Nearly everywhere flat spaces

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We discuss some spacetimes, which are flat everywhere except for a thin shell of matter or a string of matter, in the framework of the Israel formalism. First we study spherically symmetric universes with a single sheet of matter. Then we show that the construction of a cosmic string as a limit of various thin shell distributions of matter leads to identical results.

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1 Introduction

First we discuss the physical properties of some closed universes that are flat everywhere except for a spherical sheet of matter. In the first section, we closely follow the works of Langer [1] and Lynden-Bell, Katz and Redmount [2], who investigated the geometrical properties of a wider category of solutions. However, we discuss the physical properties of the dynamical solutions with a more general equation of state. Next section then explores the possibility of modeling a cosmic string as an identification of two hyper-surfaces in the Minkowski spacetime in the limit where the bent part of these surfaces is infinitely narrow. For the description of thin shells of matter the Israel formalism [3] is used.

2 Spherical sheet universes

Let us take the Minkowski spacetime. Now imagine a world-tube generated by a sphere of changing radius r_-. We assume that r_- changes with time in such a way that the resulting hypersurface is time-like everywhere. (We shall denote all quantities relating to this world-tube by index '$-$'.) Next we construct an exact copy of our world-tube (quantities relating to this world-tube are denoted by index '$+$'). We now identify the boundaries of these world-tubes in a smooth way. The resulting spacetime consists of the interiors of our two world-tubes in the Minkowski spacetime separated by a time-like hypersurface. If we take a constant-time slice of the spacetime, we obtain the *interiors* of two spheres separated by a spherical surface layer.

We apply the Israel formalism now [3]. In both world-tubes, we use spherical coordinates $r_\pm, \theta_\pm, \varphi_\pm$ and the Minkowskian time $t_\pm$. We introduce intrinsic co-

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ordinates Θ, Φ, t on the separating hypersurface itself. The embedding is defined by

$$\begin{aligned} \theta_- &= \Theta = \theta_+, \\ \varphi_- &= \Phi = \varphi_+, \\ t_- &= t = g(t_+), \\ r_\pm &= f_\pm(t). \end{aligned}$$

The junction conditions require that $f_- = f_+ \equiv f$ and $t_- = t_+ = t$ (g is identity). For the normal (n) and tangential vectors (e) we have

$$\begin{aligned} t_- = t = t_+ &\Rightarrow e^{\alpha}_{t\pm} = (1; f'; 0; 0), \\ r_- = f(t) = r_+ &\Rightarrow n^{\alpha}_{\pm} = \mp(f'; 1; 0; 0)/\sqrt{1-f'^2}, \\ \theta_- = \Theta = \theta_+ &\Rightarrow e^{\alpha}_{\Theta\pm} = (0; 0; 1; 0), \\ \varphi_- = \Phi = \varphi_+ &\Rightarrow e^{\alpha}_{\Phi\pm} = (0; 0; 0; 1). \end{aligned} \tag{1}$$

The induced metric is $\mathrm{d}s^2 = -\mathrm{d}t^2(1-f'(t)^2) + f(t)^2(\mathrm{d}\Theta^2 + \sin^2\Theta\, \mathrm{d}\Phi^2)$. Finally, we compute the extrinsic curvatures on both sides of the hyper-surface and take their difference. This simply gives twice the original value. After a simple calculation we then find the induced energy–momentum tensor to be of the form

$$S_{ij} = \frac{\sqrt{1-f'^2}}{2\pi} \begin{pmatrix} \frac{1}{f} & 0 & 0 \\ 0 & \frac{f}{2}\frac{f'^2-1-ff''}{(1-f'^2)^2} & 0 \\ 0 & 0 & \frac{f}{2}\frac{f'^2-1-ff''}{(1-f'^2)^2}\sin^2\Theta \end{pmatrix}.$$

There will be two conditions restricting the solution. The first one ensures the existence of the square root appearing throughout in the formulae and says that the expansion rate of this sphere is not super-luminal $|f'| \le 1$. The second (energy) condition ensures that the induced pressures are non-negative $f'^2 - ff'' \ge 1$. Substituting from the first condition into the second one, we have $f'' \le 0$ (this is a necessary condition although not a sufficient one). Therefore, the solution cannot be a smooth periodic function (repeated contraction and expansion of the sphere) with some minimal $f > 0$.

We would like to interpret this situation now as a sphere of particles moving freely and randomly in all directions on the sphere. First, let us consider a three-dimensional Minkowskian spacetime with Cartesian coordinates (T, X, Y). We can write the three-dimensional energy-momentum tensor of particles moving with classical velocity $v = (v_x, v_y)$ in the $z = 0$ plane $[U^i = (\mathrm{d}T/\mathrm{d}\tau, \mathrm{d}X/\mathrm{d}\tau, \mathrm{d}Y/\mathrm{d}\tau) = (\mathrm{d}T/\mathrm{d}\tau)(1, v\cos\alpha, v\sin\alpha)]$, where α is the inclination angle with respect to the x-axis) that have a rest-mass rest-density ρ_{00}

$$T^{ij} = \rho_{00}U^iU^j = \rho_{00}\left(\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)^2 \left[\begin{pmatrix} 1 \\ v\cos\alpha \\ v\sin\alpha \end{pmatrix} (1, \;\; v\cos\alpha, \;\; v\sin\alpha)\right] =$$

$$= \rho_{00} \left(\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)^2 \begin{pmatrix} 1 & v\cos\alpha & v\sin\alpha \\ v\cos\alpha & v^2\cos^2\alpha & v^2\sin\alpha\,\cos\alpha \\ v\sin\alpha & v^2\sin\alpha\,\cos\alpha & v^2\sin^2\alpha \end{pmatrix}.$$

Averaging over all possible directions ($\bar{T}^{ij} \equiv (1/2\pi)\int_0^{2\pi} T^{ij}(\alpha)\mathrm{d}\alpha$) we obtain

$$T^{ij} = \gamma^2 \rho_{00} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}v^2 & 0 \\ 0 & 0 & \frac{1}{2}v^2 \end{pmatrix}, \tag{2}$$

with $\gamma = 1/\sqrt{1-v^2}$ the usual Lorentzian factor.

Introducing locally Cartesian coordinates on the shell, we find

$$\tilde{S}_{ij} = \frac{1}{2\pi f\sqrt{1-f'^2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{f'^2-1-ff''}{2(1-f'^2)} & 0 \\ 0 & 0 & \dfrac{f'^2-1-ff''}{2(1-f'^2)} \end{pmatrix}. \tag{3}$$

Comparison with formula (2) yields

$$v^2 = \frac{f'^2-1-ff''}{1-f'^2}, \tag{4}$$
$$\rho_{00} = \frac{ff''-2f'^2+2}{2\pi f(1-f'^2)^{3/2}}.$$

If we set $v = 0$, we obtain particles on radial paths with $M_{00} \equiv 4\pi f^2 \rho_{00} = 2f/\sqrt{1-f'^2}$. However, for radial paths, M_{00} is constant as $f'^2 - 1 - ff'' = 0$. Integrating once more, we have

$$f(t) = \frac{M_{00}}{2} \sin \frac{2(t-t_0)}{M_{00}}, \tag{5}$$

where t_0 is the instant, when the sphere starts expanding from the central point. We see immediately that the solution is valid for the range $t - t_0 \in [0, \frac{1}{2}\pi M_{00}]$ and thus $f \in [0, \frac{1}{2}M_{00}]$. The upper limit is the turning point of the expansion. This is in agreement with both works cited in Introduction ([1], [2]).

The trace of tensor (3) is equal to

$$\frac{1}{2\pi f}\frac{2f'^2-ff''-2}{(1-f'^2)^{3/2}}.$$

Thus, if we want to construct this sphere of photons, we need to ensure that the equation $ff'' - 2f'^2 + 2 = 0$ is fulfilled.

Substituting $f' = p$ (where $|p| < 1$) and using f as an independent coordinate instead of t, we have $f'' = \dot{p}p < 0$ ($\dot{p} \equiv \mathrm{d}p/\mathrm{d}f$) and $\dot{p}pf - 2p^2 + 2 = 0$. From here we infer $f = 2(1-p^2)/(-p\dot{p})$. We can separate the variables yielding

$$\frac{\mathrm{d}f}{f} = \frac{1}{2}\frac{-p\mathrm{d}p}{1-p^2}.$$

This we can integrate arriving at $f = f_{\max}(1-p^2)^{1/4}$. We thus have $f' = \pm\sqrt{1-(f/f_{\max})^4}$. As we can see there always is a turning point and the sphere cannot expand beyond $f = f_{\max}$. Achieving this point it will recollapse. We can write

$$\int \frac{\mathrm{d}f}{\sqrt{1-\left(\frac{f}{f_{\max}}\right)^4}} = \pm \int \mathrm{d}t.$$

The solution to this equation can be written in an implicit form as

$$f_{\max}\, F\left(\frac{f}{f_{\max}}, I\right) = \pm(t-t_0),$$

where F is the incomplete elliptic integral of the first kind (see [4], Chapter 17). The inverse formula reads

$$f(t) = \pm f_{\max}\,\mathrm{sn}\left(\frac{t-t_0}{f_{\max}}, I\right),$$

where sn is the Jacobi elliptic function, (see [4], Chapter 16). We can only accept the plus sign as we cannot go beyond $f = 0$. Therefore, the evolution of our system is described by the first half-period of this periodic function.

In a general situation we need to specify certain equations of motion. We may, for example, fix the total number N of particles within the shell and the rest mass m_0 of the individual particles. We then have

$$4\pi f^2 \tilde{S}_{00} = M = N m_0 \gamma \equiv M_0 \gamma, \quad M_0 = \mathrm{const}.$$

We find

$$M_0 = \frac{2f}{1-f'^2}\sqrt{ff'' - 2f'^2 + 2}\,.$$

Multiplying the square of this equation by $2ff'$, we obtain

$$2ff'M_0^2 = \frac{8f^3 f'}{(1-f'^2)^2}(ff'' - 2f'^2 + 2).$$

This can be integrated immediately yielding

$$M_0^2 f^2 - 4\frac{f^4}{1-f'^2} = -\beta^4. \tag{6}$$

We need to choose a negative integration constant here since a solution with a positive one does not satisfy the condition $v^2 \geq 0$ (equivalent to the positive pressure condition).

Using formula (4) we find

$$v^2 = \frac{1}{1 + \left(\frac{M_0 f}{\beta^2}\right)^2}.$$

Therefore

$$\beta^2 = \frac{M_0}{\sqrt{1-v^2}} f v = N \frac{m_0}{\sqrt{1-v^2}} f v = N m f v.$$

We can thus interpret the conserved quantity β^2/N as the relativistic angular momentum of each particle. However, β^2 is not the total relativistic angular momentum of the sphere, which is identically zero due to the spherical symmetry. If $\beta \to 0$, the particles only move in the radial direction as expected and we end up with the solution (5). If, on the other hand, we increase the (relativistic) angular momentum ad infinitum, the trajectories of particles become null and we obtain the photonic case. Let us express the expansion velocity from (6) as

$$f'^2 = \frac{M_0^2 f^2 - 4f^4 + \beta^4}{M_0^2 f^2 + \beta^4}.$$

The solution satisfies the energy conditions for

$$f \in \left[0, \frac{M_0}{2}\sqrt{\frac{1+\sqrt{1+(2\beta/M_0)^4}}{2}}\right].$$

This situation describes a sphere expanding from the central point at time $t = t_0$, reaching the maximal size of $f = \frac{1}{2} M_0 \sqrt{\frac{1}{2}(1 + \sqrt{1 + (2\beta/M_0)^4})}$, and then recollapsing into its initial state. The implicit form of the solution is

$$t - t_0 = \int_0^{f(t)} \sqrt{\frac{M_0^2 x^2 + \beta^4}{M_0^2 x^2 - 4x^4 + \beta^4}} \mathrm{d}x.$$

As we can see the turning point is always located at $f_{\max} \geq M_0/2$. It is interesting that particles with purely radial trajectories start to recollapse after reaching a smaller distance from the point of origin as measured by their total mass M_0, than particles having tangential velocities in addition to their radial motion. This situation is counterintuitive within the Newtonian concepts that, however, are difficult to apply here as there is no Newtonian analogue to this situation.

Interpreting this as a perfect fluid with

$$T_{ij} = \begin{pmatrix} \rho_{00} & 0 & 0 \\ 0 & p_0 & 0 \\ 0 & 0 & p_0 \end{pmatrix},$$

we conclude $\rho_{00} = \tilde{S}_{00}$ and $p_0 = \tilde{S}_{11} = \tilde{S}_{22}$. To be able to find the equations of motion we would need to specify the equation of state $p = p(\rho)$ for the fluid. For example, requiring that $p = \rho/2$ (gas of photons moving randomly in all directions within a 2-dimensional surface) we again have the above photonic case.

Let us briefly review the static case. The basis vectors are

$$
\begin{aligned}
t_- &= T = t_+ \Rightarrow e^{\alpha}_{T\pm} = (1;0;0;0),\\
r_- &= R = r_+ \Rightarrow n^{\alpha}_{\pm} = (0;\mp 1;0;0),\\
\theta_- &= \Theta = \theta_+ \Rightarrow e^{\alpha}_{\Theta\pm} = (0;0;1;0),\\
\phi_- &= \Phi = \phi_+ \Rightarrow e^{\alpha}_{\Phi\pm} = (0;0;0;1).
\end{aligned}
$$

The induced metric is $\mathrm{d}s^2 = -\mathrm{d}T^2 + R^2(\mathrm{d}\Theta^2 + \sin^2\Theta\ \mathrm{d}\Phi^2)$. We now again introduce the locally Cartesian coordinates and after some manipulation we find the induced energy-momentum tensor

$$
S_{ij} = R/4\pi \begin{pmatrix} \dfrac{2}{R^2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
$$

The resulting matter density is thus positive, however, the pressures in the main directions are negative. The induced matter therefore behaves as an inflated soap bubble. The total induced mass is $M \equiv 4\pi R^2 S_{TT} = 2R$.

For a general, spherically symmetric metric $\mathrm{d}s^2 = -F(r)\mathrm{d}t^2 + \mathrm{d}r^2/F(r) + r^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2)$, we have another relevant nonzero Christoffel symbol $\Gamma^t_{rt} = \frac{1}{2}F'/F$, where $F' \equiv \mathrm{d}F(r)/\mathrm{d}r|_{r=R}$. For the induced energy–momentum tensor we find in the locally Cartesian coordinates

$$
S_{ij} = \frac{R}{4\pi} \begin{pmatrix} 2\dfrac{F(R)^{3/2}}{R^2} & 0 & 0 \\ 0 & -\dfrac{1}{2}\dfrac{2F(R) + RF'(R)}{\sqrt{F(R)}} & 0 \\ 0 & 0 & -\dfrac{1}{2}\dfrac{2F(R) + RF'(R)}{\sqrt{F(R)}} \end{pmatrix}.
$$

Therefore, the total shell mass is $M = 2RF(R)^{3/2}$. In this situation, we might succeed in keeping all the tensor components positive. For example, in the de Sitter case ($F(r) = 1 - \frac{1}{3}\Lambda r^2$), it is sufficient to fulfill the condition $\frac{3}{2} < \Lambda R^2 < 3$. In this case, the collapse of the shell is obviously prevented by the repulsive effect of the positive cosmological constant.

The thin shell model considered here can be interpreted as two maximal vacuoles in the "spherical" Robertson–Walker universe where all mater is located only on the singular surface separating the vacuoles. When constructing this model, we started with two world-tubes and we identified the external points of these tubes (Fig. 1a). However, we can also take just one world-tube and identify the external antipodal

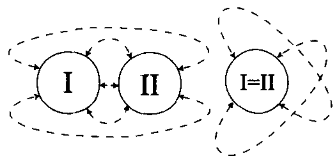

Fig. 1. Identification of world-tubes.

points on the sphere that represents a space section of this tube (Fig. 1b). In this case an observer living in this universe finds himself surrounded by a thin spherical shell of matter at any moment. After going through the shell he appears on the other side of his own flat universe.

The latter case can be considered to be the maximal vacuole in an "elliptical" world. The equations of motion remain unchanged, only the total mass of the shell is one half of the total mass in the former case.

3 Producing a cosmic string

It is clear that a cosmic string source can be achieved by taking an infinitely extended physically acceptable source with cylindrical symmetry and then decreasing its radius to zero while ensuring that the mass per unit length remains constant. Langer [5] used a hollow cylinder of matter approaching a string with the same energy-momentum tensor as later derived by Linet [6] for a general string metric.

We take a different path arriving at the same result, however. We identify two sheets in Minkowski spacetime (we use Cartesian coordinates). The sheets represent the identification surfaces at a given Minkowskian time and do not bend in the z-direction (their defining equation does not involve z). Furthermore, for both surfaces, there exists a bounded area in the x–y plane outside of which the defining equation is $x = \alpha y$. See Fig. 2.

Identifying the surfaces, we then find the induced matter to be located only within this area. The limiting procedure consists of decreasing this area while keeping the total bending angle of the two sheets constant. Whatever initial sheets we use, we always find the same form of the limiting matter distribution.

Using Cartesian coordinates in Minkowski spacetime, we now identify hypersurfaces $x_\pm = f_{2,1}(y_\pm)$. We assume that these are flat outside a compact region (i.e., $x_\pm = \alpha_\pm y_\pm$ for $y_\pm \notin [y_{1\pm}, y_{2\pm}]$). On the hypersurface, we define the inner coordinates (T, Y, Z) with $(t_\pm, x_\pm, y_\pm, z_\pm) \equiv (T, f_{2,1}(y_\pm(Y)), y_\pm(Y), Z)$. For the induced metric tensor we have

$$g_{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & y_\pm'^2\left(1 + f_{2,1}'^2\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

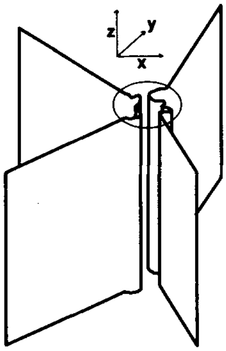

Fig. 2. Identification surfaces.

Following the same lines as above, we find the induced energy-momentum tensor to be

$$S_{ij} = \frac{1}{8\pi}\begin{pmatrix} -\frac{f_1''}{(1+f_1'^2)^{3/2}} + \frac{f_2''}{(1+f_2'^2)^{3/2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{f_1''}{(1+f_1'^2)^{3/2}} - \frac{f_2''}{(1+f_2'^2)^{3/2}} \end{pmatrix}.$$

Using the definition of the curvature radius of a function, we can rewrite this as

$$S_{ij} = \frac{1}{8\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In the expression above, $R_{1,2}$ are functions of position. However, if the two hypersurfaces are cylinders, then R's are constant and we can compare this expression with a similar one derived in a situation where we identify two coaxial cylinders in Levi-Civita spacetime (for coaxial cylinders, there is a minus sign between the inverted radii). The outer and inner cylindrical spacetimes are characterized by $\rho_\pm, m_\pm, C_\pm$, respectively, where ρ refers to the radius of the cylinder, C stands for the conicity while m is the Levi-Civita parameter related to the mass per unit length of the string. The inner Levi-Civita spacetime is flat and thus we have $C_- = 1$ and $m_- = 0$. C_+ is the outer spacetime's conicity. The outer LC parameter m_+ is set to be zero (otherwise we obtain an infinite energy density per unit length in the cosmic-string limit of $\rho_- \to 0$). Here, using coordinates (T, Φ, Z), we find

$$S_{ij} = \frac{1}{8\pi}\left(\frac{C_+ - 1}{\rho_+}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Including an additional condition $\rho_- = \rho_+/C_+$ (owing to the requirement of equal circumferences of both cylinders) we can rewrite the expression standing in front of the matrix as $1/\rho_- - 1/\rho_+$. This is an exact analogue of the situation above.

To obtain the induced mass per unit length along the z-direction, we integrate over Y:

$$M_1 \equiv \int_{Y_1}^{Y_2} S_{TT}\sqrt{g_{YY}}\mathrm{d}Y = \int_{Y_1}^{Y_2} S_{TT} y'_{\pm}\sqrt{1+f'^{\,2}_{2,1}}\mathrm{d}Y = \\ = \frac{1}{8\pi}\int_{Y_1}^{Y_2}\left[\frac{f''_2 y'_+}{1+f'^2_2} - \frac{f''_1 y'_-}{1+f'^2_1}\right]\mathrm{d}Y.$$

Let us now express the change of direction in f_1 when crossing the non-flat region $\Delta\varphi_1 = \arctan f'_1(y_2) - \arctan f'_1(y_1)$ for the first region with y_2, y_1 denoting the upper and the lower limit of the flat region, respectively. We thus write

$$\Delta\varphi_1 = \int_{Y_1}^{Y_2} \frac{\mathrm{d}\arctan f'_1(y_-)}{\mathrm{d}Y}\mathrm{d}Y = \int_{Y_1}^{Y_2} \frac{f''_1 y'_-}{1+f'^2_1}\mathrm{d}Y.$$

This is exactly our formula for the induced unit-length mass. We conclude

$$M_1 = \frac{1}{8\pi}\Delta\varphi$$

with $\Delta\varphi = \Delta\varphi_2 - \Delta\varphi_1$. This is exactly the correct formula obtained by Linet [6]. If we want to achieve a cosmic string, we need to keep $\Delta\varphi$ constant while limiting the non-flat region to zero size. This results in diverging derivatives of the $f_{1,2}$ functions. Regardless of the particular form of the non-flat region, we always get the right form of mass per unit length for a cosmic string due to a missing angle.

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