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Comments on photonic shells

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Abstract
We point out the similarities and differences between cylindrical and disc photonic shells separating regions of flat spacetime.

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Introduction

We investigate in detail the special case of an infinitely thin static cylindrical shell composed of counter-rotating photons on circular geodetical paths separating two distinct parts of Minkowski spacetimes—one inside and the other outside the shell [1]—and compare it to a static disc shell formed by null particles counter-rotating on circular geodesics within the shell located between two sections of flat spacetime [2]. One might ask whether the two cases are not, in fact, merely one.

1. The cylinder

Inside the shell we have flat spacetime metric in cylindrical coordinates

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\varphi^2. \tag{1.1}$$

Outside, we again have flat spacetime but we use accelerated Rindler coordinates:

$$\mathrm{d}s^2 = -\rho^2\,\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}\rho^2 + \mathrm{d}\varphi^2/C^2, \tag{1.2}$$

where C determines the conicity of the metric [3] ($\varphi \in (0, 2\pi]$). The relation of this coordinate system to the Minkowski coordinates is as follows

$$T = \rho\sinh t, \qquad X = \rho\cosh t, \qquad Y = \varphi/C, \qquad Z = z. \tag{1.3}$$

If we wish to obtain the entire covering Minkowski spacetime we must extend the ranges of X and Y to the entire real axis (see also [4]). A three-dimensional cylinder in coordinates (1.2) with $\rho = \rho_0$ corresponds to a planar strip $X^2 - T^2 = \rho_0^2$ with $Z \in \mathbb{R}$, $Y \in [0, 2\pi/C)$ and identified edges that moves at the speed T/X from $X = \infty$ in the distant past, $T \to -\infty$,

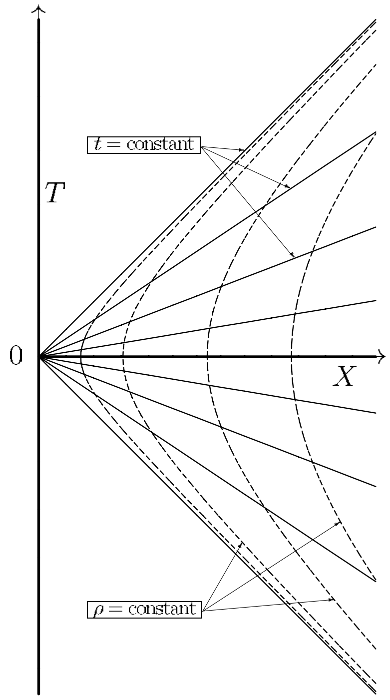

Figure 1. Rindler cylindrical coordinates as functions of Minkowskian coordinates. Each point in the diagram corresponds to a plane $Y, Z \in \mathbb{R}$.

comes to a stop at $X = \rho_0$ and $T = 0$, and speeds off to $X = \infty$ in the distant future, $T \to \infty$, as follows from figure 1. Both metrics (1.1), (1.2) are special cases of the Levi-Civita metric [5].

The two separate systems are connected through a cylindrical hypersurface located at the same coordinate radius $\rho = 1/C$, both inside and outside. The radial coordinate is thus continuous and the remaining coordinates are connected as follows,

$$t_- = T = t_+\rho = t_+/C, \qquad z_- = Z = z_+, \qquad \varphi_-\rho = Y = \varphi_+/C, \tag{1.4}$$

where T, Z, Y are coordinates on the shell itself (for clarity, coordinates inside and outside the shell are denoted by + and − indices, respectively). With this definition, the three-dimensional induced metric on the cylindrical shell is identical from both sides and it is flat[1]. We find the extrinsic curvatures, K_{AB}, of the shell inside and outside and, using the Israel formalism [6], we calculate the induced energy–momentum tensor S_{AB} on the shell (for details, refer to a more general situation in [1]):

$$S_{AB} = \frac{1}{8\pi\rho}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{1.5}$$

The resulting energy–momentum tensor is diagonal with zero trace and thus can be interpreted as streams of counter-rotating photons. Moreover, the axial component of the energy–momentum tensor is zero and so the photons only move in the φ-direction. Since they

[1] This ensures continuity of the first fundamental forms. To be able to identify the two surfaces globally, we require that the length of hoops t, z, $\rho =$ const be the same from both sides of the shell. With our choice of conicity parameter, $C = 1/\rho$, this condition is fulfilled.

have constant coordinate velocities, their trajectories are necessarily circular geodesics within the three-dimensional subspace of the cylindrical shell. A similar situation with multiple photonic shells has been discussed recently in [7] (the authors use the Kasner form of the metric[2])—we focus on a special case of a single shell formed by non-helical photons with flat regions both inside and outside.

If we calculate the mass of the shell per unit proper length, M_1, we find out that $M_1 \equiv 2\pi\rho S_{TT} = 1/4$. It might be surprising that this characteristic mass is not zero since there is flat spacetime both inside and outside the shell. However, there is no single *global* coordinate system that would enable us to describe the spacetime using the Minkowski metric. In other words, the shell *does* produce a gravitational field since there are freely moving observers who are accelerated towards one another.

Let us look at the geodesics of test particles passing through the shell. Inside and outside of the cylinder they just move along straight lines at a constant speed (if we use Minkowskian coordinates). Crossing the sheet, the projection of the particle's 4-velocity on a chosen tangential triad and the normal vector must be the same on both sides (we neglect here any interaction with the shell). This gives continuous 4-velocity components except for the temporal component with

$$U^t|_+ = U^t|_-/\rho. \tag{1.6}$$

The direction of motion of the particle thus remains the same but the magnitude of its coordinate velocity changes. Outside the shell, we find

$$\begin{aligned} t &= t_0 + \operatorname{arctanh}\left[\tau\frac{\sqrt{\alpha^2 + (\dot\rho_0)^2}}{(\rho_0 + \tau\dot\rho_0)}\right], \qquad & z &= z_0 + \dot z_0\tau, \\ \rho &= \sqrt{\rho_0^2 + 2\tau\rho_0\dot\rho_0 - \alpha^2\tau^2}, & \varphi &= \varphi_0 + \dot\varphi_0\tau, \end{aligned} \tag{1.7}$$

with $\alpha^2 = 1 + (\dot z_0)^2 + (\dot\varphi_0)^2/C^2 \geqslant 1$ for massive particles and $\alpha^2 = (\dot z_0)^2 + (\dot\varphi_0)^2/C^2 \geqslant 0$ for photons, where τ is the proper time and affine parameter, respectively. The symbols $t_0, \rho_0, z_0, \varphi_0, \dot\rho_0, \dot z_0$ and $\dot\varphi_0$ stand for the values of the coordinate time, distance from the axis, axial and angular coordinates of the particle, and its radial, axial and angular velocities at $\tau = 0$.[3] Free massive particles and photons leaving the cylinder fall back after a finite proper time/affine parameter interval under the same angle they left the shell before. After crossing the shell, they thus circumscribe always the same trajectory outside. An example of a typical trajectory in cylindrical coordinates is shown in figure 2. Only radially moving photons escape to infinity. This can be easily seen in figure 3, which uses Minkowskian coordinates both inside and outside of the shell—all geodesics (straight lines here) that leave the shell at some point and lie within the light cone intersect the cylindrical hypersurface again. Any observer outside perceives the cylinder as an infinite planar wall falling upon him at speed T/X. No matter how hard he tries, the wall always hits him. After he passes through the shell, he emerges in another section of Minkowski spacetime. He realizes that he is surrounded by a cylinder. Using a rocket, he is able to stay in this part of the spacetime indefinitely. However, he can always penetrate the wall again and reemerge in his original universe.

[2] The applied matching conditions are not generic here since the authors only obtain shells composed of null particles although it has been demonstrated that this is not the case in general [1].
[3] This is of course a straight line geodesic in the covering Minkowski spacetime.

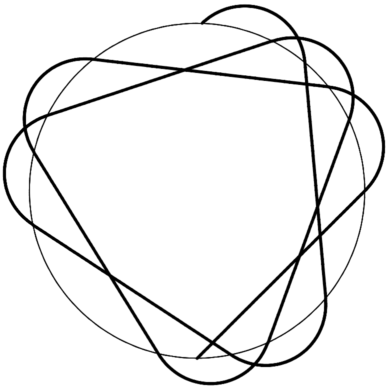

Figure 2. Cylindrical coordinates: a typical trajectory of a free massive particle entering and leaving the cylinder repeatedly.

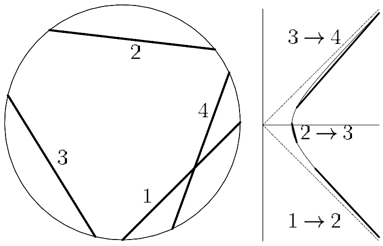

Figure 3. The same geodesic in Minkowski coordinates: the particle begins travelling along the path labelled 1 within the cylinder, then crosses the shell and appears outside on path $1 \to 2$. After a finite time it crosses the shell again and continues along the path labelled 2 eventually hitting the shell again, and so forth.

2. The disc

The metric within the flat spacetime regions above and below the disc surface reads

$$ds^2 = -\zeta^2\, d(\tau/2)^2 + d\zeta^2 + d\eta^2 + \eta^2\, d\phi^2. \tag{2.1}$$

Using the following transformation

$$T = \zeta \sinh(\tau/2), \qquad X = \zeta \cosh(\tau/2), \qquad Y = \eta \cos\phi, \qquad Z = \eta \sin\phi, \tag{2.2}$$

we find the usual Minkowski metric. An observer located above and below the disc sees it as a hypersurface defined by $X = \pm\sqrt{T^2 + Z^2 + Y^2}$, respectively. Identifying these, we find out that the disc is composed of counter-rotating null particles moving on circular geodesics around the disc centre. The surface density is positive definite everywhere. Any massive geodetical test particle penetrating the disc spirals down towards the centre of the disc in infinite proper time while photons are able to escape to infinity. A very detailed discussion is to be found in [2].

3. Comparison

The similarity between these solutions lies in the fact that a thin shell of matter separates two regions of flat spacetime and that the shell can be interpreted as free photons moving along circular orbits within the shell. One might then ask what holds the shell in place. From the point of view of the coordinate systems (1.1) and (1.2), where the shell surface is static, the centrifugal force acting upon the shell particles is balanced by the gravitational force exerted upon them by the other particles on the shell. If we adopt the perspective of two Minkowskian coordinate systems (1.1) and (1.3) then, however, the outer shell surface is not static. It is, in fact, accelerated towards free particles at rest in such a system. This effect balances the centrifugal force acting on the shell.

The difference between the two cases consists in the definition of the hypersurfaces that are identified in the two separate Minkowski spacetimes inside and outside, and above and below, respectively. If we want to compare these hypersurfaces, we need to use a single coordinate system. We begin by describing the outer cylindrical hypersurface with a coordinate radius ρ in disc coordinates (2.1). We first transform the coordinate system (1.2) into the Minkowskian coordinates and from here we go to (2.1) to obtain

$$\zeta_+ = \rho. \tag{3.1}$$

This is just a planar horizontal section of the spacetime. The inner hypersurface, as defined in coordinates (1.1), reads

$$\eta_- = \rho. \tag{3.2}$$

In this case, the cylindrical surface is again a cylinder. We identify points with

$$\begin{aligned} \zeta_- &= \sqrt{\eta_+^2 \sin^2\phi_+ - \tfrac{\tau_+^2}{4}\rho^2}, & \eta_- &= \zeta_+ = \rho, \\ \tau_- &= 2\,\mathrm{arctanh}\,\tfrac{\tau_+\rho}{2\eta_+\sin\phi_+}, & \phi_- &= \tfrac{1}{\rho}\eta_+\cos\phi_+, \end{aligned} \tag{3.3}$$

where the minus sign refers to the inner (lower) hypersurface and the plus to the outer (upper) one. The identification is not at all trivial and, moreover, it is time dependent.

In coordinates defined by (2.1), the disc surface is given by the cones

$$\zeta_\pm = \pm\eta_\pm, \tag{3.4}$$

where we identify points with

$$\zeta_- = -\zeta_+, \qquad \eta_- = \eta_+, \qquad \tau_- = \tau_+, \qquad \phi_- = \phi_+. \tag{3.5}$$

It can be seen immediately that (3.2), (3.1), (3.3) and (3.4), (3.5) are not identical.

4. Conclusions

We conclude that the two cases are different although constructed in an analogous way using two separate Minkowskian regions. One can imagine various pairs of identification surfaces, both static and time dependent, that produce infinitely thin shells separating regions of flat spacetime and that are composed of photons (another example is a thin spherical shell separating two identical regions of Minkowski spacetime of finite volume, see [8]). There are only two conditions—the induced metric must be the same on both hypersurfaces and the trace of the induced energy–momentum tensor (satisfying energy conditions) needs to be zero. If we use a coordinate system in which the shell is static and which covers both the hypersurface and the neighbouring region, geodesics will lead escaping particles back towards the shell. If, on the other hand, we prefer a simpler Minkowskian description, then test particles outside are not accelerated but the shell itself speeds towards them. In either case, the existence of the shells is enabled by the resulting balance.

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