eq. (3) can be written as $-4S^{\mu\nu}(\mathbb{H}^P)_{\beta\nu} S^\beta$, where $(\mathbb{H}^P)_{\alpha\gamma} = \star R_{\alpha\beta\gamma\delta} P^\beta P^\delta / M^2$, and that, for a radial P^α , $(\mathbb{H}^P)_{\alpha\gamma} = 0$ (cf. Eq. (50) of [15]), we see that indeed Eq. (3) yields $P^\alpha = mU^\alpha$, leading to the same solution obtained with the MP condition. Since such solution is a radial geodesic, it obviously coincides as well with a particular solution of the OKS condition, with $V^\alpha = U^\alpha$. Under other spin conditions the situation is however more complicated; the centroids that correspond to a body *whose bulk* falls radially, are, in general, shifted relative to the common centroid of the MP, TD, and OKS conditions, and do *not* move radially. Both the spin-curvature force and the derivative of the hidden momentum are in general *not* zero for such centroids, leading to a nonzero acceleration. This is the case of the centroids specified by the Corinaldesi-Papapetrou and Newton-Wigner SSC’s, which deflect as the body approaches the black hole, cf. Eqs. (4.1), (4.2), (5.1) and (5.4) of [34], and Fig. 6(c) of [12]. It is also the case of the “eccentric” centroids of the OKS SSCs, which move nearly parallel to the radial geodesic, see Eq. (45) and Fig. 6(d) of [12] (in this case the hidden momentum is zero, the accelerated “parallel” trajectory being ensured by the spin-curvature force). Conversely, if one naively prescribes an initial radial velocity \vec{U} for such centroids, then the solution will *not* correspond to a radial motion, but to another physical motion where the body’s bulk does not move radially. Therefore, without the knowledge of the radial geodesic solution (obtained either with the MP, TD or OKS conditions), it would not be clear whether a radial motion of the body’s bulk occurs, and how to prescribe the initial conditions for the corresponding centroids.

1. Helical centroids

To study the helical solutions, we consider two special cases:

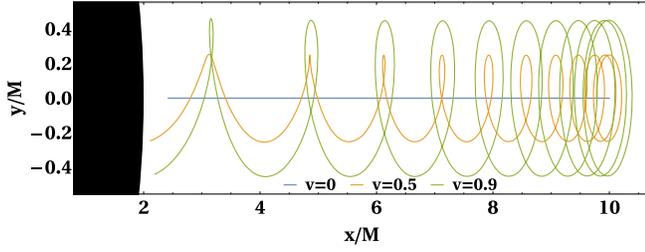


FIG. 4. The analogue of Fig. 3 for the case that the body’s spin is aligned with the polar axis, $S^\alpha = S^\theta \partial_\theta^\alpha$. The spin vector about the nonhelical centroid (blue line) has again magnitude $S = 0.5mM$, and the “helical” motions are prescribed by putting initially $v^\alpha = v^\theta \partial_\theta^\alpha$, of magnitudes $v = 0.5$ and $v = 0.9$. Their initial position is radially shifted from z_{in}^α by $\Delta \vec{x}|_{\text{in}} = (1/\mathcal{M}) \times (\vec{S} \times \vec{v})^r \partial_r$. Unlike the situation for radial spin in Fig. 4, here the “helices” are planar motions, lying in the equatorial plane $\theta = \pi/2$.

(i) radial spin $S^\alpha = S^r \partial_r^\alpha$ (Fig. 3),

(ii) polar spin $S^\alpha = S^\theta \partial_\theta^\alpha$ (Fig. 4).

In both cases we consider that the body’s bulk, which in this case is faithfully represented by its nonhelical centroid $z^\alpha(\tau)$, starts from rest. Since this baseline coincides with the TD centroid, the shift equation (43) reduces to

$$\Delta x^\alpha = -\frac{S^{\alpha\beta} v_\beta}{\mathcal{M}} \equiv -\frac{S_\star^{\alpha\beta} v_\beta}{\mathcal{M}}. \quad (52)$$

Moreover, since $z^\alpha(\tau)$ starts from rest, the initial kinematical relative velocity v^α of the centroid $\bar{z}^\alpha(\bar{\tau})$ with respect to $z^\alpha(\tau)$ coincides with the velocity with respect to the static observers. In both cases, we choose azimuthal *initial* velocities: $v^\alpha = v^\theta \partial_\theta^\alpha$, leading to shift vectors along ∂_θ and ∂_r for the radial and polar spin cases, respectively. We approximate the initial position $\bar{z}^\alpha|_{\text{in}}$ of the helical centroids as shown in Eq. (38); we also expand in Taylor series about z^α the Christoffel symbols in Eq. (40), keeping only the lowest order term: $\Gamma_{\beta\gamma}^\alpha(x) = \Gamma_{\beta\gamma}^\alpha(z) + \mathcal{O}(x - z)$. Thus, the expressions (42) for the moments parallel transported to \bar{z}^α are approximated by

$$P^\alpha|_{\bar{z}} = \bar{g}^\alpha_\beta P^\beta|_z \simeq P^\alpha|_z - \Gamma_{\beta\gamma}^\alpha(z) P^\beta|_z \Delta x^\gamma, \quad (53)$$

$$\bar{S}^{\alpha\beta}|_{\bar{z}} \simeq \bar{S}^{\alpha\beta}|_z - \Gamma_{\delta\gamma}^\alpha \bar{S}^{\delta\beta}|_z \Delta x^\gamma - \Gamma_{\delta\gamma}^\beta \bar{S}^{\alpha\delta}|_z \Delta x^\gamma. \quad (54)$$

This provides initial data for the helical solutions, which are then numerically evolved using the equations of motion (1)–(2), together with the momentum-velocity relation (19). We obtain two types of “helical” motion. On one hand, in the radial spin case of Fig. 3, they are proper helices, winding about the (geodesic) nonhelical trajectory. On the other hand, in the polar spin case, Fig. 4, the result is quite close to a superposition of an infalling radial geodesic (the

nonhelical solution) with a circular motion on the $\theta = \pi/2$ plane. The fact that, for both trajectories, the winding is about the nonhelical centroid was to be expected from the fact that such centroid coincides with the TD centroid, which is the center of the disk of the possible centroids (see [12]). Indeed, Eq. (52), whose space part, in the zero 3-momentum frame and in vector notation, reads

$$\Delta \vec{x} = \frac{\vec{S}_\star \times \vec{v}}{\mathcal{M}}, \quad (55)$$

tells us that the shift vectors corresponding to all the possible helical solutions span a disk orthogonal to both P^α and S_\star^α , of radius (“Møller radius”)

$$R_{\text{Moller}} = \frac{S_\star}{\mathcal{M}}, \quad (56)$$

in the tangent space at z^α . Such a situation resembles the behavior of a free particle in flat spacetime (see [35], Sec. I), only now the winding stretches for decreasing r (unlike in flat spacetime) due to the increase in radial velocity caused by the black hole’s gravitational field. The plots also indicate that (contrary to the flat spacetime case) the amplitude of the helices is not constant. In particular, as the particle approaches the horizon the amplitude slightly decreases in the radial spin case (Fig. 3), whilst it slightly increases in the case of polar spin (Fig. 4). The amplitude changes are however very slight in both cases.

Let us stress, however, an important difference in the dynamics comparing to the flat spacetime case: in flat spacetime, no force is exerted on any of the centroids ($F^\alpha = 0$), the helical-motion acceleration comes only from an interchange between the kinetic momentum (mU^α) and the hidden momentum P_{hid}^α (see Fig. 3 of [18]). Here, however, the spin-curvature force (13) is nonzero along all helical trajectories ($F^\alpha \neq 0$). Thus the acceleration results from the combined effects of the force and the hidden momentum variation. The role of the force, however, is actually to prevent the worldlines from diverging/converging, counteracting the tidal forces due to the curvature, and ensuring that, from the point of view of the zero 3-momentum frame, the helical motions stay close to what they would be in flat spacetime. This is what we are going to show next.

2. An analysis of the helical dynamics through the deviation of worldlines

We start by noticing that since, as discussed in Sec. III, every point within the worldtube of centroids coincides momentarily with a certain unique helical centroid, the helical solutions form locally a *congruence* of worldlines filling the worldtube of centroids. Let U^α be the unit vector field tangent to such a congruence of worldlines, and δx^α be a connecting vector between different worldlines, so that it is Lie dragged along the congruence, $\mathcal{L}_U \delta x^\alpha = 0$. The

latter expression implies that $D^2\delta x^\alpha/d\tau^2 \equiv \nabla_U \nabla_U \delta x^\alpha = \nabla_U \nabla_{\delta x} U^\alpha$, and, thus,

$$\begin{aligned} \frac{D^2\delta x^\alpha}{d\tau^2} &= \nabla_{\delta x} a^\alpha - [\nabla_{\delta x}, \nabla_U] U^\alpha \\ &= \nabla_{\delta x} a^\alpha - \mathbb{E}^\alpha{}_\gamma \delta x^\gamma, \end{aligned} \quad (57)$$

where $\mathbb{E}_{\alpha\beta} \equiv R_{\alpha\mu\beta\nu} U^\mu U^\nu$ is the electric part of the Riemann tensor. This is the deviation equation for accelerated worldlines [36], i.e. a generalization of the geodesic deviation equation to nongeodesic curves. From the relation $P_{\text{hid}}^\alpha = P^\alpha - mU^\alpha$ [cf. Eq. (24)], we have $a^\alpha = (F^\alpha - \nabla_U P_{\text{hid}}^\alpha)/m$. Substituting the latter into Eq. (57) leads to

$$\frac{D^2\delta x^\alpha}{d\tau^2} = -\mathbb{E}^\alpha{}_\gamma \delta x^\gamma + \nabla_{\delta x} \frac{F^\alpha}{m} - \nabla_{\delta x} \nabla_U \frac{P_{\text{hid}}^\alpha}{m}. \quad (58)$$

To dipole order, the covariant derivative (along δx^α) of Eq. (1) reads

$$\begin{aligned} \nabla_{\delta x} F^\alpha &\simeq -\frac{1}{2} R^\alpha{}_{\beta\mu\nu} U^\beta \nabla_{\delta x} S^{\mu\nu} \\ &\quad - \frac{1}{2} R^\alpha{}_{\beta\mu\nu} S^{\mu\nu} \nabla_U \delta x^\beta. \end{aligned} \quad (59)$$

The term $U^\beta S^{\mu\nu} \nabla_{\delta x} R^\alpha{}_{\beta\mu\nu} \equiv U^\beta R^\alpha{}_{\beta\mu\nu;\lambda} \delta x^\lambda S^{\mu\nu}$, being of order $\mathcal{O}(\delta x^\lambda S^{\mu\nu})$, was neglected, since $S^{\mu\nu} \delta x^\lambda \lesssim m\rho^2$, recall that ρ is the body's radius. The second term, however, is not negligible to dipole order, since $\nabla_U \delta x^\beta = \nabla_{\delta x} U^\beta = U^\beta{}_{;\alpha} \delta x^\alpha$ and $U^\beta{}_{;\alpha}$ is of the order of the angular velocity of the helical motions, $\Omega = \mathcal{M}/S_\star = \mathcal{O}(S^{-1})$. To compute $\nabla_{\delta x} S^{\mu\nu}$, it is convenient to use the normal coordinate system $\{x^{\hat{\alpha}}\}$ originating from z^α , where the tensor function $S^{\hat{\mu}\hat{\nu}}(x) = S^{\hat{\mu}\hat{\nu}}(z) + 2P^{[\hat{\mu}} x^{\hat{\nu}]}$ yields the angular momentum taken about any point of coordinates $x^{\hat{\alpha}}$ in terms of $x^{\hat{\alpha}}$ and the angular momentum about the origin $S^{\hat{\mu}\hat{\nu}}(z)$. At the origin of such coordinates one has therefore $S^{\hat{\mu}\hat{\nu}}(x)_{;\lambda} = S^{\hat{\mu}\hat{\nu}}(x)_{;\lambda} = 2P^{[\hat{\mu}} \delta_\lambda^{\hat{\nu}]}$, the latter expression is however a tensor, so, in an arbitrary coordinate system, we may write $S^{\mu\nu}(x)_{;\lambda} = 2P^{[\mu} \delta_\lambda^{\nu]}$ and

$$\begin{aligned} \nabla_{\delta x} S^{\mu\nu} &= S^{\mu\nu}(x)_{;\lambda} \delta x^\lambda = 2P^{[\mu} \delta_\lambda^{\nu]} \delta x^\lambda \\ &= 2P^{[\mu} \delta x^{\nu]}. \end{aligned} \quad (60)$$

We note in passing that Eq. (60) actually holds for an arbitrary infinitesimal displacement vector $\delta\mathcal{V}^\alpha$ (not necessarily the connecting vector δx^α): $\nabla_{\delta\mathcal{V}} S^{\mu\nu} = 2P^{[\mu} \delta\mathcal{V}^{\nu]}$; if one takes $\delta\mathcal{V}^\alpha = d\tau U^\alpha$, we obtain $\nabla_U S^{\mu\nu} = 2P^{[\mu} U^{\nu]}$, and hence a very simple derivation of the spin evolution equation (2).

Taking, for simplicity, the nonhelical centroid as the basis worldline, we have $\nabla_{\delta x} S^{\mu\nu} = 2mU^{[\mu} \delta x^{\nu]}$ since, as shown above, for this worldline⁸ $a^\alpha = 0 \Rightarrow P_{\text{hid}}^\alpha = 0 \Rightarrow P^\alpha = mU^\alpha$. For such a motion, Eq. (59) reads

$$\nabla_{\delta x} F^\alpha = m\mathbb{E}^\alpha{}_\nu \delta x^\nu - \frac{1}{2} R^\alpha{}_{\beta\mu\nu} S^{\mu\nu} \nabla_U \delta x^\beta, \quad (61)$$

whose first term exactly cancels out the tidal term in (58), i.e. one has then

$$\frac{D^2\delta x^\alpha}{d\tau^2} = -\frac{1}{2m} R^\alpha{}_{\beta\mu\nu} S^{\mu\nu} \nabla_U \delta x^\beta - \nabla_{\delta x} \nabla_U \frac{P_{\text{hid}}^\alpha}{m}. \quad (62)$$

These expressions have the following interpretation. The first term of the force variation (61) ensures that the worldlines move at a constant distance, by counteracting the tidal force $-\mathbb{E}^\alpha{}_\nu \delta x^\nu$ in Eq. (58), which ‘‘tries’’ to make the worldlines diverge/converge. The second term of Eq. (61), together with the hidden momentum term $\nabla_{\delta x} \nabla_U (P_{\text{hid}}^\alpha/m)$ in Eq. (58), which form Eq. (62), are responsible for the winding motion around the nonhelical centroid.

In a flat spacetime, such a winding—and hence the relative acceleration between centroids—are solely due to the hidden momentum: $D^2\delta x^\alpha/d\tau^2 = -\nabla_{\delta x} \nabla_U (P_{\text{hid}}^\alpha/m)$, cf. Eq. (62). Curvature changes both P_{hid}^α and its derivative, but such a change is nevertheless compensated by the first term of Eq. (62) [second term of Eq. (61)] ensuring that, apart from an overall motion in the radial direction, the trajectories are almost the same as in flat spacetime, as manifest in Figs. 3, 4.

In order to see how these things play out in the examples herein, we first notice that the connecting vector δx^α is simply related with the shift vector Δx^α : *for worldlines infinitesimally close*, $\Delta x^\alpha = h^\alpha{}_\beta \delta x^\beta$; that is, Δx^α is the projection of δx^α in the direction orthogonal to the basis worldline $z^\alpha(\tau)$ [see Eq. (21)]. Consider now the radial spin case of Fig. 3. The spin-curvature force exerted on the helical centroids \bar{z}^α is, to leading post-Newtonian order, $\vec{F} \simeq -3M\vec{v} \times \vec{S}/r^3$ (e.g. Eq. (56) of [12], for a radial⁹ \vec{S}), pointing *outwards*, in the direction of the shift vector $\Delta\vec{x}$. Since the force \vec{F} on the nonhelical centroid z^α is zero, we have that $\vec{F} = \vec{F} - \vec{F} \equiv \Delta\vec{F}$; and since $\Delta\vec{F} \simeq \nabla_{\Delta\vec{x}} \vec{F}$, we see that the force \vec{F} consists of two parts, which are the two terms of (61) (with $\delta\vec{x} \rightarrow \Delta\vec{x}$). The explicit expression for the first term follows from approximating $\Delta\vec{x} \simeq \vec{S} \times \vec{v}/m$,

⁸Notice however that $\nabla_{\delta x} a^\alpha \neq 0$ and $\nabla_{\delta x} P_{\text{hid}}^\alpha \neq 0$.

⁹For a helical centroid $\bar{z}^\alpha(\tau)$ the spin vector that follows from (54) is actually not exactly radial, since $\bar{z}^\alpha(\tau)$ is shifted from the radial geodesic; that however amounts to corrections of order $\mathcal{O}(S^2)$ in \vec{F} .

and using the expression for \mathbb{E}^{ij} in e.g. Eq. (88) of [15], leading to $m\mathbb{E}^{ij}\Delta x_j \simeq -M(\vec{v} \times \vec{S})^i/r^3 = \vec{F}/3$ (so it amounts to one third of the force). This term counteracts the tidal force between \bar{z}^α and z^α (first term of (58), with $\delta\vec{x} \rightarrow \Delta\vec{x}$), which is *compressive*, i.e., antiparallel to $\Delta\vec{x}$, preventing the two worldlines from converging. To compute the second term of Eq. (61), we first note that $\nabla_U \delta x^\beta$ relates to the relative velocity v^α between *infinitesimally close* worldlines $\bar{z}^\alpha(\tau)$ and $z^\alpha(\tau)$ by (cf. Eq. (4.27) of [37])

$$v^\alpha = h^\alpha_\beta \nabla_U (h^\beta_\gamma \delta x^\gamma) = h^\alpha_\gamma \nabla_U \delta x^\gamma + a^\alpha U_\gamma \delta x^\gamma,$$

reducing to $v^\alpha = h^\alpha_\gamma \nabla_U \delta x^\gamma$ in the present case that the basis worldline $z^\alpha(\tau)$ is geodesic. To leading post-Newtonian order, one can thus write, for the second term of (61), $-R^\alpha_{jkl} \epsilon^{kl} S^m v^j / 2$, yielding $-2M\vec{v} \times \vec{S} / r^3 = 2\vec{F} / 3$. Two aspects of the latter term are worth mentioning: (i) unlike the first term of the force variation (61) (which is due to the dependence of the force on the centroid's position), this one is due to the dependence of the spin-curvature force on the centroid's velocity; (ii) it is only for helical solutions of the MP SSC (where $\nabla_U \delta x^\beta \sim v \sim \mathcal{O}(\delta x/S)$) that this term is non-negligible. Under other SSCs (see [12]), $\nabla_U \delta x^\beta \sim \mathcal{O}(P_{\text{hid}}) \sim \mathcal{O}(S^n)$, with $n \geq 1$, so $R^\alpha_{\beta\mu\nu} S^{\mu\nu} \nabla_U \delta x^\beta \lesssim \mathcal{O}(m\rho^2)$ is negligible to dipole order. This means that, between centroids of other spin conditions, the difference in the forces equals minus the tidal term: $\nabla_{\delta x} F^\alpha = m\mathbb{E}^\alpha_\nu \delta x^\nu$, and thus the relative acceleration between centroids is solely down to the hidden momentum term (cf. Sec. 3.3 of [12]).

B. Circular equatorial orbits in Schwarzschild spacetime

A problem where the MP SSC is a convenient choice is that of circular equatorial orbits (CEOs) of a spinning particle in stationary axisymmetric spacetimes, where (as shall be discussed in more detail elsewhere [38]), it allows to obtain, in a very simple fashion, the *exact analytical solutions* for CEOs. Here we briefly present the procedure for the special case of the Schwarzschild spacetime.

One starts by taking the spin vector to be polar, $S^\alpha = S^\theta \partial_\theta^\alpha$. The spin evolution equation (14) ensures, in this case, that the components S^α remain constant as long as $U^r = U^\theta = 0$ (as is the case for a CEO).

For a CEO, the four-velocity has the form

$$U^\mu = U^0(\partial_0 + \omega\partial_\phi); \quad U^0 = \left[1 - \frac{2M}{r} - r^2\omega^2\right]^{-1/2} \quad (63)$$

where $\omega \equiv U^\phi/U^0$ is the (constant) angular velocity. We take this as an ansatz for the centroid 4-velocity, and shall

now show that it is compatible with the equation of motion for the centroid under the MP SSC,

$$ma^\alpha = F^\alpha - \frac{DP_{\text{hid}}^\alpha}{d\tau}, \quad (64)$$

with F^α and P_{hid}^α given by Eqs. (13) and (25). The acceleration corresponding to (63) has only radial component,

$$a^r = -\frac{(r-2M)[r^3\omega^2 - M](U^0)^2}{r^3}. \quad (65)$$

Now, for such 4-velocity and acceleration, and a polar spin vector $S^\alpha = S^\theta \partial_\theta^\alpha$, both the spin-curvature force (13) and the covariant derivative of the hidden momentum along U^α have, as only nonvanishing components,

$$F^r = \frac{3M(r-2M)S^\theta\omega(U^0)^2}{r^2}, \quad (66)$$

$$\frac{DP_{\text{hid}}^r}{d\tau} = \frac{(r-2M)(3M-r)(M-r^3\omega^2)S^\theta\omega(U^0)^4}{r^3}. \quad (67)$$

They are purely radial, just like the acceleration; it then follows from (65) that finding CEOs reduces to solving for ω the radial equation

$$ma^r + \frac{DP_{\text{hid}}^r}{d\tau} - F^r = 0. \quad (68)$$

This is a fourth order equation for ω , leading to four distinct solutions. Their explicit (lengthy) expressions, obtained using *Mathematica*, are given in [39]. Two of the solutions are spurious and do not reduce to the circular geodesics for $S=0$. One of them (or both, depending on r and the parameters M and S) is unphysical, as its speed is supra-luminal. The other is a “giant” highly-relativistic helical motion of radius r , whose speed approaches the speed of light as $r \rightarrow \infty$. It does not correspond to an orbital motion of the physical body: its velocity remains nonzero and highly relativistic even for $M \rightarrow 0$ (Minkowski spacetime), when it becomes an helical solution of a giant body *at rest* in the static frame (i.e., $\vec{P}=0$ in such frame), similar to those depicted in body 1 of Fig. 1. More details on these solutions shall be given in [38].

The remaining two solutions are the physically relevant ones, corresponding to “prograde” and “retrograde” orbits (i.e., positive or negative angular velocity $\dot{\phi}$, respectively), which reduce to circular geodesics when $S=0$ (and whose velocity appropriately reduces to zero when $M \rightarrow 0$). All four solutions match numerical results in [40].

Finally, concerning the problem of CEO's under other spin conditions, to our knowledge, exact, analytical solutions, have so far been obtained only with the TD condition [41], through lengthier computations.¹⁰

1. Helical centroids

To study helical centroids $\bar{z}^\alpha(\bar{\tau})$ corresponding initially (within the realm of the pole-dipole approximation) to the same physical body as the one described by a given circular orbit $z^\alpha(\tau)$, we shall consider radial shifts. This is achieved by demanding in Eq. (49) the *initial* relative velocity of $\bar{z}^\alpha(\bar{\tau})$ with respect to $z^\alpha(\tau)$ to be azimuthal, $v^\mu = v^t \partial_t^\mu + v^\phi \partial_\phi^\mu$. In practice one needs only to prescribe its magnitude $v \equiv \sqrt{v^\alpha v_\alpha} < 1$ and sign; orthogonality to U^α then yields the explicit components

$$\begin{aligned} v^\phi &= \pm \frac{v}{\sqrt{g_{\phi\phi} + \omega^2 (g_{\phi\phi})^2 / g_{00}}}, \\ v^0 &= -\frac{\omega g_{\phi\phi}}{g_{00}} v^\phi. \end{aligned} \quad (69)$$

This leads to $\bar{U}^\mu|_z = \bar{U}^0|_z \partial_0^\mu + \bar{U}^\phi|_z \partial_\phi^\mu$ and to a radial shift vector, cf. Eq. (43),

$$\Delta x^\alpha|_{\text{in}} = -\frac{S^\alpha_\beta \bar{U}^\beta|_z}{\bar{m}} = \gamma \frac{\epsilon^\alpha_{\beta\gamma\delta} S^\gamma U^\delta v^\beta}{\bar{m}} = \Delta x^r \delta_r^\alpha, \quad (70)$$

where $\gamma \equiv -U_\beta \bar{U}^\beta|_z$, cf. Eq. (49).

Herein no approximation will be made, so the starting point \bar{z}^μ is obtained by exact application of the exponential map $\bar{z}^\mu = \exp^\mu_z(\Delta x)$. For that, we first note that the radial lines $\theta = \text{const}$, $\phi = \text{const}$, $t = \text{const}$ are spatial geodesics in Schwarzschild's spacetime.¹¹ Along such a geodesic, the line element is $dl^2 = g_{rr} dr^2$. Hence, being z^α and \bar{z}^α points along that curve, by definition of the exponential map, the arclength of the segment between z^α and \bar{z}^α equals the magnitude of Δx^α : $\|\Delta x^\alpha\| = \mp \int_r^{\bar{r}} \sqrt{g_{rr}} dr$, the + (−) sign applying when $\bar{r} > r$ ($\bar{r} < r$). Integrating this leads to

$$\begin{aligned} \bar{r} \sqrt{1 - \frac{2M}{\bar{r}}} + M \ln \left[\bar{r} - M + \bar{r} \sqrt{1 - \frac{2M}{\bar{r}}} \right] \\ = \pm \|\Delta x^\alpha\| + r \sqrt{1 - \frac{2M}{r}} + M \ln \left[r - M + r \sqrt{1 - \frac{2M}{r}} \right] \end{aligned} \quad (71)$$

which is an equation that yields \bar{r} (thus \bar{z}^α), given the values of $\|\Delta x^\alpha\|$ and r , to be solved numerically.

The parallel transport of the moments P^α and the transformed spin tensor $\bar{S}^{\alpha\beta}|_z$, Eqs. (42)–(41), shall also be calculated exactly. Let $\eta^\alpha = \delta_r^\alpha dr/ds$ denote the tangent vector to the spatial geodesic $c^\alpha(s)$ connecting z^α to \bar{z}^α . It is easy to check that the parallel transport conditions $\nabla_\eta P^\alpha = 0$ and $\nabla_\eta \bar{S}^{\alpha\beta} = 0$ are satisfied if these tensors have constant components in the orthonormal tetrad $\mathbf{e}_{\hat{\alpha}}$, tied to the Schwarzschild basis vectors ∂_α , defined by

$$\mathbf{e}_{\hat{i}} = (1/\sqrt{g_{ii}})\partial_i, \quad \mathbf{e}_{\hat{0}} = (1/\sqrt{-g_{00}})\partial_0.$$

That is, when one has

$$\begin{aligned} P^0 &= P^{\hat{0}}/\sqrt{-g_{00}}, & P^i &= P^{\hat{i}}/\sqrt{g_{ii}}, \\ \bar{S}^{ij} &= \bar{S}^{\hat{i}\hat{j}}/\sqrt{g_{ii}g_{jj}}, & \bar{S}^{i0} &= \bar{S}^{\hat{i}\hat{0}}/\sqrt{-g_{00}g_{ii}}, \end{aligned}$$

with $P^{\hat{\alpha}}$ and $\bar{S}^{\hat{\alpha}\hat{\beta}}$ constant. This leads to the relations

$$P^i|_{\bar{z}} = P^i|_z \frac{\sqrt{g_{ii}}|_z}{\sqrt{g_{ii}}|_{\bar{z}}}, \quad P^0|_{\bar{z}} = P^0|_z \frac{\sqrt{-g_{00}}|_z}{\sqrt{-g_{00}}|_{\bar{z}}}, \quad (72)$$

$$\bar{S}^{ij}|_{\bar{z}} = \bar{S}^{ij}|_z \frac{\sqrt{g_{ii}g_{jj}}|_z}{\sqrt{g_{ii}g_{jj}}|_{\bar{z}}}, \quad \bar{S}^{i0}|_{\bar{z}} = \bar{S}^{i0}|_z \frac{\sqrt{-g_{ii}g_{00}}|_z}{\sqrt{-g_{ii}g_{00}}|_{\bar{z}}}. \quad (73)$$

For given values of z^α , P^α , and $S^{\alpha\beta}$ of the basis centroid, and of the magnitude v of the initial relative velocity of \bar{z}^α with respect to z^α , Eqs. (69)–(73) yield the initial data needed for helical solutions. The initial data are then numerically evolved by the equations of motion (27) [i.e., Eqs. (1)–(2) plus the momentum-velocity relation (19)].

They are plotted, together with the corresponding non-helical solutions, in Figs. 5–6, for $r = 7M$, $r = 10M$ and $r = 30M$. Initially the helices are winding about, approximately, the nonhelical worldline, much like in the way they do in flat spacetime (see Sec. I of [35]), as one would expect for different worldlines of the same body (recall discussion in Sec. VA). However, as the motion progresses, the plots show that the helices start detaching one from another, in the sense that the “peaks” do not meet. The effect is larger the closer the orbit is to the horizon (i.e., the stronger the field). For $r = 7M$ they visibly diverge outside the spatial tube swept by the body's minimum size, which is the size of its disk of centroids, of radius $R_{\text{Moller}} = S_*/\mathcal{M} \simeq S/m$,

¹⁰CEO's are obtained from the results in [41] through the following algorithm: one expresses P_0 and P_ϕ in terms of the conserved “energy” E and “angular momentum” J [Eqs. (37)–(38), (40)–(41) therein]; then use the $U - P$ relation (13), (22)–(24) to express U^α also terms of E and J . The condition $U^r = 0$ eventually leads to Eq. (47) on the quantity (44), which is solved for E and J . Substituting back into Eqs. (40)–(41), (13), (22)–(24) therein, yields P^α , and, finally, the 4-velocity U^α of the circular orbits.

¹¹This is seen from the geodesic equation which, for a radial tangent vector $\eta^\alpha \equiv \delta_r^\alpha dr/ds$, reads $d\eta^r/ds + \Gamma^r_{rr}(\eta^r)^2 = 0 \Leftrightarrow d\eta^r/dr + \Gamma^r_{rr}\eta^r = 0$. This is a first order ODE with solution $\eta^r = C\sqrt{1 - \frac{2M}{r}}$, being C an arbitrary constant ($C = 1$ if s is chosen as the arclength of the curve, case in which $\eta^\alpha \eta_\alpha = 1$).

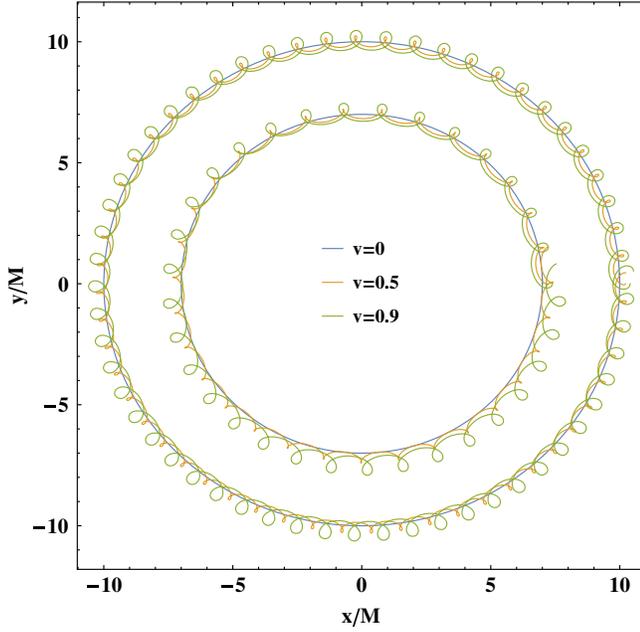


FIG. 5. Circular “prograde” orbits for $r = 7M$ and $r = 10M$ (blue lines) and the helical solutions representing the same physical motions, corresponding to two different values of the relative velocity in Eq. (69) ($v = 0.5$ and $v = 0.9$). All the trajectories start at $\phi = 0$, and only the first lap about the black hole is plotted. The spin angular momentum of the body, measured about the nonhelical centroid, has magnitude $S = 0.5mM$. As the motion progresses, the helices start “detaching” one from another; for $r = 7M$ the trajectories visibly diverge outside the body’s minimal worldtube (of radius $R_{\text{Møller}} = S_*/\mathcal{M} \simeq S/m = 0.5M$). This signals a breakdown of the approximation scheme for these centroids.

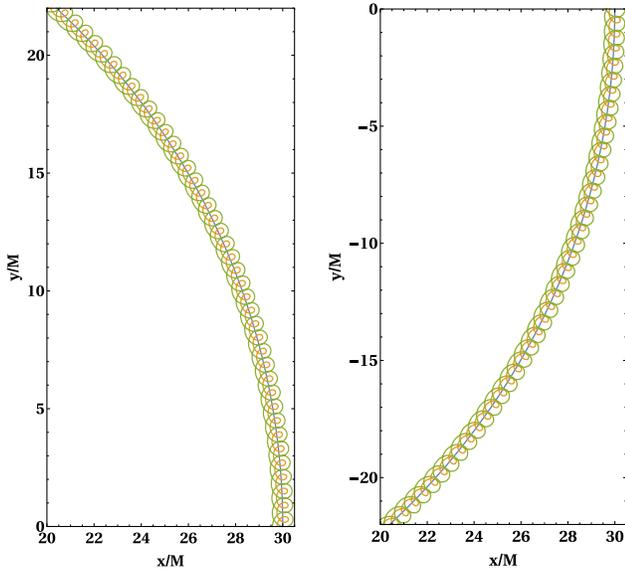


FIG. 6. Similar plots to Fig. 5, now for $r = 30M$. (Due to size constraints, only the initial and final segments of the first lap are shown. For the full plot, see [35]).

cf. Eq. (56). Here $m = -P^\alpha U_\alpha$ and S are, respectively, the mass and spin corresponding to the nonhelical centroid z^α , and in the approximate equality $S_*/\mathcal{M} \simeq S/m$ we noted that $m = \gamma(P, U)\mathcal{M}$ [where $\gamma(P, U)$ is the Lorentz factor between U^α and P^α/\mathcal{M}], used the flat spacetime relation (31) to estimate $S \simeq S_*/\gamma(P, U)$, and finally noted that $\gamma(P, U) \approx 1$, since U^α is very nearly parallel to P^α for a nonhelical centroid (e.g. for $r = 10M$, $\gamma(P, U) - 1 = 10^{-8}$). For $r = 30M$ and $r = 10M$, the effect is less pronounced, and the trajectories of the helical centroids stay contained within a spatial tube seemingly consistent with the size of the body’s disk of centroids. Although in Figs. 5–6 only one lap is depicted, the situation does not change significantly after several laps (see additional plots in [35]). Nevertheless, even in these cases, simultaneous points (in the sense of having the same coordinate time t) on different worldlines become separated, after some time, by “illegal” shifts, larger than the body’s Møller radius $R_{\text{Møller}}$. This is shown by the space-time plot of position versus coordinate time t in Fig. 7. The plot also reveals that the helical orbits have an overall orbital velocity slightly smaller than the nonhelical centroid. The effect grows with the radius of the helix, and it is not affected on whether the initial shift points inwards or outwards (cf. additional plots in [35]).

Now, the transition rules between centroids devised in Sec. IV, as discussed therein, require $\lambda = \|\mathbf{R}\|\rho^2 \ll 1$. As mentioned above, in order to have a finite spin S , a body must have a minimum radius $\rho \geq R_{\text{Møller}} \simeq S/m$; estimating the Riemann tensor magnitude by $\|\mathbf{R}\| \simeq M/r^3$, we have

$$\lambda \simeq \|\mathbf{R}\| R_{\text{Møller}}^2 \sim \frac{S^2 M}{m^2 r^3} = \left(\frac{S}{mM}\right)^2 \left(\frac{r}{M}\right)^{-3}.$$

We are using $S = 0.5mM$, so, for $r = 30M$ this yields $\lambda \sim 10^{-5}$, and, for $r = 10M$ and $r = 7M$, $\lambda \sim 10^{-4}$, which

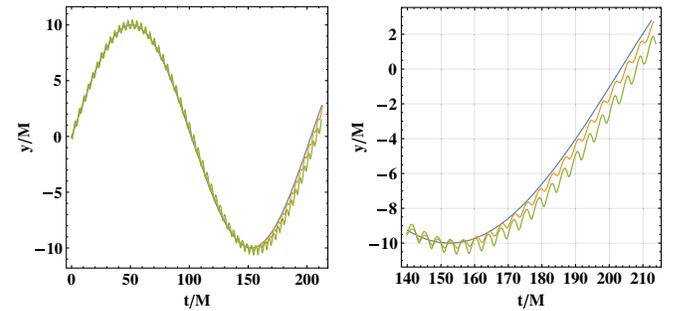


FIG. 7. The left panel shows how the y/M evolves during the first lap with respect to the coordinate time t for the trajectories in Fig. 5 corresponding to $r = 10M$. The color scheme is the same as in Fig. 5. The right panel is a closeup of the final part of the first lap. The separation between points of the different worldlines for the same instant t becomes larger than the body’s disk of centroids (of radius $R_{\text{Møller}} = S_*/\mathcal{M} \simeq S/m = 0.5M$), signaling a breakdown of the approximation scheme for these centroids.

well satisfies the restriction $\lambda \ll 1$. The illegal shifts are then likely down to a breakdown of the pole-dipole approximation itself—more precisely, *of the assumption that one can represent the same body through different centroids, while at the same time keeping a (dipole order) cutoff in the multipole expansions*. This is an unavoidable, basic feature, that arises already in Newtonian mechanics (or electromagnetism), when one describes an extended body through different representative points.

Let us recall the Newtonian problem which is enlightening for the problem at hand. Consider a homogeneous spherical body in Newtonian mechanics. It is exactly a monopole body only with respect to one point (the center of mass z^i); with respect to any other point z'^i , it will have dipole, quadrupole, and (infinite) higher order moments. Under a nonuniform gravitational field $\vec{G}(x)$, the monopole force $m\vec{G}(z')$ with respect to $z'^i = z^i + \Delta x^i$ is different from the one at z^i , $m\vec{G}(z)$. That difference is, however, exactly compensated by the dipole, quadrupole, ... n -pole forces that arise at z'^i , so that the total Newtonian force is the same in both cases, see Sec. 3.3 of [12] for more details. The larger part of the compensation comes from the dipole force $\vec{F}_{\text{dip}} = -m\Delta\vec{x} \cdot \nabla\vec{G}$, and a smaller part from the higher order moments. When one truncates the expansion at a finite order, the compensation is not perfect. Then the forces on the two points will no longer be exactly the same, and the trajectories obtained generically will end up diverging.

The relativistic problem herein is analogous, only now the two points z^α and z'^α are both centers of mass, and instead of the gradient of the monopole force $m\vec{G}$ (which has no place in general relativity) we talk about tidal forces, cf. Sec. VA. Let us assume that the body is well approximated by a pole-dipole particle *with respect to the nonhelical centroid*; i.e., it is nearly “spherical” [15], and centered at z^α . When we shift to the helical centroid z'^α via Eqs. (70)–(71) and (72)–(73), only the momentum P^α (of monopole order) and $S^{\alpha\beta}$ (dipole order) are adjusted. Thus, we are neglecting the quadrupole and higher order moments that such shift generates. For a free particle in flat spacetime this has no consequence in the dynamics. In a curved spacetime however the gravitational field couples to such moments, and the corresponding forces are needed for a full consistency of the solutions.

For the nonhelical centroid having $v = 0.9$ in Figs. 5–6 (the one for which the shift from z^α is larger, $\Delta x = 0.9R_{\text{Moller}}$), the quadrupole force is of the order

$$F_Q \sim m \|R_{\alpha\beta\gamma\delta,\lambda}\| R_{\text{Moller}}^2 \sim mMR_{\text{Moller}}^2/r^4$$

(cf. e.g., Eq. (43) of [29], Eq. (7.4) of [5]). The change in the spin-curvature force in shifting from the nonhelical centroid to the helical centroid for $v = 0.9$ is of the order

$$\Delta F \sim m \|E^\alpha{}_\nu\| R_{\text{Moller}} \sim mMR_{\text{Moller}}/r^3,$$

cf. Eq. (61). Thus,

$$\frac{F_Q}{\Delta F} \sim \frac{R_{\text{Moller}}}{r} \left(\sim \frac{S}{mr} \right).$$

In most astrophysical systems $R_{\text{Moller}} \ll r$, so the quadrupole-force correction is negligible compared to the spin-curvature one (ΔF), and it is therefore appropriate to shift between worldlines ignoring quadrupole and higher moments, through the method proposed herein.

In the examples of Figs. 5–6, we are considering a spin magnitude $S = 0.5mM$, so $F_Q/\Delta F \sim 0.5M/r$. For $r = 30M$, we have $F_Q/\Delta F \sim 0.01$, and for $r = 7M$, $F_Q/\Delta F \sim 0.1$, i.e. the quadrupole force is only one order of magnitude smaller than ΔF and the spin-curvature force itself. Given these orders of magnitude, the neglect of the quadrupole order correction F_Q is expected to be reflected on the orbits, and is likely¹² the cause for the detaching of the helices and the inconsistent separation between centroids in Figs. 5 and 7.

Finally, we note that in the examples of radial fall in Sec. VA this effect also arose, but much less pronounced. Namely, there is only a slight misalignment, close to the horizon, in the “peaks” of the helices in Figs. 3 (right bottom panel) and 4. The likely reason is that these orbits are too short-lived, especially in the stronger field region, for the effect to manifest itself. (One can infer about the duration of the motion, in comparison with the progress of the circular orbits, from the number of helical loops, since the frequency of the helices is roughly the same in both settings.)

VI. CONCLUSIONS

This paper concerns the role of the spin supplementary condition in the spinning-particle problem, focusing mainly on the Mathisson-Pirani (MP) version of the condition, $S^{\alpha\beta}U_\beta = 0$. We start by showing that the MP SSC has an explicit, and very simple, momentum-velocity relation. This result was long-sought in the literature, and once even thought not to exist. We clarify the apparent paradox between such definite relation and the fact that this SSC is degenerate, solving the apparent conflicts in the literature. We also explain the differences from other SSCs regarding the initial data required to uniquely specify a solution. These differences are seen to stem from MP’s peculiar momentum-velocity relation, and a thorough physical interpretation of this feature is provided. Then,

¹²The neglect of the quadrupole force in the pole-dipole approximation seems also possibly the cause for the eventual divergence of the centroids of different spin conditions for the same body outside its “minimal worldtube,” that have been found in [8].

we explicitly show how, for a given body, this SSC yields infinitely many possible representative worldlines, generalizing, for a curved spacetime, the flat spacetime analysis made in [18]. In the process we establish a method for transition between different representative worldlines corresponding to the same body in a curved spacetime.

To illustrate these features, we considered settings, in Schwarzschild spacetime, where this SSC is a convenient choice. Namely, we consider (i) the case of a body (whose bulk is) initially at rest, in which case it makes immediately clear that the body moves radially, as its nonhelical centroid follows a radial geodesic; and (ii) the case of the circular equatorial orbits, where it yields a very simple way of showing that such orbits are possible, and to obtain them *analytically*. We then compare the evolution of different centroids (helical and nonhelical) given by the MP SSC. Such a comparison, for different solutions corresponding to *the same body*, is done here for the first time. In the radial motions case, we have found that (apart from an overall increase in radial velocity as the body approaches the black hole, due to its gravitational field), the helices are very similar to their flat spacetime counterparts, even though their description is substantially different (e.g. due to the spin-curvature force). This we physically interpreted using the worldline deviation equation for the congruence formed by the worldlines of the centroids obeying this SSC.

A centroid shift implies a change in the body's multipole moments; but in a pole-dipole approximation only moments up to the dipole order (i.e. P^α and $S^{\alpha\beta}$) are adjusted. In flat spacetime this has no consequences. In a curved spacetime, however, curvature couples to the higher order moments, so ignoring them leads to the trajectories that can no longer be *exactly* consistent. Given this fact, the results (Figs. 3–4) show that the pole-dipole approximation holds surprisingly well in the radial motion examples. On the other hand, the CEOs provide trajectories lasting long enough, in a strong field region, to seemingly reveal these limitations.

An important point to emphasize, regarding the helical motions, is their nature as *pure gauge* effects (in other words, “noise”). Contrary to some suggestions made in the literature, they are not wrong or unphysical, but they do not contain any new physics either, nor are they down to any mysterious forces: *the physical body they represent does not undergo any helical motion* (so no experiment could ever detect it), which is but a spurious motion of the representative worldlines that this SSC does not exclude. This is so in flat spacetime as shown in [18]; herein we show that the same principle naturally holds in a curved spacetime, just requiring a more subtle treatment. In particular, by using proper transition rules to ensure that one is dealing with solutions corresponding to a given body, the different solutions (helical or nonhelical) remain close and describe, within the scope of the pole-dipole approximation, the same physics.

It is crucial to distinguish the physical, measurable effects (i.e., those that reflect in the actual motion of the body's bulk), from the pure gauge ones: spin effects in general relativity are typically small, frequently within the same order of magnitude as the superfluous motions induced by some spin conditions. For instance, the pure gauge centroid acceleration induced by the CP or NW SSC's is of the same order of magnitude as that originating from the actual spin-curvature force [12,34]. In the case of the helical solutions of the MP SSC, such as those exemplified in Sec. V, it is even typically much larger [42].

Concerning the practicality of the MP SSC, the situation is ambivalent. In those special cases where it is easy, e.g., thanks to the symmetries of the problem, to prescribe the nonhelical solution, such as the cases in Sec. V, or the ones treated in [15], this SSC can be of advantage. It is also suitable for some approximate treatments, namely linear in spin approximations, where setting the nonhelical centroid amounts to simply additionally demanding the centroid's 4-velocity U^α to be parallel to the body's momentum P^α . This can be seen by noticing, from Eq. (3), that, *for the centroid fixed by the Tulczyjew-Dixon SSC* ($S^{\alpha\beta}P_\beta = 0$), one has $P^\alpha = mU^\alpha + \mathcal{O}(S^2)$, implying that, to such accuracy, it satisfies as well the MP SSC, and therefore coincides with a centroid of the latter (the nonhelical one, since the hidden momentum, which is a necessary ingredient for the helical motions, cf. Sec. VA 2, vanishes in this case by definition). By definition, it also coincides with a centroid of the OKS SSC (the one set up by initially choosing $V^\alpha = P^\alpha/\mathcal{M}$). One may actually argue that such an approximation is inherent to the spirit of the pole-dipole approximation [43]. The same method can be applied in post-Newtonian schemes. However, in the framework of an “exact” approach, and in the generic case when it is not clear how to set the initial conditions for a nonhelical motion, the MP SSC should rather be avoided, because the helices are superfluous. They are just an *unnecessarily* complicated description of motions that can be made simpler using other SSCs. Thus, future prospects for a wider applicability of the MP SSC crucially relies on finding a generic method for singling out its nonhelical solution [44].

ACKNOWLEDGMENTS

We thank J. Natário for useful discussions. L. F. C. is funded by FCT/Portugal through Grant No. SFRH/BDP/85664/2012. G. L-G is supported by Grant No. GACR-17-06962Y of the Czech Science Foundation. O. S. was supported from the Grant No. GACR-17-13525S of the Czech Science Foundation which is acknowledged gratefully. This work was partially supported by FCT/Portugal through the project No. UID/MAT/04459/2013.

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 [43] As shown by Eqs. (3) or (12), under the TD condition, the component of P^α orthogonal to U^α (hidden momentum) is quadratic in the body’s spin. However, for a generic body, such terms are of the same typical order of magnitude as terms appearing in the quadrupole order equations of motion [4,15]; in particular, as the quadrupole contributions for the hidden momentum itself (see Supplement in [15] for a demonstration). In some models, the mass quadrupole is actually an explicit spin-squared function [17,19]. Hence, unless an assumption is made that the (traceless) mass quadrupole about the TD centroid is small (e.g., that the body has a nearly spherical shape), such $\mathcal{O}(S^2)$ terms are to be neglected in a dipole-order approximation.
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