Geodesic Chaos in Perturbed Black-Hole Fields

Petra Suková and Oldřich Semerák

Abstract Dynamics of time-like geodesics in the static and axially symmetric field of a black hole surrounded by a thin disc is studied by two recurrence methods, the recurrence plots (RPs) and the average of directional vectors (ADVs). Their results supplement the information obtained before from Poincaré surfaces of section and from phase-variable evolutions and the corresponding power spectra. The occurrence of chaos due to the presence of ambient matter may be important for evolution and appearance of astrophysical black-hole systems.

Inspired by models of accreting astrophysical black holes, we consider a simple, static and axisymmetric exact configuration of a black hole surrounded by a concentric thin disc or ring. Due to the presence of the additional source, the geodesic dynamics – originally completely integrable in the Schwarzschild field – generally becomes chaotic. Having illustrated this on Poincaré sections and on phase-variable time series and their power spectra [1], we have now turned to two recurrence methods which are based on statistics (i) over recurrences of the orbits to cells of the phase space (the recurrence plots [2, 3], RPs) and (ii) over direction in which the orbits recurrently pass through the cells (the average of directional vectors [4], ADVs). Here just a short glimpse on the results is given obtained for time-like geodesics in the field of a black hole (of mass M) surrounded by the inverted 1st Morgan-Morgan disc with mass $\mathscr{M} = 0.5M$ and inner Schwarzschild radius $r = 15M$; see [5, 6, 7] for details of our earlier results.

RPs consist in recording the recurrence matrix whose components (zeros or ones) indicate (non-)recurrences of a given orbit to selected cells. This symmetric matrix itself reveals the nature of dynamics, but here we rather show several examples of useful "quantifiers" which can be computed from the recurrence data. The simplest of them is the recurrence rate RR, given by the ratio of the recurrence points within all points of the matrix. Another one called DET is given by ratio of the points which form a diagonal line longer than a certain minimum $l_{\min}$ within all the recur-

Petra Suková and Oldřich Semerák
Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University in Prague, Czech Republic, e-mail: lvicekps@seznam.cz, oldrich.semerak@mff.cuni.cz

rence points. The inverse of the longest diagonal $DIV = 1/L_{\max}$ has been shown to provide a rough estimate of the largest Lyapunov exponent. A lower estimate of the sum of positive Lyapunov exponents is given by the correlation (or Rényi's) entropy $\hat{K}_2$, determined by a slope of the cumulative histogram plotted (in log scale) against the diagonal length l (for large l). Statistics over vertical (or horizontal) lines of the matrix brings similar quantifiers, for example, LAM is a counter-part of DET and $V\,ENTROPY$ is obtained from probability that a chosen vertical line has a prescribed length l. Note that the recurrence matrix and all the quantifiers of course depend on the chosen size of spatial cells and on time step with which the orbits are processed.

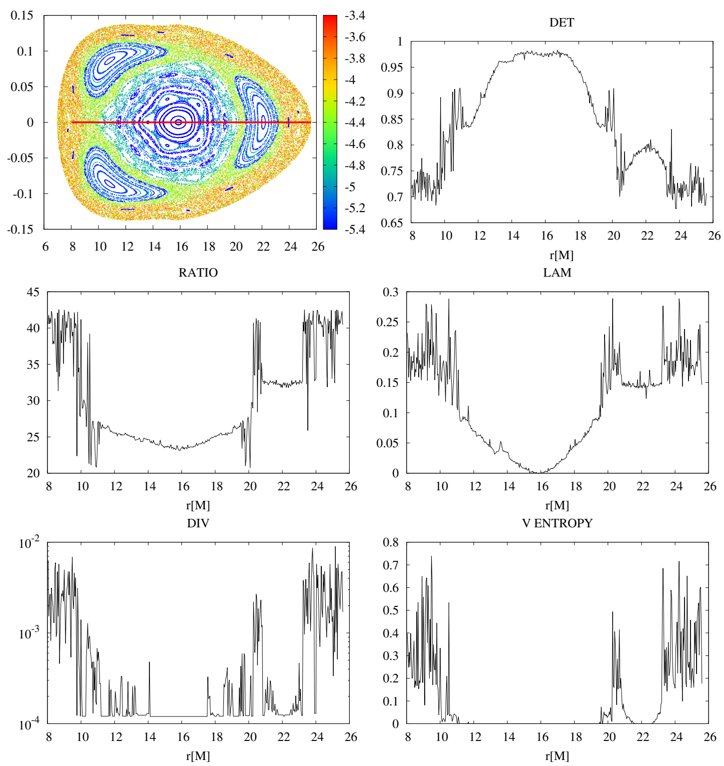

Fig. 1 Poincaré section (top left) with each orbit coloured according to the value of DIV in log scale. The red line indicates starting points. The other plots show several recurrence quantifiers as functions of the initial radius of the orbits.

We launch geodesic particles with zero radial velocity from different radii between $r = 8M$ and $r = 25M$ with specific energy $E = 0.955$ and specific angular momentum $L = 4M$. Poincaré section of several hundreds of such orbits coloured according to the value of DIV is depicted in Fig. 1 together with the behavior of several RP quantifiers.

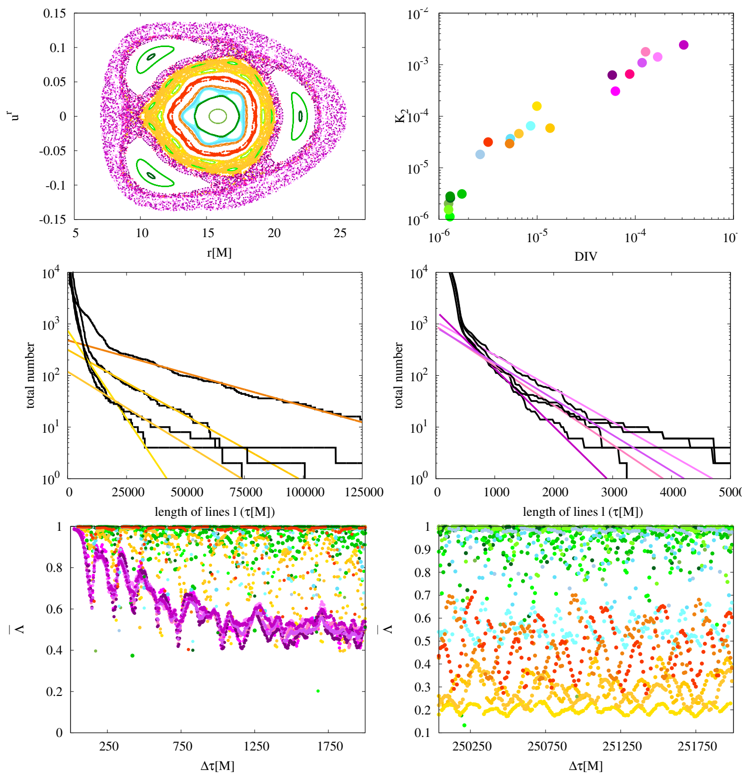

Fig. 2 Poincaré surface of section with each orbit plotted in different colour (top left); comparison of K_2 against *DIV* (top right); cumulative histograms of diagonal lines in RP for several strongly chaotic (middle left) and weakly chaotic (middle right) with the linear regression yielding K_2; the ADV parameter $\bar{\Lambda}$ for small (bottom left) and big (bottom right) time lags.

The other method, ADVs, is based on monitoring the evolution of tangent to the trajectory in selected boxes of the phase space "reconstructed", for a given data series $x(\tau)$, by taking the delayed replicas $x(\tau), x(\tau-\Delta\tau), x(\tau-2\Delta\tau), \ldots, x(\tau-d\Delta\tau)$ as its axes, where $\Delta\tau$ is a chosen time shift. The vectors obtained for a large number of transits through the j-th box are summed, the resulting vector is normalised and the norm is averaged over all boxes which were crossed n-times. Finally, the dependence of this average on n and also of the averaged quadratic difference from random-walk case on time lag, $\bar{\Lambda}(\Delta\tau)$, are analysed. For random data the average decreases more quickly than for a deterministic signal; for a regular orbit, it remains 1 theoretically. The ADV method was originally proposed to distinguish between deterministic and random data, but we have found it is also sensitive to different degrees of chaoticity.

The comparison of the above methods is illustrated in Fig. 2. Several trajectories have been followed there for a very long time: the most chaotic trajectories up to $\tau_{\max} = 250000M$ while other to $\tau_{\max} = 800000M$ (one orbital period represents some $300M$ of proper time). The particle position is recorded with the step $\Delta\tau = 0.25M$.[1] Each trajectory is plotted in a different colour; regular orbits are green, weakly chaotic orbits range from light blue to orange and yellow and the most chaotic ones are purple. It is seen that the character of orbits is revealed by Poincaré diagram as well as by the values of DIV or K_2, and also from the behaviour of $\bar{\Lambda}(\Delta\tau)$. We pay particular attention to comparison of the quantities DIV or K_2 (bottom left). Namely, although K_2 is considered a more precise and reliable estimate of orbital divergence, its correct determination is tricky and less suitable for an automatic procedure (see [5] for details). The DIV quantifier is obtained much more easily and reliably. The plot shows that the log values of DIV and K_2 are roughly proportional and that different types of motion (regular, weakly chaotic and strongly chaotic) yield clearly different values of both quantities.

In the last two plots of Fig. 2, the ADV main result $\bar{\Lambda}$ is shown against the time lag $\Delta\tau$. At bottom left, the time lag ranges from $10M$ to $2000M$ (more than 6 orbital periods); $\bar{\Lambda}(\Delta\tau)$ for strongly chaotic trajectories (purple) is rapidly decreasing, while weakly chaotic and regular trajectories look quite similar. At bottom right, $\bar{\Lambda}(\Delta\tau)$ is given for much bigger time lags ($250000M \div 252000M$); here the difference between weakly chaotic and regular orbits becomes obvious.

We would like to thank the projects GACR-205/09/H033, SVV-267301 and GAUK-428011 (PS), GACR-202/09/0772 and MSM0021620860 (OS).

References

1. O. Semerák, P. Suková, *Free motion around black holes with discs or rings: between integrability and chaos – I*, Mon. Not. R. Astron. Soc. **404**, 545 (2010)
2. J.P. Eckmann, S. Oliffson Kamphorst, D. Ruelle, *Recurrence plots of dynamical systems*, Europhys. Lett. **4**, 973 (1987)
3. N. Marwan, M.C. Romano, M. Thiel, J. Kurths, *Recurrence plots for the analysis of complex systems*, Phys. Rep. **438**, 237 (2007)
4. D.T. Kaplan, L. Glass, *Direct test for determinism in a time series*, Phys. Rev. Lett. **68**(4), 427 (1992)
5. O. Semerák, P. Suková, *Free motion around black holes with discs or rings: between integrability and chaos – II*, Mon. Not. R. Astron. Soc. **425**, 2455 (2012)
6. P. Suková, *Chaotic geodesic motion around a black hole and disc*, J. Phys.: Conf. Ser. **314**, 012087 (2011)
7. P. Suková, O. Semerák, *Recurrence of geodesics in a black-hole-disc field*, in *Towards New Paradigms: Proceeding of the Spanish Relativity Meeting 2011*, *AIP Conference Proceedings*, vol. 1458, ed. by J. Beltrán Jiménez, J.A.R. Cembranos, A. Dobado, A. López Maroto, A. De la Cruz Dombriz (AIP, Melville, NY, 2012), pp. 523–526. doi: 10.1063/1.4734475

[1] ADVs require a sufficient number of orbital points in each phase-space box (and the boxes must be small enough, naturally), which leads to quite large data sets. In comparison, for the RPs more sparse data are sufficient (time step $30M \div 50M$).