Newman-Penrose formalism in quadratic gravity

R. Švarc, [1,*] A. Pravdová, [2,†] and D. Miškovský [1,‡]

[1] *Institute of Theoretical Physics, Charles University, Prague, Faculty of Mathematics and Physics, V Holešovičkách 2, 180 00 Praha 8, Czech Republic*

[2] *Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic*

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The quadratic gravity constraints are reformulated in terms of the Newman-Penrose-like quantities. In such a frame language, the field equations represent a linear algebraic system for the traceless Ricci tensor components. In principle, a procedure for the combination of the Ricci components with standard geometric identities can be applied in a similar way as in the case of general relativity. These results could serve in various subsequent analyses and physical interpretations of admitted solutions to quadratic gravity. Here, we demonstrate the utility of such an approach by proving general propositions restricting the spacetime geometry under assumptions on a specific algebraic type of curvature tensors.

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I. INTRODUCTION

In 1915, Albert Einstein finished his theory of general relativity (GR) [1], which provided a geometric description of gravity in terms of curved four-dimensional spacetime. Einstein's theory brought dozens of surprising implications during more than a century of its analyses and astrophysical applications. However, simultaneously with its formulation, there appeared concerns about the possibility of solving its highly-complicated nonlinear field equations. These doubts were allayed almost immediately by Karl Schwarzschild and his famous spherically symmetric solution [2]. Unfortunately, the Schwarzschild spacetime also uncovered difficulties related to physical interpretation and insecurity of employing a particular coordinate choice. In the following decades, the construction of coordinate-independent quantities, revealing the true nature of a given gravitational field, become crucial. The conceptually important step within the coordinate-independent analysis was to express studied quantities in terms of their frame components. The privileged role within the frame approaches to Einstein's theory plays the Newman-Penrose (NP) formalism [3] employing a null vector basis, see Sec. I B for its summary. The spacetime description in terms of the frame projections allows one to invariantly define the ansatz geometry, e.g., admitting specific null congruences or special algebraic structure of related tensors, and then try to find and discuss its explicit form restricted by the field equations and geometric identities. Importantly, this formalism naturally reflects distinct parts of a gravitational field and its peeling properties. The generalization of NP formalism becomes important also in studies of higher-dimensional GR [4–7].

Even though GR has beyond doubt proved its ability to describe various strong-field gravitational situations and processes, there remain important theoretical issues unclarified. One can think, e.g., about its combination with quantum field theory or the nature of singularities that necessarily occurs in its solutions. Attempts to solve these open problems typically consider additional fields or various modifications of GR extending the Einstein-Hilbert action, see e.g., reviews [8–12]. Alternatively and more pragmatically, one can study modified gravities to analyze the unique position of GR in the space of general metric theories of gravity. From this perspective, the simplest extension of GR corresponds to the quadratic gravity (QG) [13–16] including all possible curvature squares into the Einstein-Hilbert action (see also Sec. I A). Such a class of theories may directly solve some of the open problems (however, simultaneously it introduces new ones), or it may be understood as a higher-order correction to GR induced by some unknown final theory.

To better understand QG and its implications on a geometric level, the exact solution analysis becomes important, see, e.g., QG counterparts to the classic Schwarzschild black hole [17–20] or algebraically special geometries [21]. It is extremely interesting to compare solutions to QG with those to GR. However, to do so one has to invariantly define the same geometric ansatz, and therefore, the extension of the Newman-Penrose formalism for the case of quadratic gravity seems to be very natural starting point. This is thus the main aim of our contribution.

The paper is organized as follows. In Introduction we summarize concepts of QG and NP formalism. In Sec. II, the NP form of the QG field equations is derived which

*robert.svarc@mff.cuni.cz
†pravdova@math.cas.cz
‡davidmiskovsky@icloud.com

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represents our main result. Two simple examples of its use are subsequently discussed (see Sec. III). Finally, in Appendix A the geometric NP identities are summarized, in Appendixes B and C we compare various conventions related to the NP formalism, and in Appendix D, the QG field equations are listed in the fully explicit form.

A. Quadratic gravity

The vacuum quadratic gravity can be introduced via its action as

$$S = \int \left[\frac{1}{\mathsf{k}}(R - 2\Lambda) - \mathfrak{a} C_{abcd} C^{abcd} + \mathfrak{b} R^2\right] \sqrt{-g}\, \mathrm{d}^4 x, \quad (1)$$

where R is the Ricci scalar, C_{abcd} is the Weyl tensor, k, $\mathfrak{a}$, and $\mathfrak{b}$ are coupling constants of the theory, and Λ stands for the cosmological term, see, e.g., [13–16]. Due to the Gauss-Bonnet theorem, this represents the most general class of four-dimensional quadratic theories. Subsequently, the least action principle $\delta S = 0$ leads to the fourth-order field equations in the form

$$\frac{1}{\mathsf{k}}\left(R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab}\right) - 4\mathfrak{a} B_{ab} + 2\mathfrak{b}\left(R_{ab} - \frac{1}{4} R g_{ab} + g_{ab}\Box - \nabla_a \nabla_b\right) R = 0, \quad (2)$$

where B_{ab} is the Bach tensor defined as

$$B_{ab} = \left(\nabla^c \nabla^d + \frac{1}{2} R^{cd}\right) C_{acbd}, \quad (3)$$

which is symmetric, traceless, divergence-free, and conformally rescaled, i.e.,

$$B_{ab} = B_{ba}, \qquad B_{ab} g^{ab} = 0, \qquad B_{ab;c} g^{bc} = 0,$$
$$\tilde{g}_{ab} = \Omega^2 g_{ab} \Rightarrow \tilde{B}_{ab} = \Omega^{-2} B_{ab}. \quad (4)$$

The field equations trace yields immediately the condition for the scalar curvature, namely

$$R = 6\mathfrak{b}\mathsf{k}\Box R + 4\Lambda. \quad (5)$$

To employ the Newman-Penrose-like approach to the discussion of admissible gravitational fields in quadratic gravity we separate the Ricci tensor contribution in (2). Substituting the Bach tensor (3) and grouping all terms with the Ricci tensor we thus get

$$\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b} R\right) R_{ab} - 2\mathfrak{a} R^{cd} C_{acbd} + Z_{ab} = 0, \quad (6)$$

where Z_{ab} is a shorthand for

$$Z_{ab} = -\frac{1}{\mathsf{k}}\left(\frac{1}{2} R g_{ab} - \Lambda g_{ab}\right) - 4\mathfrak{a} \nabla^c \nabla^d C_{acbd} - 2\mathfrak{b}\left(\frac{1}{4} R g_{ab} - g_{ab}\Box + \nabla_a \nabla_b\right) R. \quad (7)$$

B. The Newman-Penrose quantities

To set up the notation and fix the conventions, we summarize essential definitions of the Newman-Penrose formalism.[1] Subsequently, the geometric constraints including the commutation relations, the Ricci and Bianchi identities are listed in Appendix A. Let us introduce the null orthonormal frame $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$, where $\boldsymbol{k}, \boldsymbol{l}$ are real null vectors and $\boldsymbol{m}$, $\bar{\boldsymbol{m}}$ are complex null vectors, respectively. They are normalized as

$$\boldsymbol{k} \cdot \boldsymbol{l} = -1, \qquad \boldsymbol{m} \cdot \bar{\boldsymbol{m}} = 1, \quad (8)$$

with other combinations being vanishing. The metric thus becomes

$$g_{ab} = -2k_{(a} l_{b)} + 2m_{(a} \bar{m}_{b)}. \quad (9)$$

Freedom in a choice of the frame is given by the Lorentz transformations, namely:

(i) boost in the plane of null vectors $\boldsymbol{k}$ and $\boldsymbol{l}$ with a positive parameter A:

$$k^a \mapsto A k^a, \qquad l^a \mapsto A^{-1} l^a, \qquad m^a \mapsto m^a, \quad (10)$$

(ii) rotation in the transverse space of vectors $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ encoded in a real parameter Θ:

$$k^a \mapsto k^a, \qquad l^a \mapsto l^a, \qquad m^a \mapsto e^{\mathrm{i}\Theta} m^a, \quad (11)$$

(iii) null rotation with $\boldsymbol{k}$ fixed given by a complex parameter B:

$$k^a \mapsto k^a, \qquad m^a \mapsto m^a + B k^a,$$
$$l^a \mapsto l^a + \bar{B} m^a + B \bar{m}^a + |B|^2 k^a, \quad (12)$$

(iv) null rotation with $\boldsymbol{l}$ fixed given by a complex parameter E:

$$l^a \mapsto l^a, \qquad m^a \mapsto m^a + E l^a,$$
$$k^a \mapsto k^a + \bar{E} m^a + E \bar{m}^a + |E|^2 l^a. \quad (13)$$

[1] Here we follow the notation of classic book [22] while its relation to other common sources [23,24] is summarized in Appendix B.

We will say that a quantity q has a boost weight (b.w.) b if it transforms under the boost (10) as

$$q \mapsto A^b q. \tag{14}$$

The covariant derivative components in the frame vector directions are denoted by

$$\mathrm{D} = k^a \nabla_a, \qquad \Delta = l^a \nabla_a, \qquad \delta = m^a \nabla_a, \qquad \bar{\delta} = \bar{m}^a \nabla_a. \tag{15}$$

To characterize the above derivatives action on the frame vectors we define the spin coefficients as

$$\begin{aligned}
\kappa &= -k_{a;b} m^a k^b, & \nu &= l_{a;b} \bar{m}^a l^b, & \epsilon &= \frac{1}{2}(m_{a;b} \bar{m}^a k^b - k_{a;b} l^a k^b),\\
\rho &= -k_{a;b} m^a \bar{m}^b, & \mu &= l_{a;b} \bar{m}^a m^b, & \beta &= \frac{1}{2}(m_{a;b} \bar{m}^a m^b - k_{a;b} l^a m^b),\\
\sigma &= -k_{a;b} m^a m^b, & \lambda &= l_{a;b} \bar{m}^a \bar{m}^b, & \gamma &= \frac{1}{2}(l_{a;b} k^a l^b - \bar{m}_{a;b} m^a l^b),\\
\tau &= -k_{a;b} m^a l^b, & \pi &= l_{a;b} \bar{m}^a k^b, & \alpha &= \frac{1}{2}(l_{a;b} k^a \bar{m}^b - \bar{m}_{a;b} m^a \bar{m}^b).
\end{aligned} \tag{16}$$

The Weyl tensor null tetrad independent components are

$$\begin{aligned}
\Psi_0 &= C_{abcd} k^a m^b k^c m^d,\\
\Psi_1 &= C_{abcd} k^a l^b k^c m^d,\\
\Psi_2 &= C_{abcd} k^a m^b \bar{m}^c l^d = \frac{1}{2} C_{abcd} k^a l^b (k^c l^d - m^c \bar{m}^d),\\
\Psi_3 &= C_{abcd} l^a k^b l^c \bar{m}^d,\\
\Psi_4 &= C_{abcd} l^a \bar{m}^b l^c \bar{m}^d,
\end{aligned} \tag{17}$$

and the projections of the Ricci tensor (or equivalently its traceless part $S_{ab} = R_{ab} - \frac{1}{4} R g_{ab}$) can be introduced as

$$\begin{aligned}
\Phi_{00} &= \frac{1}{2} R_{ab} k^a k^b, & & & &\\
\Phi_{01} &= \frac{1}{2} R_{ab} k^a m^b, & \Phi_{10} &= \frac{1}{2} R_{ab} k^a \bar{m}^b, & &\\
\Phi_{11} &= \frac{1}{4} R_{ab} (k^a l^b + m^a \bar{m}^b), & \Phi_{02} &= \frac{1}{2} R_{ab} m^a m^b, & \Phi_{20} &= \frac{1}{2} R_{ab} \bar{m}^a \bar{m}^b,\\
\Phi_{12} &= \frac{1}{2} R_{ab} l^a m^b, & \Phi_{21} &= \frac{1}{2} R_{ab} l^a \bar{m}^b, & &\\
\Phi_{22} &= \frac{1}{2} R_{ab} l^a l^b, & & & &
\end{aligned} \tag{18}$$

with the trace $R = 2R_{ab}(m^a \bar{m}^b - k^a l^b)$ which implies

$$R_{ab} k^a l^b = -\frac{1}{4} R + 2\Phi_{11}, \qquad R_{ab} m^a \bar{m}^b = \frac{1}{4} R + 2\Phi_{11}. \tag{19}$$

II. QUADRATIC GRAVITY CONSTRAINTS

The quadratic gravity field equations (6), expressed in terms of the null frame $\{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m},\bar{\boldsymbol{m}}\}$, take the form

$$0 = -4\mathfrak{a}[\Phi_{20}\Psi_0 + \Phi_{02}\bar{\Psi}_0 - 2\Phi_{10}\Psi_1 - 2\Phi_{01}\bar{\Psi}_1 + \Phi_{00}(\Psi_2 + \bar{\Psi}_2)] + 2\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\Phi_{00} + Z_{(0)(0)}, \tag{20}$$

$$0 = -4\mathfrak{a}[\Phi_{21}\Psi_1 + \Phi_{12}\bar{\Psi}_1 - 2\Phi_{11}(\Psi_2 + \bar{\Psi}_2) + \Phi_{01}\Psi_3 + \Phi_{10}\bar{\Psi}_3] + \left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\left(2\Phi_{11} - \frac{R}{4}\right) + Z_{(0)(1)}, \tag{21}$$

$$0 = -4\mathfrak{a}[\Phi_{21}\Psi_0 - 2\Phi_{11}\Psi_1 + \Phi_{02}\bar{\Psi}_1 + \Phi_{01}(\Psi_2 - 2\bar{\Psi}_2) + \Phi_{00}\bar{\Psi}_3] + 2\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\Phi_{01} + Z_{(0)(2)}, \tag{22}$$

$$0 = -4\mathfrak{a}[\Phi_{22}(\Psi_2 + \bar{\Psi}_2) - 2\Phi_{12}\Psi_3 - 2\Phi_{21}\bar{\Psi}_3 + \Phi_{02}\Psi_4 + \Phi_{20}\bar{\Psi}_4] + 2\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\Phi_{22} + Z_{(1)(1)}, \tag{23}$$

$$0 = -4\mathfrak{a}[\Phi_{22}\Psi_1 - \Phi_{12}(2\Psi_2 - \bar{\Psi}_2) + \Phi_{02}\Psi_3 - 2\Phi_{11}\bar{\Psi}_3 + \Phi_{10}\bar{\Psi}_4] + 2\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\Phi_{12} + Z_{(1)(2)}, \tag{24}$$

$$0 = -4\mathfrak{a}[\Phi_{22}\Psi_0 - 2\Phi_{12}\Psi_1 + \Phi_{02}(\Psi_2 + \bar{\Psi}_2) - 2\Phi_{01}\bar{\Psi}_3 + \Phi_{00}\bar{\Psi}_4] + 2\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\Phi_{02} + Z_{(2)(2)}, \tag{25}$$

$$0 = -4\mathfrak{a}[\Phi_{21}\Psi_1 + \Phi_{12}\bar{\Psi}_1 - 2\Phi_{11}(\Psi_2 + \bar{\Psi}_2) + \Phi_{01}\Psi_3 + \Phi_{10}\bar{\Psi}_3] + \left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\left(2\Phi_{11} + \frac{R}{4}\right) + Z_{(2)(3)}, \tag{26}$$

where components of the Weyl and Ricci tensors are defined by (17) and (18), respectively. The symbols $Z_{(c)(d)} = Z_{ab}e_{(c)}^{\ a}e_{(d)}^{\ b}$ stand for the frame components of Z_{ab} given by (7), e.g., $Z_{(0)(0)} = Z_{ab}k^ak^b$ and $Z_{(1)(2)} = Z_{ab}l^am^b$ etc. In principle, the above system of equations can be understood as algebraic constraints on the traceless Ricci tensor components which have to be further combined with the geometric conditions[2] listed in Appendix A.

Finally, to be fully explicit we express all relevant projections of the Z_{ab} tensor, i.e.,

$$Z_{(0)(0)} = -4\mathfrak{a}B^Z_{(0)(0)} + 2\mathfrak{b}[(\epsilon + \bar{\epsilon})\mathrm{D}R - \mathrm{D}\mathrm{D}R - \bar{\kappa}\delta R - \kappa\bar{\delta}R], \tag{27}$$

$$Z_{(0)(1)} = -4\mathfrak{a}B^Z_{(0)(1)} + \frac{1}{2\mathsf{k}}(R - 2\Lambda) + 2\mathfrak{b}\Big[\frac{1}{4}R^2 - (\gamma + \bar{\gamma} - \mu - \bar{\mu})\mathrm{D}R - (\rho + \bar{\rho})\Delta R + \Delta\mathrm{D}R + (\alpha - \bar{\beta} + \bar{\tau})\delta R - \delta\bar{\delta}R + (\bar{\alpha} - \beta + \tau)\bar{\delta}R - \bar{\delta}\delta R\Big], \tag{28}$$

$$Z_{(0)(2)} = -4\mathfrak{a}B^Z_{(0)(2)} + 2\mathfrak{b}[\bar{\pi}\mathrm{D}R - \mathrm{D}\delta R - \kappa\Delta R + (\epsilon - \bar{\epsilon})\delta R], \tag{29}$$

$$Z_{(1)(1)} = -4\mathfrak{a}B^Z_{(1)(1)} + 2\mathfrak{b}[-(\gamma + \bar{\gamma})\Delta R - \Delta\Delta R + \nu\delta R + \bar{\nu}\,\bar{\delta}\,R], \tag{30}$$

$$Z_{(1)(2)} = -4\mathfrak{a}B^Z_{(1)(2)} + 2\mathfrak{b}[\bar{\nu}\mathrm{D}R - \tau\Delta R - \Delta\delta R + (\gamma - \bar{\gamma})\delta R], \tag{31}$$

$$Z_{(2)(2)} = -4\mathfrak{a}B^Z_{(2)(2)} + 2\mathfrak{b}[\bar{\lambda}\mathrm{D}R - \sigma\Delta R - (\bar{\alpha} - \beta)\delta R - \delta\delta R], \tag{32}$$

$$Z_{(2)(3)} = -4\mathfrak{a}B^Z_{(2)(3)} - \frac{1}{2\mathsf{k}}(R - 2\Lambda) + 2\mathfrak{b}\Big[-\frac{1}{4}R^2 + (\gamma + \bar{\gamma} - \bar{\mu})\mathrm{D}R - \mathrm{D}\Delta R + (\rho - \epsilon - \bar{\epsilon})\Delta R - \Delta\mathrm{D}R - (\alpha - \bar{\beta} - \pi + \bar{\tau})\delta R + (\bar{\pi} - \tau)\bar{\delta}R + \bar{\delta}\delta R\Big], \tag{33}$$

[2]In fact, the same approach is applied in the context of vacuum Einstein's general relativity, where the Ricci tensor components are also directly restricted by the field equations. However, in such a case ($\mathfrak{a} = 0 = \mathfrak{b}$) the constraints are very simple with all components (18) vanishing and $R = 4\Lambda$.

where $B^Z_{(c)(d)} = B^Z_{ab} e^a_{(c)} e^b_{(d)}$ represents the Ricci-independent part of the Bach tensor (3) corresponding to the second covariant derivative of the Weyl tensor, namely

$$B^Z_{ab} = \nabla^c \nabla^d C_{acbd}. \tag{34}$$

The frame components become

$$\begin{aligned} B^Z_{(0)(0)} = {} & \bar{\delta}\,\bar{\delta}\,\Psi_0 - \mathrm{D}\bar{\delta}\Psi_1 - \bar{\delta}\mathrm{D}\Psi_1 + \mathrm{DD}\Psi_2 + \lambda\mathrm{D}\Psi_0 + \bar{\sigma}\Delta\Psi_0 + (2\pi - 7\alpha - \bar{\beta})\bar{\delta}\Psi_0 \\ & + (5\alpha + \bar{\beta} - 3\pi)\mathrm{D}\Psi_1 - \bar{\kappa}\Delta\Psi_1 - \bar{\sigma}\delta\Psi_1 + (3\epsilon + \bar{\epsilon} + 7\rho)\bar{\delta}\Psi_1 - (\epsilon + \bar{\epsilon} + 6\rho)\mathrm{D}\Psi_2 + \bar{\kappa}\delta\Psi_2 - 5\kappa\bar{\delta}\Psi_2 + 4\kappa\mathrm{D}\Psi_3 \\ & + \Psi_0[\bar{\kappa}\nu + 4\alpha(3\alpha + \bar{\beta}) - (\epsilon + \bar{\epsilon} + 3\rho)\lambda + \pi(\pi - 7\alpha - \bar{\beta}) + \bar{\sigma}(\mu - 4\gamma) + \mathrm{D}\lambda - 4\bar{\delta}\alpha + \bar{\delta}\pi] \\ & + 2\Psi_1[2\kappa\lambda + \bar{\kappa}(\gamma - \mu) + \rho(5\pi - 9\alpha - 2\bar{\beta}) + \bar{\sigma}(\beta + 2\tau) + \epsilon(2\pi - 4\alpha - \bar{\beta}) + \bar{\epsilon}(\pi - \alpha) + \mathrm{D}\alpha - \mathrm{D}\pi + \bar{\delta}\epsilon + 2\bar{\delta}\rho] \\ & + 3\Psi_2[\kappa(3\alpha + \bar{\beta} - 3\pi) - \bar{\kappa}\tau + \rho(\epsilon + \bar{\epsilon} + 3\rho) - \sigma\bar{\sigma} - \mathrm{D}\rho - \bar{\delta}\kappa] + 2\Psi_3[\kappa(\epsilon - \bar{\epsilon} - 5\rho) + \bar{\kappa}\sigma + \mathrm{D}\kappa] + 2\Psi_4\kappa^2 + \text{c.c.}, \end{aligned} \tag{35}$$

$$\begin{aligned} B^Z_{(0)(1)} = {} & \bar{\delta}\Delta\Psi_1 - \mathrm{D}\Delta\Psi_2 - \bar{\delta}\delta\Psi_2 + \mathrm{D}\delta\Psi_3 - \lambda\Delta\Psi_0 - \nu\bar{\delta}\Psi_0 + 2\nu\mathrm{D}\Psi_1 + (2\pi - \alpha + \bar{\beta})\Delta\Psi_1 + \lambda\delta\Psi_1 \\ & + (2\mu - \bar{\mu} - 2\gamma)\bar{\delta}\Psi_1 + (\bar{\mu} - 3\mu)\mathrm{D}\Psi_2 + (2\rho - \epsilon - \bar{\epsilon})\Delta\Psi_2 + (\alpha - \bar{\beta} - 2\pi)\delta\Psi_2 + (\bar{\pi} + 3\tau)\bar{\delta}\Psi_2 \\ & + (2\beta - \bar{\pi} - 2\tau)\mathrm{D}\Psi_3 - \kappa\Delta\Psi_3 + (\epsilon + \bar{\epsilon} - 2\rho)\delta\Psi_3 - 2\sigma\bar{\delta}\Psi_3 + \sigma\mathrm{D}\Psi_4 + \kappa\delta\Psi_4 + \Psi_0[\lambda(4\gamma - \mu + \bar{\mu}) + \nu(\alpha - \bar{\beta} - 2\pi) - \bar{\delta}\nu] \\ & + 2\Psi_1[\gamma(\alpha - \bar{\beta} - 2\pi) - \lambda(\beta + \bar{\pi} + 2\tau) + \mu(\bar{\beta} - \alpha + 2\pi) + \bar{\mu}(\alpha - \pi) + \nu(\epsilon + \bar{\epsilon} - 2\rho) + \mathrm{D}\nu - \bar{\delta}\gamma + \bar{\delta}\mu] \\ & + 3\Psi_2[\kappa\nu + \mu(2\rho - \epsilon - \bar{\epsilon}) - \bar{\mu}\rho + \pi\bar{\pi} + \lambda\sigma + \tau(2\pi - \alpha + \bar{\beta}) - \mathrm{D}\mu + \bar{\delta}\tau] + 2\Psi_3[\kappa(\bar{\mu} - 2\mu - \gamma) \\ & + \epsilon(\beta - \tau - \bar{\pi}) + \bar{\epsilon}(\beta - \tau) + \rho(\bar{\pi} - 2\beta + 2\tau) + \sigma(\alpha - \bar{\beta} - 2\pi) + \mathrm{D}\beta - \mathrm{D}\tau - \bar{\delta}\sigma] \\ & + \Psi_4[\kappa(4\beta - \bar{\pi} - \tau) + \sigma(\epsilon + \bar{\epsilon} - 2\rho) + \mathrm{D}\sigma] + \text{c.c.}, \end{aligned} \tag{36}$$

$$\begin{aligned} B^Z_{(0)(2)} = {} & \bar{\delta}\Delta\Psi_0 - \mathrm{D}\Delta\Psi_1 - \bar{\delta}\delta\Psi_1 + \mathrm{D}\delta\Psi_2 + \nu\mathrm{D}\Psi_0 + (\pi - 3\alpha + \bar{\beta})\Delta\Psi_0 + (\mu - \bar{\mu} - 4\gamma)\bar{\delta}\Psi_0 \\ & + (2\gamma - 2\mu + \bar{\mu})\mathrm{D}\Psi_1 + (\epsilon - \bar{\epsilon} + 3\rho)\Delta\Psi_1 + (3\alpha - \bar{\beta} - \pi)\delta\Psi_1 + (2\beta + \bar{\pi} + 4\tau)\bar{\delta}\Psi_1 \\ & - (\bar{\pi} + 3\tau)\mathrm{D}\Psi_2 - 2\kappa\Delta\Psi_2 - (\epsilon - \bar{\epsilon} + 3\rho)\delta\Psi_2 - 3\sigma\bar{\delta}\Psi_2 + 2\sigma\mathrm{D}\Psi_3 + 2\kappa\delta\Psi_3 \\ & + \Psi_0[(4\gamma - \mu)(3\alpha - \bar{\beta} - \pi) + \bar{\mu}(4\alpha - \pi) + \nu(\bar{\epsilon} - \epsilon - 3\rho) - \lambda\bar{\pi} + \mathrm{D}\nu - 4\bar{\delta}\gamma + \bar{\delta}\mu] \\ & + 2\Psi_1[2\kappa\nu + (\mu - \gamma)(\epsilon - \bar{\epsilon} + 3\rho) - \bar{\mu}(2\rho + \epsilon) + (\beta + 2\tau)(\pi - 3\alpha + \bar{\beta}) + \bar{\pi}(\pi - \alpha) \\ & + \mathrm{D}\gamma - \mathrm{D}\mu + \bar{\delta}\beta + 2\bar{\delta}\tau] + 3\Psi_2[\kappa(\bar{\mu} - 2\mu) + \bar{\pi}\rho + \sigma(3\alpha - \bar{\beta} - \pi) + \tau(\epsilon - \bar{\epsilon} + 3\rho) - \mathrm{D}\tau - \bar{\delta}\sigma] \\ & + 2\Psi_3[\kappa(2\beta - \bar{\pi} - 2\tau) + \sigma(\bar{\epsilon} - \epsilon - 3\rho) + \mathrm{D}\sigma] + 2\Psi_4\kappa\sigma + \delta\delta\bar{\Psi}_1 - \delta\mathrm{D}\bar{\Psi}_2 - \mathrm{D}\delta\bar{\Psi}_2 + \mathrm{DD}\bar{\Psi}_3 \\ & - 2\bar{\lambda}\delta\bar{\Psi}_0 + 3\bar{\lambda}\mathrm{D}\bar{\Psi}_1 + \sigma\Delta\bar{\Psi}_1 + (4\bar{\pi} - 3\bar{\alpha} - \beta)\delta\bar{\Psi}_1 + (\bar{\alpha} + \beta - 5\bar{\pi})\mathrm{D}\bar{\Psi}_2 - \kappa\Delta\bar{\Psi}_2 + (\epsilon - \bar{\epsilon} + 5\bar{\rho})\delta\bar{\Psi}_2 - \sigma\bar{\delta}\bar{\Psi}_2 \\ & + (3\bar{\epsilon} - \epsilon - 4\bar{\rho})\mathrm{D}\bar{\Psi}_3 - 3\bar{\kappa}\delta\bar{\Psi}_3 + \kappa\bar{\delta}\bar{\Psi}_3 + 2\bar{\kappa}\mathrm{D}\bar{\Psi}_4 + \bar{\Psi}_0[\bar{\lambda}(5\bar{\alpha} + \beta - 3\bar{\pi}) - \bar{\nu}\sigma - \delta\bar{\lambda}] \\ & + 2\bar{\Psi}_1[\kappa\bar{\nu} + \bar{\alpha}(\bar{\alpha} + \beta) + \bar{\pi}(2\bar{\pi} - 3\bar{\alpha} - \beta) - \bar{\lambda}(4\bar{\rho} + \epsilon) + \sigma(\bar{\mu} - \bar{\gamma}) + \mathrm{D}\bar{\lambda} - \delta\bar{\alpha} + \delta\bar{\pi}] \\ & + 3\bar{\Psi}_2[2\bar{\kappa}\,\bar{\lambda} - \kappa\bar{\mu} + \bar{\pi}(\epsilon - \bar{\epsilon}) + \bar{\rho}(4\bar{\pi} - \bar{\alpha} - \beta) + \sigma\bar{\tau} - \mathrm{D}\bar{\pi} + \delta\bar{\rho}] + 2\bar{\Psi}_3[\kappa(\bar{\beta} - \bar{\tau}) \\ & + \bar{\kappa}(\beta - 4\bar{\pi}) - \sigma\bar{\sigma} + (\bar{\rho} - \bar{\epsilon})(\epsilon - \bar{\epsilon} + 2\bar{\rho}) + \mathrm{D}\bar{\epsilon} - \mathrm{D}\bar{\rho} - \delta\bar{\kappa}] + \bar{\Psi}_4[\bar{\kappa}(5\bar{\epsilon} - \epsilon - 3\bar{\rho}) + \kappa\bar{\sigma} + \mathrm{D}\bar{\kappa}], \end{aligned} \tag{37}$$

$$
\begin{aligned}
B^{Z}_{(1)(2)} =\ & \Delta\Delta\Psi_1 - \Delta\delta\Psi_2 - \delta\Delta\Psi_2 + \delta\delta\Psi_3 - 2\nu\Delta\Psi_0 + (4\mu - 3\gamma + \bar{\gamma})\Delta\Psi_1 + 3\nu\delta\Psi_1 - \bar{\nu}\,\bar{\delta}\,\Psi_1 + \bar{\nu}\mathrm{D}\Psi_2 + (5\tau - \bar{\alpha} - \beta)\Delta\Psi_2 \\
& + (\gamma - \bar{\gamma} - 5\mu)\delta\Psi_2 + \bar{\lambda}\,\bar{\delta}\,\Psi_2 - \bar{\lambda}\mathrm{D}\Psi_3 - 3\sigma\Delta\Psi_3 + (\bar{\alpha} + 3\beta - 4\tau)\delta\Psi_3 + 2\sigma\delta\Psi_4 + \Psi_0[\nu(5\gamma - \bar{\gamma} - 3\mu) + \lambda\bar{\nu} - \Delta\nu] \\
& + 2\Psi_1[\nu(\bar{\alpha} - 4\tau) + \bar{\nu}(\alpha - \pi) - \lambda\bar{\lambda} + (\gamma - \mu)(\gamma - \bar{\gamma} - 2\mu) - \Delta\gamma + \Delta\mu + \delta\nu] + 3\Psi_2[\mu(4\tau - \bar{\alpha} - \beta) \\
& + \bar{\lambda}\pi - \bar{\nu}\rho + 2\nu\sigma + \tau(\bar{\gamma} - \gamma) + \Delta\tau - \delta\mu] + 2\Psi_3[\kappa\bar{\nu} - \sigma(\bar{\gamma} + 4\mu) + \tau(2\tau - \bar{\alpha} - 3\beta) + \beta(\bar{\alpha} + \beta) \\
& + \bar{\lambda}(\rho - \epsilon) - \Delta\sigma + \delta\beta - \delta\tau] + \Psi_4[-\kappa\bar{\lambda} + \sigma(\bar{\alpha} + 5\beta - 3\tau) + \delta\sigma] - \Delta\mathrm{D}\bar{\Psi}_3 + \Delta\delta\bar{\Psi}_2 + \bar{\delta}\mathrm{D}\bar{\Psi}_4 - \bar{\delta}\delta\bar{\Psi}_3 \\
& - 2\bar{\lambda}\Delta\bar{\Psi}_1 - 2\bar{\nu}\delta\bar{\Psi}_1 + 2\bar{\nu}\mathrm{D}\bar{\Psi}_2 + (3\bar{\pi} + \tau)\Delta\bar{\Psi}_2 + (\bar{\gamma} - \gamma + 3\bar{\mu})\delta\bar{\Psi}_2 + 3\bar{\lambda}\,\bar{\delta}\,\bar{\Psi}_2 + (\gamma - \bar{\gamma} - 3\bar{\mu})\mathrm{D}\bar{\Psi}_3 \\
& + (2\bar{\rho} - \rho - 2\bar{\epsilon})\Delta\bar{\Psi}_3 + (\alpha - 3\bar{\beta} + \bar{\tau})\delta\bar{\Psi}_3 - (2\bar{\alpha} + 4\bar{\pi} + \tau)\bar{\delta}\bar{\Psi}_3 + (3\bar{\beta} - \alpha - \bar{\tau})\mathrm{D}\bar{\Psi}_4 - \bar{\kappa}\Delta\bar{\Psi}_4 + (4\bar{\epsilon} + \rho - \bar{\rho})\bar{\delta}\bar{\Psi}_4 \\
& + 2\bar{\Psi}_0\bar{\lambda}\,\bar{\nu} + 2\bar{\Psi}_1[\bar{\lambda}(\gamma - \bar{\gamma} - 3\bar{\mu}) + \bar{\nu}(2\bar{\alpha} - 2\bar{\pi} - \tau) - \Delta\bar{\lambda}] + 3\bar{\Psi}_2[\bar{\lambda}(3\bar{\beta} - \bar{\tau} - \alpha) + \bar{\pi}(3\bar{\mu} - \gamma + \bar{\gamma}) + \bar{\nu}(\rho - 2\bar{\rho}) \\
& + \bar{\mu}\tau + \Delta\bar{\pi} + \bar{\delta}\,\bar{\lambda}] + 2\bar{\Psi}_3[2\bar{\kappa}\,\bar{\nu} + (\bar{\epsilon} - \bar{\rho})(\gamma - \bar{\gamma} - 3\bar{\mu}) - \rho(\bar{\gamma} + 2\bar{\mu}) + \tau(\bar{\tau} - \bar{\beta}) + (\bar{\alpha} + 2\bar{\pi})(\alpha - 3\bar{\beta} + \bar{\tau}) \\
& - \Delta\bar{\epsilon} + \Delta\bar{\rho} - \bar{\delta}\,\bar{\alpha} - 2\bar{\delta}\,\bar{\pi}] + \bar{\Psi}_4[\bar{\kappa}(\gamma - \bar{\gamma} - 3\bar{\mu}) + \rho(4\bar{\beta} - \bar{\tau}) + \bar{\rho}(\alpha - 3\bar{\beta} + \bar{\tau}) + 4\bar{\epsilon}(3\bar{\beta} - \bar{\tau} - \alpha) - \bar{\sigma}\tau - \Delta\bar{\kappa} + 4\bar{\delta}\,\bar{\epsilon} - \bar{\delta}\,\bar{\rho}],
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
B^{Z}_{(2)(2)} =\ & \Delta\Delta\Psi_0 - \Delta\delta\Psi_1 - \delta\Delta\Psi_1 + \delta\delta\Psi_2 + (2\mu - 7\gamma + \bar{\gamma})\Delta\Psi_0 + \nu\delta\Psi_0 - \bar{\nu}\,\bar{\delta}\,\Psi_0 + \bar{\nu}\mathrm{D}\Psi_1 + (7\tau - \bar{\alpha} + 3\beta)\Delta\Psi_1 \\
& + (5\gamma - \bar{\gamma} - 3\mu)\delta\Psi_1 + \bar{\lambda}\,\bar{\delta}\,\Psi_1 - \bar{\lambda}\mathrm{D}\Psi_2 - 5\sigma\Delta\Psi_2 + (\bar{\alpha} - \beta - 6\tau)\delta\Psi_2 + 4\sigma\delta\Psi_3 + \Psi_0[\mu(\mu - 7\gamma + \bar{\gamma}) \\
& + \nu(\bar{\alpha} - \beta - 3\tau) + \bar{\nu}(4\alpha - \pi) + 4\gamma(3\gamma - \bar{\gamma}) - \lambda\bar{\lambda} - 4\Delta\gamma + \Delta\mu + \delta\nu] + 2\Psi_1[2\nu\sigma - \bar{\nu}(\epsilon + 2\rho) \\
& + \bar{\lambda}(\pi - \alpha) + (\bar{\gamma} - 2\gamma)(\beta + 2\tau) + (\mu - \gamma)(5\tau - \bar{\alpha} + 2\beta) + \Delta\beta + 2\Delta\tau + \delta\gamma - \delta\mu] + 3\Psi_2[\kappa\bar{\nu} + \bar{\lambda}\rho \\
& + \sigma(3\gamma - \bar{\gamma} - 3\mu) + \tau(3\tau - \bar{\alpha} + \beta) - \Delta\sigma - \delta\tau] + 2\Psi_3[-\kappa\bar{\lambda} + \sigma(\bar{\alpha} + \beta - 5\tau) + \delta\sigma] + 2\Psi_4\sigma^2 \\
& + \mathrm{D}\mathrm{D}\bar{\Psi}_4 - \mathrm{D}\delta\bar{\Psi}_3 - \delta\mathrm{D}\bar{\Psi}_3 + \delta\delta\bar{\Psi}_2 - 4\bar{\lambda}\delta\bar{\Psi}_1 + 5\bar{\lambda}\mathrm{D}\bar{\Psi}_2 + \sigma\Delta\bar{\Psi}_2 + (\bar{\alpha} - \beta + 6\bar{\pi})\delta\bar{\Psi}_2 \\
& + (\beta - 3\bar{\alpha} - 7\bar{\pi})\mathrm{D}\bar{\Psi}_3 - \kappa\Delta\bar{\Psi}_3 + (\epsilon - 5\bar{\epsilon} + 3\bar{\rho})\delta\bar{\Psi}_3 - \sigma\bar{\delta}\bar{\Psi}_3 + (7\bar{\epsilon} - \epsilon - 2\bar{\rho})\mathrm{D}\bar{\Psi}_4 - \bar{\kappa}\delta\bar{\Psi}_4 + \kappa\bar{\delta}\bar{\Psi}_4 \\
& + 2\bar{\Psi}_0\bar{\lambda}^2 + 2\bar{\Psi}_1[\bar{\lambda}(\bar{\alpha} + \beta - 5\bar{\pi}) - \bar{\nu}\sigma - \delta\bar{\lambda}] + 3\bar{\Psi}_2[\kappa\bar{\nu} + \bar{\lambda}(3\bar{\epsilon} - \epsilon - 3\bar{\rho}) + \bar{\mu}\sigma + \bar{\pi}(\bar{\alpha} - \beta + 3\bar{\pi}) + \mathrm{D}\bar{\lambda} + \delta\bar{\pi}] \\
& + 2\bar{\Psi}_3[2\bar{\kappa}\,\bar{\lambda} - \kappa(2\bar{\mu} + \bar{\gamma}) + \sigma(\bar{\tau} - \bar{\beta}) + (\bar{\rho} - \bar{\epsilon})(2\bar{\alpha} - \beta + 5\bar{\pi}) + (\epsilon - 2\bar{\epsilon})(2\bar{\pi} + \bar{\alpha}) - \mathrm{D}\bar{\alpha} - 2\mathrm{D}\bar{\pi} - \delta\bar{\epsilon} + \delta\bar{\rho}] \\
& + \bar{\Psi}_4[\kappa(4\bar{\beta} - \bar{\tau}) + \bar{\kappa}(\beta - \bar{\alpha} - 3\bar{\pi}) + (\bar{\rho} - 4\bar{\epsilon})(\epsilon - 3\bar{\epsilon} + \bar{\rho}) - \sigma\bar{\sigma} + 4\mathrm{D}\bar{\epsilon} - \mathrm{D}\bar{\rho} - \delta\bar{\kappa}],
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
B^{Z}_{(1)(1)} =\ & \Delta\Delta\Psi_2 - \Delta\delta\Psi_3 - \delta\Delta\Psi_3 + \delta\delta\Psi_4 - 4\nu\Delta\Psi_1 + (\gamma + \bar{\gamma} + 6\mu)\Delta\Psi_2 + 5\nu\delta\Psi_2 - \bar{\nu}\,\bar{\delta}\,\Psi_2 \\
& + \bar{\nu}\mathrm{D}\Psi_3 + (3\tau - \bar{\alpha} - 5\beta)\Delta\Psi_3 - (3\gamma + \bar{\gamma} + 7\mu)\delta\Psi_3 + \bar{\lambda}\,\bar{\delta}\,\Psi_3 - \bar{\lambda}\mathrm{D}\Psi_4 - \sigma\Delta\Psi_4 \\
& + (\bar{\alpha} + 7\beta - 2\tau)\delta\Psi_4 + 2\Psi_0\nu^2 + 2\Psi_1[\nu(\gamma - \bar{\gamma} - 5\mu) + \lambda\bar{\nu} - \Delta\nu] + 3\Psi_2[\mu(\gamma + \bar{\gamma} + 3\mu) \\
& + \nu(\bar{\alpha} + 3\beta - 3\tau) - \lambda\bar{\lambda} - \bar{\nu}\pi + \Delta\mu + \delta\nu] + 2\Psi_3[\bar{\nu}(\epsilon - \rho) + \bar{\lambda}(\alpha + 2\pi) + \gamma(2\tau - \bar{\alpha} - 4\beta) \\
& + \bar{\gamma}(\tau - \beta) + \mu(5\tau - 2\bar{\alpha} - 9\beta) + 2\nu\sigma - \Delta\beta + \Delta\tau - \delta\gamma - 2\delta\mu] + \Psi_4[\kappa\bar{\nu} + \bar{\lambda}(\rho - 4\epsilon) \\
& - \sigma(\gamma + \bar{\gamma} + 3\mu) + 4\beta(3\beta + \bar{\alpha}) + \tau(\tau - \bar{\alpha} - 7\beta) - \Delta\sigma + 4\delta\beta - \delta\tau] + \mathrm{c.c.},
\end{aligned} \tag{40}
$$

where c.c. denotes the complex conjugation. Backwards, the Bach tensor can be constructed as

$$B_{(0)(0)} = B^{Z}_{(0)(0)} + \Phi_{20}\Psi_0 + \Phi_{02}\bar{\Psi}_0 - 2\Phi_{10}\Psi_1 - 2\Phi_{01}\bar{\Psi}_1 + \Phi_{00}(\Psi_2 + \bar{\Psi}_2), \tag{41}$$

$$B_{(0)(1)} = B^{Z}_{(0)(1)} + \Phi_{21}\Psi_1 + \Phi_{12}\bar{\Psi}_1 - 2\Phi_{11}(\Psi_2 + \bar{\Psi}_2) + \Phi_{01}\Psi_3 + \Phi_{10}\bar{\Psi}_3, \tag{42}$$

$$B_{(0)(2)} = B^{Z}_{(0)(2)} + \Phi_{21}\Psi_0 - 2\Phi_{11}\Psi_1 + \Phi_{01}(\Psi_2 - 2\bar{\Psi}_2) + \Phi_{02}\bar{\Psi}_1 + \Phi_{00}\bar{\Psi}_3, \tag{43}$$

$$B_{(1)(2)} = B^{Z}_{(1)(2)} + \Phi_{22}\Psi_1 - \Phi_{12}(2\Psi_2 - \bar{\Psi}_2) + \Phi_{02}\Psi_3 - 2\Phi_{11}\bar{\Psi}_3 + \Phi_{10}\bar{\Psi}_4, \tag{44}$$

$$B_{(2)(2)} = B^{Z}_{(2)(2)} + \Phi_{22}\Psi_0 - 2\Phi_{12}\Psi_1 + \Phi_{02}(\Psi_2 + \bar{\Psi}_2) - 2\Phi_{01}\bar{\Psi}_3 + \Phi_{00}\bar{\Psi}_4, \tag{45}$$

$$B_{(1)(1)} = B^{Z}_{(1)(1)} + \Phi_{22}(\Psi_2 + \bar{\Psi}_2) - 2\Phi_{12}\Psi_3 - 2\Phi_{21}\bar{\Psi}_3 + \Phi_{02}\Psi_4 + \Phi_{20}\bar{\Psi}_4. \tag{46}$$

Since $\boldsymbol{m}$ is a complex vector, we have, e.g., $\bar{B}_{(0)(2)} = B_{(0)(3)}$, and since the Bach tensor is traceless, it holds that $B_{(0)(1)} = B_{(2)(3)}$, and actually also $B^Z_{(0)(1)} = B^Z_{(2)(3)}$.

III. APPLICATIONS

To illustrate efficiency of the above general approach we analyze particular scenarios corresponding to special algebraic properties of the Ricci and Weyl tensors, respectively. Such assumptions are related to a specific behavior of privileged null geodesic congruence defining the Kundt and/or Robinson-Trautman classes in terms of its twist, shear, and expansion.[3] In these important cases we discuss algebraic structure of the Bach tensor.

A. Restrictions following from a special form of the Ricci tensor

In this section we use NP formalism to prove Propositions 1.2 and 1.1 in [29] in four dimensions. The original proof using higher-dimensional NP formalism was based on the analysis of dominant boost weights, however, here we can proceed fully explicitly.

Let us briefly summarize the algebraic classification of tensors in higher dimensions used in [29] and introduced in [4,30] (see also [6] for a review) in the case of rank-2 symmetric tensors.

In a null frame $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}^{(i)}\}$, employed in Appendix C, a rank-2 symmetric tensor

$$R_{ab} = \overbrace{R_{00} n_a n_b}^{\text{b.w. 2, type G}} + \overbrace{2R_{0i} n_{(a} m^{(i)}_{b)}}^{1,\text{I}} + \overbrace{2R_{01} n_{(a}\ell_{b)} + R_{ij} m^{(i)}_a m^{(j)}_b}^{0,\text{II,D}} + \overbrace{2R_{1i}\ell_{(a} m^{(i)}_{b)}}^{-1,\text{III}} + \overbrace{R_{11}\ell_a \ell_b}^{-2,\text{N}} \tag{47}$$

has components R_{00}, R_{0i}, (R_{01}, R_{ij}), R_{1i}, and R_{11} of boost weights $+2$, $+1$, 0, -1, and -2 [see (14)].[4] Symbolically, it can be expressed as

$$\boldsymbol{R} = (\boldsymbol{R})_{(+2)} + (\boldsymbol{R})_{(+1)} + (\boldsymbol{R})_{(0)} + (\boldsymbol{R})_{(-1)} + (\boldsymbol{R})_{(-2)}. \tag{48}$$

For various algebraically special classes of rank-2 symmetric tensors, it is possible to transform away components of certain boost weights by choosing an aligned null frame. In generic case this is not possible (type G). If b.w. $+2$ components can be transformed away the tensor is of type I, if b.w. $+2$ and $+1$ components it is of type II etc (see Table I).

[3]By definition, the Kundt family of geometries admits a nontwisting, shear-free, and nonexpanding null geodesic congruence [25,26], while additionally, the Robinson-Trautman class allows a nontrivial expansion [27,28].

[4]In Appendix C, the reader can find boost weights of all Ricci and Weyl components, and Ricci rotation coefficients.

TABLE I. Algebraic types of a rank-2 symmetric tensor.

Algebraic type	Conditions
Type G	General
Type I	$(\boldsymbol{R})_{(+2)} = 0$
Type II	$(\boldsymbol{R})_{(+2)} = (\boldsymbol{R})_{(+1)} = 0$
Type D	Only $(\boldsymbol{R})_{(0)} \neq 0$
Type III	Only $(\boldsymbol{R})_{(-1)}, (\boldsymbol{R})_{(-2)} \neq 0$
Type N	Only $(\boldsymbol{R})_{(-2)} \neq 0$

Similarly, using the frame $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$, a symmetric rank-2 tensor, e.g., the Bach tensor, has components of the following b.w.

b.w.	$+2$	$+1$	0	-1	-2
	$B_{(0)(0)}$	$B_{(0)(2)}$; $B_{(0)(3)}$	$B_{(0)(1)}$; $B_{(2)(3)}$	$B_{(1)(2)}$; $B_{(1)(3)}$	$B_{(1)(1)}$

1. Traceless Ricci type III

Propositions 1.2 in [29] reads:

Proposition 1. A vacuum solution to quadratic gravity (2) with the Ricci tensor of the form

$$R_{ab} = \Lambda g_{ab} + \psi'_i(\ell_a m^{(i)}_b + m^{(i)}_a \ell_b) + \omega' \ell_a \ell_b, \quad \psi'_i \psi'_i \neq 0,$$

[using the frame $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}^{(i)}\}$, see Appendix C] and aligned Weyl tensor of Petrov type II, or more special, is necessarily Kundt.

Proof.—Using the NP formalism notation, $\Psi_0 = \Psi_1 = 0$ and the Ricci tensor is of the form (using the frame $\{k^a, l^a, m^a, \bar{m}^a\}$)

$$R_{ab} = 2\Phi_{22} k_a k_b - 2\Phi_{12}(k_a \bar{m}_b + \bar{m}_a k_b) - 2\Phi_{21}(k_a m_b + m_a k_b) + \Lambda g_{ab}, \tag{49}$$

where $\Lambda = \text{const}$, i.e., $\Phi_{00} = \Phi_{01} = \Phi_{10} = \Phi_{11} = \Phi_{02} = \Phi_{20} = 0$.

Considering the above assumption on the Petrov type II or III, the Bianchi identities imply $\kappa\Psi_2 = 0$ (A23) or $\kappa\Psi_3 = 0$ (A24), respectively. For type III or N and $\Phi_{12} \neq 0$, the Bianchi equations imply $\kappa\Phi_{12} = 0$ (A27), while for $\Phi_{12} = 0$ it follows that $\kappa\Phi_{22} = 0$ (A33). Therefore, in all possible cases we obtain

$$\kappa = 0, \tag{50}$$

and the multiple principal null direction (PND) congruence generated by $\boldsymbol{k}$ is necessary *geodetic*.

Further, let us assume that the congruence is affinely parametrized and the frame is parallelly propagated along this congruence, i.e.,

$$\epsilon = 0, \qquad \pi = 0. \tag{51}$$

Now, it is convenient to discuss specific Petrov types separately:

(i) Type II ($\Psi_0 = \Psi_1 = 0$): the QG field equation (20) simplifies to $-4\mathfrak{a}B^Z_{(0)(0)} = 0$,

$$B^Z_{(0)(0)} = \mathrm{D}\mathrm{D}\Psi_2 - 6\rho\mathrm{D}\Psi_2 + 3\Psi_2(3\rho^2 - \sigma\bar{\sigma} - \mathrm{D}\rho) + \text{c.c.} = 0, \tag{52}$$

and using the Ricci and Bianchi identities (A6) and (A24) for type II it implies

$$3\sigma\bar{\sigma}(\Psi_2 + \bar{\Psi}_2) = 0, \tag{53}$$

and therefore

$$\sigma = 0. \tag{54}$$

Equivalently, it immediately follows from (A27) that $3\sigma\Psi_2 = 0$ and thus $\sigma = 0$.

The field equation (22) reduces to $-4\mathfrak{a}B^Z_{(0)(2)} = 0$ which gives

$$\begin{aligned} B^Z_{(0)(2)} = {}& \mathrm{D}\delta\Psi_2 - 3\tau\mathrm{D}\Psi_2 - 3\rho\delta\Psi_2 - 3\Psi_2(\mathrm{D}\tau - 3\rho\tau) \\ & - \delta\mathrm{D}\bar{\Psi}_2 - \mathrm{D}\delta\bar{\Psi}_2 + \mathrm{D}\mathrm{D}\bar{\Psi}_3 + (\bar{\alpha} + \beta)\mathrm{D}\bar{\Psi}_2 \\ & + 5\bar{\rho}\delta\bar{\Psi}_2 - 4\bar{\rho}\mathrm{D}\bar{\Psi}_3 + 3\bar{\Psi}_2(\delta\bar{\rho} - (\bar{\alpha} + \beta)\bar{\rho}) \\ & + 2\bar{\Psi}_3(2\bar{\rho}^2 - \mathrm{D}\bar{\rho}) = 0. \end{aligned} \tag{55}$$

Using geometric identities (A6), (A7), (A15), (A24), (A25), (A28), and (A33) we obtain

$$-4\rho\bar{\rho}\Phi_{12} = 0, \tag{56}$$

and therefore

$$\rho = 0. \tag{57}$$

The spacetime has to belong necessarily to the *Kundt* class.

(ii) Type III ($\Psi_0 = \Psi_1 = \Psi_2 = 0$): the QG field equation (20) is automatically satisfied, while Eq. (20), namely $-4\mathfrak{a}B^Z_{(0)(2)} = 0$, reads

$$B^Z_{(0)(2)} = 2\sigma\mathrm{D}\Psi_3 + 2\Psi_3(\mathrm{D}\sigma - 3\rho\sigma) + \mathrm{D}\mathrm{D}\bar{\Psi}_3 - 4\bar{\rho}\mathrm{D}\bar{\Psi}_3 + 2\bar{\Psi}_3(2\bar{\rho}^2 - \sigma\bar{\sigma} - \mathrm{D}\bar{\rho}) = 0. \tag{58}$$

Using (A6), (A5), (A25), (A28), (A33), it implies (together with its complex conjugate)

$$(\sigma\bar{\sigma} + \rho\bar{\rho})\Phi_{12} + \sigma\rho\Phi_{21} = 0, \tag{59}$$

$$(\sigma\bar{\sigma} + \rho\bar{\rho})\Phi_{21} + \bar{\sigma}\,\bar{\rho}\,\Phi_{12} = 0. \tag{60}$$

To have $\Phi_{12} \neq 0$, the determinant should be vanishing, i.e.,

$$(\sigma\bar{\sigma})^2 + (\rho\bar{\rho})^2 + \sigma\bar{\sigma}\rho\bar{\rho} = 0. \tag{61}$$

We thus get $\sigma = 0$ and $\rho = 0$ and the spacetime has to be *Kundt*.

(iii) Type N ($\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$): the Bianchi identity (A28) simplifies to $-2\rho\Phi_{12} = 0$ and we immediately get either $\rho = 0$ or $\Phi_{12} = 0$. Taking $\rho = 0$, the Ricci identity (A6), i.e., $D\rho = \rho^2 + \sigma\bar{\sigma}$, implies $\sigma = 0$. In the case $\Phi_{12} = 0$ combined with the Weyl type N, the QG field equations (20) and (22) are automatically satisfied, while Eq. (25) becomes $-4\mathfrak{a}B^Z_{(2)(2)} = 0$, namely

$$B^Z_{(2)(2)} = 2\Psi_4\sigma^2 + \mathrm{D}\mathrm{D}\bar{\Psi}_4 - 2\bar{\rho}\mathrm{D}\bar{\Psi}_4 + \bar{\Psi}_4(\bar{\rho}^2 - \sigma\bar{\sigma} - \mathrm{D}\bar{\rho}) = 0. \tag{62}$$

Using (A6), (A5), (A26), (A29), (A32) it takes the form

$$-4\sigma^2\Psi_4 = 0 \tag{63}$$

and we get $\sigma = 0$. Moreover, a combination of the Bianchi identities (A29) and (A32) leads to $\rho\Phi_{22} = \sigma\Psi_4$ that gives

$$\rho = 0. \tag{64}$$

Therefore, in both cases the resulting spacetime has to be *Kundt*.

■

2. Traceless Ricci type N

Propositions 1.1 in [29] reads:

Proposition 2. A vacuum solution to quadratic gravity (2) with the Ricci tensor of the form

$$R_{ab} = \Lambda g_{ab} + \omega' \ell_a \ell_b, \qquad \omega' \neq 0, \qquad \ell^a \ell_a = 0,$$

and aligned Weyl tensor of any Petrov type is necessarily Kundt.

Proof.—Using the NP formalism, $\Psi_0 = 0$ and the Ricci tensor is of the form (using the frame $\{k^a, l^a, m^a, \bar{m}^a\}$)

$$R_{ab} = 2\Phi_{22}k_a k_b + \Lambda g_{ab}, \tag{65}$$

where $\Lambda = \text{const}$, i.e., $\Phi_{00} = \Phi_{01} = \Phi_{10} = \Phi_{11} = \Phi_{02} = \Phi_{20} = \Phi_{12} = \Phi_{21} = 0$.

To prove this proposition let us begin with the Bianchi identity (A33) which gives $\kappa\Phi_{22} = 0$ and therefore

$$\kappa = 0, \tag{66}$$

and the congruence is *geodetic*. Further, let us assume that the congruence is affinely parametrized and the tetrad is paralelly propagated, i.e.,

$$\epsilon = 0, \qquad \pi = 0. \tag{67}$$

(i) Type I: interestingly, in combination with geometric identities, the QG field equations (20) and (22) are identically satisfied. Equation (21) reduces to $-4\mathfrak{a}B^Z_{(0)(1)} = 0$ and (36) explicitly gives

$$\begin{aligned} B^Z_{(0)(1)} = {} & \bar{\delta}\Delta\Psi_1 - \mathrm{D}\Delta\Psi_2 - \bar{\delta}\delta\Psi_2 + \mathrm{D}\delta\Psi_3 + 2\nu\mathrm{D}\Psi_1 - (\alpha - \bar{\beta})\Delta\Psi_1 + \lambda\delta\Psi_1 - (2\gamma - 2\mu + \bar{\mu})\bar{\delta}\Psi_1 \\ & - (3\mu - \bar{\mu})\mathrm{D}\Psi_2 + 2\rho\Delta\Psi_2 + (\alpha - \bar{\beta})\delta\Psi_2 + 3\tau\bar{\delta}\Psi_2 + (2\beta - 2\tau)\mathrm{D}\Psi_3 - 2\rho\delta\Psi_3 - 2\sigma\bar{\delta}\Psi_3 + \sigma\mathrm{D}\Psi_4 \\ & + \Psi_1[2\gamma(\alpha - \bar{\beta}) - 2\lambda(\beta + 2\tau) + 2\mu(\bar{\beta} - \alpha) + 2\alpha\bar{\mu} - 4\nu\rho + 2\mathrm{D}\nu - 2\bar{\delta}\gamma + 2\bar{\delta}\mu] - 3\Psi_2[\rho(\bar{\mu} - 2\mu) \\ & - \lambda\sigma + \tau(\alpha - \bar{\beta}) + \mathrm{D}\mu - \bar{\delta}\tau] + 2\Psi_3[2\rho(\tau - \beta) + \sigma(\alpha - \bar{\beta}) + \mathrm{D}\beta - \mathrm{D}\tau - \bar{\delta}\sigma] + \Psi_4(\mathrm{D}\sigma - 2\rho\sigma) + \text{c.c.}, \end{aligned} \tag{68}$$

which can be significantly simplified to

$$(\rho\bar{\rho} + \sigma\bar{\sigma})\Phi_{22} = 0. \tag{69}$$

This condition obviously implies

$$\rho = 0, \quad \sigma = 0, \tag{70}$$

and the resulting spacetime has to be necessarily *Kundt*.

(ii) Type II: employing $\Psi_0 = \Psi_1 = 0$ the Bianchi identity (A27) reduces to

$$3\sigma\Psi_2 = 0 \tag{71}$$

and therefore $\sigma = 0$. Alternatively, the QG field equation (20) becomes

$$\begin{aligned} B^Z_{(0)(0)} = {} & \mathrm{D}\mathrm{D}\Psi_2 - 6\rho\mathrm{D}\Psi_2 + 3\Psi_2(3\rho^2 - \sigma\bar{\sigma} - \mathrm{D}\rho) \\ & + \text{c.c.} = 0, \end{aligned} \tag{72}$$

which gives

$$3\sigma\bar{\sigma}(\Psi_2 + \bar{\Psi}_2) = 0, \tag{73}$$

and therefore we get $\sigma = 0$ again. The QG field equations (22) and (25) are identically satisfied. However, the QG field equation (21) implies

$$-\rho\bar{\rho}(\Phi_{22} + \bar{\Phi}_{22}) = 0 \tag{74}$$

and therefore also $\rho = 0$. The spacetime has to be *Kundt*. Since Eq. (74) does not contain Ψ_2 it holds also for more algebraically special Petrov types III and N.

(iii) Type III: the Bianchi identity (A28) implies

$$2\sigma\Psi_3 = 0 \tag{75}$$

and therefore $\sigma = 0$. Employing Eq. (74), which does not contain Ψ_2 and holds also for Petrov type III, we end up with the *Kundt* spacetime.

(iv) Type N: in this case, the last part of the Proposition 1.2 proof (discussing the subcase $\Phi_{12} = 0$) can be used and therefore the spacetime is *Kundt* again. ■

B. The Bach tensor for Robinson-Trautman geometries of specific Weyl type

Due to their geometrical and physical importance, various particular Robinson-Trautman (RT) and Kundt spacetimes have been already studied within quadratic gravity. For example in [29], solutions to quadratic gravity were constructed by a conformal transformation from Kundt seed metrics. Starting with a vanishing Bach tensor, the resulting spacetimes then became Kundt or RT preserving a vanishing Bach tensor. There was also a specific case discussed, where all tensors (Weyl, Ricci, and Bach) were of type N. All spherical solutions discussed in [19,20,31,32] are of Petrov type D and general Ricci and Bach type. Here, the simplifying assumption of constant scalar curvature was employed. Thus the Bach tensor became the only nontrivial higher-curvature correction in the field equations (2) and therefore its properties crucially affected the resulting geometry. Its contribution was directly reflected in analysis of geodesic deviation representing, in principle, observable effects. All these cases are in agreement with Table II.

Here, let us examine possible Bach types for different Petrov types of the RT metrics

$$\begin{aligned} \mathrm{d}s^2 = {} & g_{uu}(u,r,x^k)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + 2g_{ui}(u,r,x^k)\mathrm{d}u\mathrm{d}x^i \\ & + g_{ij}(u,r,x^k)\mathrm{d}x^i\mathrm{d}x^j, \end{aligned} \tag{76}$$

admitting geodetic ($\kappa = 0$) shear-free ($\sigma = 0$), twist-free ($\rho = \bar{\rho}$), and expanding ($\rho \neq 0$) null congruence generated by

TABLE II. Possible Bach types depending on the Petrov type for Robinson-Trautman spacetimes. The privileged RT null vector field $\boldsymbol{k}$ is taken as the Weyl PND.

Petrov type	Vanishing Bach components	Possible Bach types
N	$B_{(0)(0)}$; $B_{(0)(2)}$; $B_{(0)(1)}$	II/III/N/O
III	$B_{(0)(0)}$	I/II/III/N/O
II/D		G/I/II/III/III/N/O

$$\boldsymbol{k} = \partial_r, \tag{77}$$

which is affinely parametrized ($\epsilon + \bar{\epsilon} = 0$). The coordinate r is the affine parameter along the congruence, u labels null hypersurfaces with $\boldsymbol{k}$ tangent (normal), and x^2, x^3 cover the transverse Riemannian 2-space. Moreover, without loss of generality, we employ a parallelly propagated frame, i.e.,

$$\pi = 0, \qquad \epsilon = 0. \tag{78}$$

In what follows, the Ricci equations (A6)–(A10), (A15) will be useful, namely

$$D\rho = \rho^2 + \Phi_{00}, \tag{79}$$

$$D\tau = \rho\tau + \Psi_1 + \Phi_{01}, \tag{80}$$

$$D\alpha = \rho\alpha + \Phi_{10}, \tag{81}$$

$$D\beta = \rho\beta + \Psi_1, \tag{82}$$

$$D\gamma = \alpha\tau + \beta\bar{\tau} + \Psi_2 + \Phi_{11} - \frac{R}{24}, \tag{83}$$

$$\delta\rho = \rho(\bar{\alpha} + \beta) - \Psi_1 + \Phi_{01}. \tag{84}$$

1. Petrov type N

Let us start with the Petrov type N (with $\boldsymbol{k}$ being PND) represented by the Weyl components

$$\Psi_4 \neq 0, \qquad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0. \tag{85}$$

Within this setting, the Bach tensor components (41)–(46) simplify to

$$B_{(0)(0)} = 0, \tag{86}$$

$$B_{(0)(2)} = \bar{B}_{(0)(2)} = B_{(0)(3)} = 0, \tag{87}$$

$$B_{(0)(1)} = B_{(2)(3)} = 0, \tag{88}$$

$$B_{(2)(2)} = DD\bar{\Psi}_4 - 2\rho D\bar{\Psi}_4, \tag{89}$$

$$B_{(1)(2)} = \bar{B}_{(1)(3)} = \bar{\delta}D\bar{\Psi}_4 - (\alpha - 3\bar{\beta} + \bar{\tau})D\bar{\Psi}_4 + \Phi_{10}\bar{\Psi}_4 + \bar{\Psi}_4[\rho(\alpha + \bar{\beta}) - \bar{\delta}\rho], \tag{90}$$

$$B_{(1)(1)} = \delta\delta\Psi_4 - \bar{\lambda}D\Psi_4 + (\bar{\alpha} + 7\beta - 2\tau)\delta\Psi_4 + \Phi_{02}\Psi_4 + \Psi_4[4\beta(\bar{\alpha} + 3\beta) + \bar{\lambda}\rho + \tau(\tau - \bar{\alpha} - 7\beta) + 4\delta\beta - \delta\tau] + \text{c.c.} \tag{91}$$

Since the b.w. *zero* component $B_{(2)(2)}$ is nonvanishing, the Bach tensor is in general of type II. In a special case with $B_{(2)(2)} = 0$, e.g., if $D\Psi_4 = 0$, then the Bach tensor becomes of type III or more special.

2. Petrov type III

For the Petrov type III, with the Weyl components

$$\Psi_3 \neq 0, \qquad \Psi_4 \neq 0, \qquad \Psi_0 = \Psi_1 = \Psi_2 = 0, \tag{92}$$

the non-negative boost-weight components of the Bach tensor (41)–(46) simplify to

$$B_{(0)(0)} = 0, \tag{93}$$

$$B_{(0)(2)} = DD\bar{\Psi}_3 - 4\rho D\bar{\Psi}_3 - \bar{\Psi}_3(\Phi_{00} - 2\rho^2), \tag{94}$$

$$B_{(0)(1)} = D\delta\Psi_3 + 2(\beta - \tau)D\Psi_3 - 2\rho\delta\Psi_3 - \Psi_3[\Phi_{01} + 2\rho(\beta - \tau)] + \text{c.c.}, \tag{95}$$

$$B_{(2)(2)} = DD\bar{\Psi}_4 - D\delta\bar{\Psi}_3 - \delta D\bar{\Psi}_3 + (\beta - 3\bar{\alpha})D\bar{\Psi}_3 + 3\rho\delta\bar{\Psi}_3 - 2\rho D\bar{\Psi}_4 - 2\bar{\Psi}_3(\Phi_{01} - 2\rho\bar{\alpha}), \tag{96}$$

where we use the Ricci equations (79)–(82) and (84). Since the b.w. +1 component $B_{(0)(2)}$ is nonvanishing, the Bach tensor is of type I.

3. Petrov type II/D

For the Petrov type D, defined by

$$\Psi_2 \neq 0, \qquad \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \tag{97}$$

even the highest b.w. +2 component, namely

$$B_{(0)(0)} = DD\Psi_2 - 6\rho D\Psi_2 - 2\Psi_2(\Phi_{00} - 3\rho^2) + \text{c.c.}, \tag{98}$$

is nonvanishing and therefore, the Bach tensor is of general type G.

IV. SUMMARY

In the introduction, we briefly described a formulation of quadratic gravity (1) and a suitable form of its field equations (6), and summarized basic definitions of the Newman-Penrose formalism (see Sec. I). In Sec. II, we immediately proceed to our main result that is reformulation of the quadratic gravity field equations in terms of the NP quantities, see expressions (20)–(26) with the substitution from (27)–(33) and (35)–(40), or Appendix D. Importantly, the traceless Ricci tensor contribution to the field equations is only linear within QG. Therefore, the procedure combining (20)–(26) with the geometric constraints, listed in Appendix A, is similar as in the case of classic general relativity, i.e., we deal with the linear system of algebraic equations for the Ricci tensor frame components. The main aim of these results is to provide a tool for a systematic analysis of (exact) solutions to the quadratic gravity, where the invariant assumptions on the algebraic properties of curvature tensors, or e.g., specific behavior of null geodesics, can be simply made. This should allow one to compare four-dimensional quadratic gravity with other theories of gravity, primarily with Einstein's general relativity, on the level of admitted solutions where the initial ansatz is introduced in terms of purely geometric conditions. Needless to say that such prominent GR concepts as a study of peeling properties of the gravitational field and related results on spacetime asymptotic structure are also formulated in the NP language, and their extension to QG is still an open problem. From the family of exact spacetimes, more involved gravitational waves models possibly belonging to the Robinson-Trautman family, and nonstatic Vaidya-like generalizations of the Schwarzschild geometry, should be studied. The ultimate goal, also naturally defined in terms of the NP quantities, is the investigation of rotating Kerr-like solutions. One can also go beyond exact models and employ the NP approach within a discussion of perturbations to a given background solution, see e.g., [33]. Such Teukolsky-like analysis constrained by the QG field equations is of our recent interest.

In the subsequent Sec. III, we present two simple examples of applicability of the above mentioned general expressions. In particular, its Sec. III A contains explicit calculations proving pair of propositions previously formulated in [29], where the original proofs were based on the highest boost-weights discussion which does not need knowledge of the complete Bach tensor. In the Sec. III B, we analyze possible algebraic structure of the Bach tensor in the case of Robinson-Trautman geometries (76). The Weyl tensor is assumed to be of algebraically special Petrov type with respect to the frame associated with the privileged nontwisting, shear-free, and expanding null geodesic congruence. Under such conditions the admitted structure of the Bach tensor is discussed. These new results are summarized in Table II.

Moreover, the standard geometric Ricci and Bianchi identities of the Newman-Penrose formalism are summarized in Appendix A using unified notation of [22]. For the readers convenience, subsequent Appendix B compares this notation and conventions with those used in other common textbooks [23,24]. Two decades ago the arbitrary-dimensional version of the Newman-Penrose formalism was introduced, and, almost immediately, it became a useful tool with many applications, where the algebraic structure of curvature tensors or specific behavior of null geodesics play crucial role within spacetime analysis, see e.g., [34–39]. Therefore, we present relation of such a real formalism, in the case of four spacetime dimensions, to the classic complex NP quantities used within this paper in Appendix C. Finally, Appendix D presents a fully explicit form of the quadratic gravity field equations, expressed in terms of the null frame quantities, which do not require any additional substitutions.

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APPENDIX A: GEOMETRIC CONSTRAINTS ON THE FRAME COMPONENTS

In Secs. I and II we have introduced frame components of crucial tensor quantities and constraints implied by the quadratic gravity field equations, respectively. In addition, these NP objects have to satisfy conditions directly arising from their purely geometric properties. In particular, we have commutation relations of the frame derivatives, the Ricci identities defining the Riemann tensor, and the Bianchi identities coming from the covariant derivatives of the Riemann tensor. For more details see [22].

1. Commutation relations

Expressing the Lie bracket of all possible combinations of the frame vectors, which are understood as the directional derivatives, and simultaneously, rewriting covariant derivatives in terms of the Ricci rotation coefficients we obtain the commutation relations, namely

$$\Delta \mathrm{D} - \mathrm{D}\Delta = (\gamma + \bar{\gamma})\mathrm{D} + (\epsilon + \bar{\epsilon})\Delta - (\bar{\tau} + \pi)\delta - (\tau + \bar{\pi})\bar{\delta}, \tag{A1}$$

$$\delta \mathrm{D} - \mathrm{D}\delta = (\bar{\alpha} + \beta - \bar{\pi})\mathrm{D} + \kappa\Delta - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta - \sigma\bar{\delta}, \tag{A2}$$

$$\delta\Delta - \Delta\delta = -\bar{\nu}\mathrm{D} + (\tau - \bar{\alpha} - \beta)\Delta + (\mu - \gamma + \bar{\gamma})\delta + \bar{\lambda}\bar{\delta}, \tag{A3}$$

$$\bar{\delta}\delta - \delta\bar{\delta} = (\bar{\mu} - \mu)\mathrm{D} + (\bar{\rho} - \rho)\Delta + (\alpha - \bar{\beta})\delta + (\beta - \bar{\alpha})\bar{\delta}. \tag{A4}$$

2. Ricci identities

Using the notation of Ricci spin coefficients (16) the Riemann tensor nonzero components can be expressed as

$$\mathrm{D}\sigma - \delta\kappa = \sigma(3\epsilon - \bar{\epsilon} + \rho + \bar{\rho}) + \kappa(\bar{\pi} - \tau - 3\beta - \bar{\alpha}) + \Psi_0, \tag{A5}$$

$$\mathrm{D}\rho - \bar{\delta}\kappa = \rho^2 + \sigma\bar{\sigma} + \rho(\epsilon + \bar{\epsilon}) - \bar{\kappa}\tau + \kappa(\pi - 3\alpha - \bar{\beta}) + \Phi_{00}, \tag{A6}$$

$$\mathrm{D}\tau - \Delta\kappa = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + \tau(\epsilon - \bar{\epsilon}) - \kappa(3\gamma + \bar{\gamma}) + \Psi_1 + \Phi_{01}, \tag{A7}$$

$$\mathrm{D}\alpha - \bar{\delta}\epsilon = \alpha(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\epsilon + \rho) + \Phi_{10}, \tag{A8}$$

$$\mathrm{D}\beta - \delta\epsilon = \sigma(\alpha + \pi) + \beta(\bar{\rho} - \bar{\epsilon}) - \kappa(\mu + \gamma) + \epsilon(\bar{\pi} - \bar{\alpha}) + \Psi_1, \tag{A9}$$

$$\mathrm{D}\gamma - \Delta\epsilon = \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - \gamma(\epsilon + \bar{\epsilon}) - \epsilon(\gamma + \bar{\gamma}) + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \frac{1}{24}R, \tag{A10}$$

$$\mathrm{D}\lambda - \bar{\delta}\pi = \rho\lambda + \bar{\sigma}\mu + \pi(\pi + \alpha - \beta) - \nu\bar{\kappa} + \lambda(\bar{\epsilon} - 3\epsilon) + \Phi_{20}, \tag{A11}$$

$$\mathrm{D}\mu - \delta\pi = \bar{\rho}\mu + \sigma\lambda + \pi(\bar{\pi} - \bar{\alpha} + \beta) - \mu(\epsilon + \bar{\epsilon}) - \nu\kappa + \Psi_2 + \frac{1}{12}R, \tag{A12}$$

$$\mathrm{D}\nu - \Delta\pi = \mu(\pi + \bar{\tau}) + \lambda(\bar{\pi} + \tau) + \pi(\gamma - \bar{\gamma}) - \nu(3\epsilon + \bar{\epsilon}) + \Psi_3 + \Phi_{21}, \tag{A13}$$

$$\Delta\lambda - \bar{\delta}\nu = \lambda(\bar{\gamma} - 3\gamma - \mu - \bar{\mu}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau}) - \Psi_4, \tag{A14}$$

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) + \sigma(\bar{\beta} - 3\alpha) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \Psi_1 + \Phi_{01}, \tag{A15}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} + \frac{1}{24}R, \tag{A16}$$

$$\delta\lambda - \bar{\delta}\mu = \nu(\rho - \bar{\rho}) + \pi(\mu - \bar{\mu}) + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21}, \tag{A17}$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} + \mu(\gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 3\beta - \bar{\alpha}) + \Phi_{22}, \tag{A18}$$

$$\delta\gamma - \Delta\beta = \gamma(\tau - \bar{\alpha} - \beta) + \mu\tau - \sigma\nu - \epsilon\bar{\nu} + \beta(\mu - \gamma + \bar{\gamma}) + \alpha\bar{\lambda} + \Phi_{12}, \tag{A19}$$

$$\delta\tau - \Delta\sigma = \mu\sigma + \bar{\lambda}\rho + \tau(\tau + \beta - \bar{\alpha}) + \sigma(\bar{\gamma} - 3\gamma) - \kappa\bar{\nu} + \Phi_{02}, \tag{A20}$$

$$\Delta\rho - \bar{\delta}\tau = -(\rho\bar{\mu} + \sigma\lambda) + \tau(\bar{\beta} - \alpha - \bar{\tau}) + \rho(\gamma + \bar{\gamma}) + \nu\kappa - \Psi_2 - \frac{1}{12}R, \tag{A21}$$

$$\Delta\alpha - \bar{\delta}\gamma = \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3. \tag{A22}$$

3. Bianchi identities

The projection of the Riemann tensor covariant derivative with cyclic exchange of indices leads to the first Bianchi identities,

$$\begin{aligned} 0 = &-\bar{\delta}\Psi_0 + \mathrm{D}\Psi_1 + (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 \\ &- \mathrm{D}\Phi_{01} + \delta\Phi_{00} + 2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} \\ &- \bar{\kappa}\Phi_{02} + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00}, \end{aligned} \tag{A23}$$

$$\begin{aligned}0 = &+\bar{\delta}\Psi_1 - \mathrm{D}\Psi_2 - \lambda\Psi_0 + 2(\pi-\alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 \\ &+ \bar{\delta}\Phi_{01} - \Delta\Phi_{00} - 2(\alpha+\bar{\tau})\Phi_{01} + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02} \\ &+ (2\gamma + 2\bar{\gamma} - \bar{\mu})\Phi_{00} - 2\tau\Phi_{10} - \frac{1}{12}\mathrm{D}R,\end{aligned} \tag{A24}$$

$$\begin{aligned}0 = &-\bar{\delta}\Psi_2 + \mathrm{D}\Psi_3 + 2\lambda\Psi_1 - 3\pi\Psi_2 + 2(\epsilon-\rho)\Psi_3 + \kappa\Psi_4 \\ &- \mathrm{D}\Phi_{21} + \delta\Phi_{20} + 2(\bar{\rho}-\epsilon)\Phi_{21} - 2\mu\Phi_{10} + 2\pi\Phi_{11} \\ &- \bar{\kappa}\Phi_{22} + (2\beta - 2\bar{\alpha} + \bar{\pi})\Phi_{20} - \frac{1}{12}\bar{\delta}R,\end{aligned} \tag{A25}$$

$$\begin{aligned}0 = &+\bar{\delta}\Psi_3 - \mathrm{D}\Psi_4 - 3\lambda\Psi_2 + 2(2\pi+\alpha)\Psi_3 + (\rho - 4\epsilon)\Psi_4 \\ &- \Delta\Phi_{20} + \bar{\delta}\Phi_{21} + 2(\alpha-\bar{\tau})\Phi_{21} + 2\nu\Phi_{10} + \bar{\sigma}\Phi_{22} \\ &- 2\lambda\Phi_{11} + (2\bar{\gamma} - 2\gamma - \bar{\mu})\Phi_{20},\end{aligned} \tag{A26}$$

$$\begin{aligned}0 = &-\Delta\Psi_0 + \delta\Psi_1 + (4\gamma-\mu)\Psi_0 - 2(2\tau+\beta)\Psi_1 + 3\sigma\Psi_2 \\ &- \mathrm{D}\Phi_{02} + \delta\Phi_{01} + 2(\bar{\pi}-\beta)\Phi_{01} - 2\kappa\Phi_{12} - \bar{\lambda}\Phi_{00} \\ &+ 2\sigma\Phi_{11} + (\bar{\rho} + 2\epsilon - 2\bar{\epsilon})\Phi_{02},\end{aligned} \tag{A27}$$

$$\begin{aligned}0 = &-\Delta\Psi_1 + \delta\Psi_2 + \nu\Psi_0 + 2(\gamma-\mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 \\ &+ \Delta\Phi_{01} - \bar{\delta}\Phi_{02} + 2(\bar{\mu}-\gamma)\Phi_{01} - 2\rho\Phi_{12} - \bar{\nu}\Phi_{00} \\ &+ 2\tau\Phi_{11} + (\bar{\tau} - 2\bar{\beta} + 2\alpha)\Phi_{02} + \frac{1}{12}\delta R,\end{aligned} \tag{A28}$$

$$\begin{aligned}0 = &-\Delta\Psi_2 + \delta\Psi_3 + 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta-\tau)\Psi_3 + \sigma\Psi_4 \\ &- \mathrm{D}\Phi_{22} + \delta\Phi_{21} + 2(\bar{\pi}+\beta)\Phi_{21} - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} \\ &+ 2\pi\Phi_{12} + (\bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} - \frac{1}{12}\Delta R,\end{aligned} \tag{A29}$$

$$\begin{aligned}0 = &-\Delta\Psi_3 + \delta\Psi_4 + 3\nu\Psi_2 - 2(\gamma+2\mu)\Psi_3 + (4\beta-\tau)\Psi_4 \\ &+ \Delta\Phi_{21} - \bar{\delta}\Phi_{22} + 2(\bar{\mu}+\gamma)\Phi_{21} - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} \\ &+ 2\lambda\Phi_{12} + (\bar{\tau} - 2\alpha - 2\bar{\beta})\Phi_{22},\end{aligned} \tag{A30}$$

and the contraction gives the second Bianchi identities, namely

$$\begin{aligned}&\bar{\delta}\Phi_{01} + \delta\Phi_{10} - D\left(\Phi_{11} + \frac{R}{8}\right) - \Delta\Phi_{00} \\ &= \bar{\kappa}\Phi_{12} + \kappa\Phi_{21} + (2\alpha + 2\bar{\tau} - \pi)\Phi_{01} + (2\bar{\alpha} + 2\tau - \bar{\pi})\Phi_{10} \\ &\quad - 2(\rho+\bar{\rho})\Phi_{11} - \bar{\sigma}\Phi_{02} - \sigma\Phi_{20} + [\mu + \bar{\mu} - 2(\gamma+\bar{\gamma})]\Phi_{00},\end{aligned} \tag{A31}$$

$$\begin{aligned}&\bar{\delta}\Phi_{12} + \delta\Phi_{21} - \Delta\left(\Phi_{11} + \frac{R}{8}\right) - D\Phi_{22} \\ &= -\nu\Phi_{01} - \bar{\nu}\Phi_{10} + (\bar{\tau} - 2\bar{\beta} - 2\pi)\Phi_{12} + (\tau - 2\beta - 2\bar{\pi})\Phi_{21} \\ &\quad + 2(\mu+\bar{\mu})\Phi_{11} + (2\epsilon + 2\bar{\epsilon} - \rho - \bar{\rho})\Phi_{22} + \lambda\Phi_{02} + \bar{\lambda}\Phi_{20},\end{aligned} \tag{A32}$$

$$\begin{aligned}&\delta\left(\Phi_{11} - \frac{R}{8}\right) - D\Phi_{12} - \Delta\Phi_{01} + \bar{\delta}\Phi_{02} \\ &= \kappa\Phi_{22} - \bar{\nu}\Phi_{00} + (\bar{\tau} - \pi + 2\alpha - 2\bar{\beta})\Phi_{02} - \sigma\Phi_{21} + \bar{\lambda}\Phi_{10} \\ &\quad + 2(\tau-\bar{\pi})\Phi_{11} + (2\bar{\epsilon} - 2\rho - \bar{\rho})\Phi_{12} + (2\bar{\mu} + \mu - 2\gamma)\Phi_{01}.\end{aligned} \tag{A33}$$

APPENDIX B: COMPARISON OF NP NOTATION IN CLASSICAL TEXTBOOKS

Within the geometric formulation of general relativity, several different conventions have appeared which typically affect signs of particular expressions. Here we follow the classic reference book [22], however, it is useful to compare our notation with other canonical sources [23,24]. The differences in notation[5] are summarized in Table III. However, the NP equations are the same in all three books [22–24]. To compare actual values of different quantities the subsequent Table IV can be used.

From Table IV it follows that all scalars as defined in Chandrasekhar/Penrose books, appearing in the NP equations, have the opposite/same as in Stephani, respectively, and thus all NP equations have the same form.

TABLE III. Notation comparison for the definitions of crucial geometric quantities.

Quantity	Stephani [22]	Chandrasekhar [23]	Penrose, Rindler [24]
Signature	$+++-$	$+---$	$+---$
Frame	$m^a, \bar{m}^a, l^a, k^a$	$l^a, n^a, m^a, \bar{m}^a$	$l^a, n^a, m^a, \bar{m}^a$
Riemann t.	$R^a{}_{bcd} = 2\Gamma^a{}_{b[d,c]} + 2\Gamma^a{}_{e[c}\Gamma^e{}_{d]b}$	$R^a{}_{bcd} = 2\Gamma^a{}_{b[d,c]} + 2\Gamma^a{}_{e[c}\Gamma^e{}_{d]b}$	$R^a{}_{bcd} = -2\Gamma^a{}_{b[d,c]} - 2\Gamma^a{}_{e[c}\Gamma^e{}_{d]b}$
Einstein equations	$G_{ab} = \kappa T_{ab}$	$G_{ab} = \pm\kappa T_{ab}$	$G_{ab} = -\kappa T_{ab}$
NP scalars Ψ_i	$\Psi_0 = C_{abcd}k^a m^b k^c m^d$, etc.	$\Psi_0 = -C_{abcd}l^a m^b l^c m^d$, etc.	$\Psi_0 = C_{abcd}l^a m^b l^c m^d$, etc.
NP scalars Φ_i	$\Phi_{00} = \frac{1}{2}S_{ab}k^a k^b$, etc.	$\Phi_{00} = -\frac{1}{2}S_{ab}l^a l^b$, etc.	$\Phi_{00} = -\frac{1}{2}S_{ab}l^a l^b$, etc.
Ricci rotation coefficients	$\kappa = -k_{a;b}m^a k^b$, etc.	$\kappa = l_{a;b}m^a l^b$, etc.	$\kappa = l_{a;b}m^a l^b$, etc.

TABLE IV. Values comparison summarized according to classic books by Stephani (S), Chandrasekhar (C), and Penrose, Rindler (P). Note that the definition of ${}^C T_{ab}$ is not clear, see footnote 4. Also the sign in front of ${}^C\Phi_{ij}$ does not seem to be correct since the NP equations in all three books are same, i.e., the correct sign should be $+$.

Quantity	Values
Metric	${}^S g_{ab} = -{}^C g_{ab} = -{}^P g_{ab}$
Contravariant frame	${}^S k^a = +{}^C l^a = +{}^P l^a$, ${}^S l^a = +{}^C n^a = +{}^P n^a$, ${}^S m^a = +{}^C m^a = +{}^P m^a$
Covariant frame	${}^S k_a = -{}^C l_a = -{}^P l_a$, ${}^S l_a = -{}^C n_a = -{}^P n_a$, ${}^S m_a = -{}^C m_a = -{}^P m_a$
Christoffel symbols	${}^S\Gamma^a{}_{bc} = +{}^C\Gamma^a{}_{bc} = +{}^P\Gamma^a{}_{bc}$
Riemann tensor	${}^S R^a{}_{bcd} = +{}^C R^a{}_{bcd} = -{}^P R^a{}_{bcd}$
Weyl tensor	${}^S C^a{}_{bcd} = +{}^C C^a{}_{bcd} = -{}^P C^a{}_{bcd}$
Ricci tensor	${}^S R_{ab} = +{}^C R_{ab} = -{}^P R_{ab}$
Ricci scalar	${}^S R = -{}^C R = +{}^P R$
Einstein tensor	${}^S G_{ab} = +{}^C G_{ab} = -{}^P G_{ab}$
cosmological constant	${}^S\Lambda = -{}^C\Lambda = +{}^P\Lambda$
stress-energy tensor	${}^S T_{ab} = \pm{}^C T_{ab} = {}^P T_{ab}$
NP scalars Ψ_i	${}^S\Psi_i = +{}^C\Psi_i = +{}^P\Psi_i$
NP scalars Φ_i	${}^S\Phi_{ij} = -{}^C\Phi_{ij} = +{}^P\Phi_{ij}$
Ricci rotation coefficients	${}^S\kappa = +{}^C\kappa = +{}^P\kappa$, etc.

[5]The definition of the energy-momentum tensor could be misleading in the Chandrasekhar book [23], namely, page 34/ Eq. (236) gives $G_{ij} = \frac{8\pi G}{c^4}T_{ij}$ or, alternatively, page 34/ Eq. (236′) is $R_{ij} = \frac{8\pi G}{c^4}(T_{ij} - \frac{1}{2}Tg_{ij})$, however, page 51/ Eq. (323) claims $R_{ij} = -\frac{8\pi G}{c^4}(T_{ij} - \frac{1}{2}Tg_{ij})$ [for electromagnetic field see also page 205/ Eq. (3) or page 564/ Eq. (11)].

APPENDIX C: COMPARISON WITH HD NP FORMALISM

Since the computer implementation of symbolical calculation within classic Newman-Penrose formalism may become more difficult due to the presence of complex quantities, it can be useful to employ its real version following from the real higher-dimensional (HD) NP formalism introduced in [4,5], see also [6] for a review.[6] Therefore, as a byproduct, we derived relations between complex four-dimensional and real higher-dimensional NP formalisms. These identifications are presented in the form of Tables V–XV. As abbreviation for the frame components of an arbitrary tensor $T_{\dots a\dots}$ let us use $T_{\dots(0)\dots} = T_{\dots a\dots}\ell^a$, $T_{\dots(1)\dots} = T_{\dots a\dots}n^a$, and $T_{\dots(i)\dots} = T_{\dots a\dots}m^a_{(i)}$. Moreover, each index $T_{\dots(0)\dots}$, $T_{\dots(1)\dots}$, and $T_{\dots(i)\dots}$ contributes $+1$, -1, and 0 to the boost weight of a component, respectively.

TABLE V. Relation between real HD frame vectors $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}^{(i)}\}$ with $i = 2, 3$, satisfying $\ell_a n^a = 1$, $m^a_{(i)} m^{(j)}_a = \delta^j_i$, and standard four-dimensional NP frame vectors $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$, see (8).

b.w.	$+1$	0	-1
	$\ell^a = k^a$	$m^{(2)a} = \frac{1}{\sqrt{2}}(m^a + \bar{m}^a)$ $m^{(3)a} = \frac{i}{\sqrt{2}}(m^a - \bar{m}^a)$	$n^a = -l^a$
		$m^a = \frac{1}{\sqrt{2}}(m^{(2)a} - im^{(3)a})$ $\bar{m}^a = \frac{1}{\sqrt{2}}(m^{(2)a} + im^{(3)a})$	

TABLE VI. Definition of the directional derivatives in the HD NP notation.

b.w.	$+1$	0	-1
	$D = \ell^a \nabla_a$	$\delta^i = m^{a(i)} \nabla_a$	$\Delta = n^a \nabla_a$

TABLE VII. Relation between directional derivatives in the classic NP formalism, i.e., $D = k^a \nabla_a$, $\Delta = l^a \nabla_a$, $\delta = m^a \nabla_a$, $\bar{\delta} = \bar{m}^a \nabla_a$, and in its HD reformulation.

b.w.	$+1$	0	-1
	$D = D$	$\delta_2 = \frac{\bar{\delta}+\delta}{\sqrt{2}}$ $\delta_3 = \frac{i(\delta-\bar{\delta})}{\sqrt{2}}$	$\Delta = -\Delta$
		$\delta = \frac{1}{\sqrt{2}}(\delta_2 - i\delta_3)$ $\bar{\delta} = \frac{1}{\sqrt{2}}(\delta_2 + i\delta_3)$	

TABLE VIII. Definition of the HD Ricci components (frame $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}^{(i)}\}$).

b.w.	$+2$	$+1$	0	-1	-2
	$\omega = R_{(0)(0)}$	$\psi_i = R_{(0)(i)}$	$\phi_{ij} = R_{(i)(j)}$ $\phi = R_{(0)(1)}$	$\psi'_i = R_{(1)(i)}$	$\omega' = R_{(1)(1)}$

[6]Note that in [7], there are some sigh errors whenever there is an odd number of $\boldsymbol{n}$'s in the expression, i.e., involving Δ, ϕ, ψ'_i, Ψ_i, Φ_{ij}, Φ^A_{ij}, Ψ'_{ijk}, τ_i, τ'_i, ρ'_{ij}, L_{10}, L_{1i}, and $M^i{}_{j1}$.

TABLE IX. Relation between Ricci components in the classic four-dimensional NP formalism and their HD counterparts, see also Table VIII.

b.w.	+2	+1	0	−1	−2
	$\omega = 2\Phi_{00}$	$\psi_2 = (\Phi_{01} + \bar{\Phi}_{01})\sqrt{2}$ $\psi_3 = i(\Phi_{01} - \bar{\Phi}_{01})\sqrt{2}$	$\phi_{22} = \Phi_{02} + \bar{\Phi}_{02} + 2\Phi_{11} + \frac{R}{4}$ $\phi_{33} = -\Phi_{02} - \bar{\Phi}_{02} + 2\Phi_{11} + \frac{R}{4}$ $\phi_{23} = i(\Phi_{02} - \bar{\Phi}_{02})$ $\phi = -2\Phi_{11} + \frac{R}{4}$	$\psi'_2 = -(\Phi_{12} + \bar{\Phi}_{12})\sqrt{2}$ $\psi'_3 = -i(\Phi_{12} - \bar{\Phi}_{12})\sqrt{2}$	$\omega' = 2\Phi_{22}$
	$\Phi_{00} = \frac{\omega}{2}$	$\Phi_{01} = \frac{\psi_2 - i\psi_3}{2\sqrt{2}}$	$\Phi_{02} = \frac{1}{4}(\phi_{22} - \phi_{33} - 2i\phi_{23})$ $\Phi_{11} = \frac{1}{8}(-2\phi + \phi_{22} + \phi_{33})$ $R = 2\phi + \phi_{ii}$	$\Phi_{12} = -\frac{1}{2\sqrt{2}}(\psi'_2 - i\psi'_3)$	$\Phi_{22} = \frac{\omega'}{2}$

TABLE X. Definition of the HD Weyl components [frame $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}^{(i)}\}$]. In four dimensions, the Weyl tensor symmetries imply $\Omega_{33} = -\Omega_{22}$, $\Omega'_{33} = -\Omega'_{22}$, $\Phi^S_{22} = -\frac{1}{2}\Phi_{2323} = \Phi^S_{33}$, $\Phi^S_{23} = 0 = \Phi^S_{32}$, $\Psi_2 = \Psi_{323}$, $\Psi_3 = \Psi_{232}$, $\Psi'_2 = \Psi'_{323}$, $\Psi'_3 = \Psi'_{232}$.

b.w.	+2	+1	0	−1	−2
	$\Omega_{ij} = C_{(0)(i)(0)(j)}$ $\Omega_{ii} = 0$	$\Psi_i = C_{(0)(1)(0)(i)}$ $\Psi_{ijk} = C_{(0)(i)(j)(k)}$ $\Psi_i = \Psi_{kik}$	$\Phi_{ij} = C_{(0)(i)(1)(j)}$ $\Phi^A_{ij} = \frac{1}{2}C_{(0)(1)(i)(j)}$ $\Phi = C_{(0)(1)(0)(1)} = \Phi_{ii}$ $\Phi^S_{ij} = -\frac{1}{2}C_{(i)(k)(j)(k)}$	$\Psi'_i = C_{(1)(0)(1)(i)}$ $\Psi'_{ijk} = C_{(1)(i)(j)(k)}$ $\Psi'_i = \Psi'_{kik}$	$\Omega'_{ij} = C_{(1)(i)(1)(j)}$ $\Omega'_{ii} = 0$

TABLE XI. Relation between complex Weyl components in the four-dimensional NP formalism and their real HD counterparts, see also Table X.

b.w.	+2	+1	0	−1	−2
	$\Omega_{22} = \frac{1}{2}(\Psi_0 + \bar{\Psi}_0)$ $\Omega_{33} = -\frac{1}{2}(\Psi_0 + \bar{\Psi}_0)$ $\Omega_{23} = \frac{i}{2}(\Psi_0 - \bar{\Psi}_0)$	$\Psi_2 = -\frac{1}{\sqrt{2}}(\Psi_1 + \bar{\Psi}_1)$ $\Psi_3 = -\frac{i}{\sqrt{2}}(\Psi_1 - \bar{\Psi}_1)$	$\Phi^S_{22} = +\frac{1}{2}(\Psi_2 + \bar{\Psi}_2)$ $\Phi^S_{33} = \Phi^S_{22}$ $\Phi^S_{23} = 0 = \Phi^S_{32}$ $\Phi^A_{23} = -\frac{i}{2}(\Psi_2 - \bar{\Psi}_2)$ $\Phi = \bar{\Psi}_2 + \Psi_2$	$\Psi'_2 = \frac{1}{\sqrt{2}}(\Psi_3 + \bar{\Psi}_3)$ $\Psi'_3 = -\frac{i}{\sqrt{2}}(\Psi_3 - \bar{\Psi}_3)$	$\Omega'_{22} = \frac{1}{2}(\Psi_4 + \bar{\Psi}_4)$ $\Omega'_{33} = -\frac{1}{2}(\bar{\Psi}_4 + \Psi_4)$ $\Omega'_{23} = -\frac{i}{2}(\Psi_4 - \bar{\Psi}_4)$
	$\Psi_0 = \Omega_{22} - i\Omega_{23}$	$\Psi_1 = -\frac{\Psi_2 - i\Psi_3}{\sqrt{2}}$	$\Psi_2 = \Phi^S_{22} + i\Phi^A_{23}$	$\Psi_3 = \frac{\Psi'_2 + i\Psi'_3}{\sqrt{2}}$	$\Psi_4 = \Omega'_{22} + i\Omega'_{23}$

TABLE XII. Definition of the Ricci rotation coefficients in HD NP formalism with a specific boost weight.

b.w.	+2	+1	0	−1	−2
	$\kappa_i = \ell_{(i);(0)}$	$\rho_{ij} = \ell_{(i);(j)}$	$\tau_i = \ell_{(i);(1)}$ $\tau'_i = n_{(i);(0)}$	$\rho'_{ij} = n_{(i);(j)}$	$\kappa'_i = n_{(i);(1)}$

TABLE XIII. Definition of the Ricci rotation coefficients in HD NP formalism that have a boost weight only under constant boosts.

b.w.	+1	0	−1
	$L_{10} = \ell_{(1);(0)}$	$L_{1i} = \ell_{(1);(i)}$	$L_{11} = \ell_{(1);(1)}$
	$M^i{}_{j0} = m^{(i)}_{(j);(0)}$	$M^i{}_{jk} = m^{(i)}_{(j);(k)}$	$M^i{}_{j1} = m^{(i)}_{(j);(1)}$

TABLE XIV. Relation between Ricci rotation coefficients in the four-dimensional NP formalism and their real HD counterparts that transforms with a specific boost weight.

b.w.	+2	+1	0	−1	−2
	$\kappa_2 = -\frac{1}{\sqrt{2}}(\kappa+\bar\kappa)$	$\rho_{22} = -\frac{1}{2}(\rho+\sigma+\bar\rho+\bar\sigma)$	$\tau_2 = \frac{1}{\sqrt{2}}(\tau+\bar\tau)$	$\rho'_{22} = -\frac{1}{2}(\lambda+\mu+\bar\lambda+\bar\mu)$	$\kappa'_2 = \frac{1}{\sqrt{2}}(\nu+\bar\nu)$
	$\kappa_3 = -\frac{i}{\sqrt{2}}(\kappa-\bar\kappa)$	$\rho_{33} = \frac{1}{2}(\sigma-\rho+\bar\sigma-\bar\rho)$	$\tau_3 = \frac{i}{\sqrt{2}}(\tau-\bar\tau)$	$\rho'_{33} = \frac{1}{2}(\lambda-\mu+\bar\lambda-\bar\mu)$	$\kappa'_3 = -\frac{i}{\sqrt{2}}(\nu-\bar\nu)$
		$\rho_{23} = \frac{i}{2}(\rho-\sigma-\bar\rho+\bar\sigma)$	$\tau'_2 = -\frac{1}{\sqrt{2}}(\pi+\bar\pi)$	$\rho'_{23} = \frac{i}{2}(\lambda-\mu-\bar\lambda+\bar\mu)$	
		$\rho_{32} = -\frac{i}{2}(\rho+\sigma-\bar\rho-\bar\sigma)$	$\tau'_3 = \frac{i}{\sqrt{2}}(\pi-\bar\pi)$	$\rho'_{32} = \frac{i}{2}(\lambda+\mu-\bar\lambda-\bar\mu)$	
	$\kappa = -\frac{1}{\sqrt{2}}(\kappa_2 - i\kappa_3)$	$\rho = -\frac{1}{2}(\rho_{22}+\rho_{33}+i(\rho_{23}-\rho_{32}))$	$\tau = \frac{1}{\sqrt{2}}(\tau_2 - i\tau_3)$	$\mu = -\frac{1}{2}(\rho'_{22}+\rho'_{33}+i(\rho'_{32}-\rho'_{23}))$	$\nu = \frac{1}{\sqrt{2}}(\kappa'_2 + i\kappa'_3)$
		$\sigma = -\frac{1}{2}(\rho_{22}-\rho_{33}-i(\rho_{23}+\rho_{32}))$	$\pi = -\frac{1}{\sqrt{2}}(\tau'_2 + i\tau'_3)$	$\lambda = -\frac{1}{2}(\rho'_{22}-\rho'_{33}+i(\rho'_{23}+\rho'_{32}))$	

TABLE XV. Relation between Ricci rotation coefficients in the four-dimensional NP formalism and their real HD counterparts that transforms with a specific boost weight only under constant boosts.

b.w.	+1	0	−1
	$L_{10} = \varepsilon+\bar\varepsilon$	$L_{12} = \frac{1}{\sqrt{2}}(\alpha+\beta+\bar\alpha+\bar\beta)$	$L_{11} = -(\gamma+\bar\gamma)$
	$M^2{}_{30} = -i(\varepsilon-\bar\varepsilon)$	$L_{13} = \frac{i}{\sqrt{2}}(\beta-\alpha+\bar\alpha-\bar\beta)$	$M^2{}_{31} = i(\gamma-\bar\gamma)$
		$M^2{}_{33} = \frac{1}{\sqrt{2}}(\beta-\alpha-\bar\alpha+\bar\beta)$	
		$M^2{}_{32} = -\frac{i}{\sqrt{2}}(\alpha+\beta-\bar\alpha-\bar\beta)$	
	$\varepsilon = \frac{1}{2}(L_{10}+iM^2{}_{30})$	$\alpha = \frac{1}{2\sqrt{2}}(L_{12}+iL_{13}-M^2{}_{33}+iM^2{}_{32})$	$\gamma = -\frac{1}{2}(L_{11}+iM^2{}_{31})$
		$\beta = \frac{1}{2\sqrt{2}}(L_{12}-iL_{13}+M^2{}_{33}+iM^2{}_{32})$	

APPENDIX D: COMPLETE SET OF QUADRATIC GRAVITY FIELD EQUATIONS

Finally for the readers convenience and direct applicability, we list the fully explicit set of the quadratic gravity field equations (2) expressed in terms of the null frame $\{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m},\bar{\boldsymbol{m}}\}$, see (8). In fact, the following equations correspond to (20)–(26) with $Z_{(a)(b)}$ substituted from (27)–(33), where the quantities $B^Z_{(a)(b)}$ are substituted from (35)–(40).

The $\boldsymbol{kk}$-projection is

$$
\begin{aligned}
0 = {} & -4\mathfrak{a}[\Phi_{20}\Psi_0+\Phi_{02}\bar\Psi_0-2\Phi_{10}\Psi_1-2\Phi_{01}\bar\Psi_1+\Phi_{00}(\Psi_2+\bar\Psi_2)]+2\left(\frac{1}{\mathsf{k}}+2\mathfrak{b}R\right)\Phi_{00} \\
& +2\mathfrak{b}[(\epsilon+\bar\epsilon)\mathrm{D}R-\mathrm{D}\mathrm{D}R-\bar\kappa\delta R-\kappa\bar\delta R]-4\mathfrak{a}[\bar\delta\,\bar\delta\,\Psi_0-\mathrm{D}\bar\delta\Psi_1-\bar\delta\mathrm{D}\Psi_1+\mathrm{D}\mathrm{D}\Psi_2+\lambda\mathrm{D}\Psi_0 \\
& +\bar\sigma\Delta\Psi_0+(2\pi-7\alpha-\bar\beta)\bar\delta\Psi_0+(5\alpha+\bar\beta-3\pi)\mathrm{D}\Psi_1-\bar\kappa\Delta\Psi_1-\bar\sigma\delta\Psi_1+(3\epsilon+\bar\epsilon+7\rho)\bar\delta\Psi_1 \\
& -(\epsilon+\bar\epsilon+6\rho)\mathrm{D}\Psi_2+\bar\kappa\delta\Psi_2-5\kappa\bar\delta\Psi_2+4\kappa\mathrm{D}\Psi_3+\Psi_0[\bar\kappa\nu+4\alpha(3\alpha+\bar\beta)-(\epsilon+\bar\epsilon+3\rho)\lambda \\
& +\pi(\pi-7\alpha-\bar\beta)+\bar\sigma(\mu-4\gamma)+\mathrm{D}\lambda-4\bar\delta\alpha+\bar\delta\pi]+2\Psi_1[2\kappa\lambda+\bar\kappa(\gamma-\mu)+\rho(5\pi-9\alpha-2\bar\beta) \\
& +\bar\sigma(\beta+2\tau)+\epsilon(2\pi-4\alpha-\bar\beta)+\bar\epsilon(\pi-\alpha)+\mathrm{D}\alpha-\mathrm{D}\pi+\bar\delta\epsilon+2\bar\delta\rho]+3\Psi_2[\kappa(3\alpha+\bar\beta-3\pi) \\
& -\bar\kappa\tau+\rho(\epsilon+\bar\epsilon+3\rho)-\sigma\bar\sigma-\mathrm{D}\rho-\bar\delta\kappa]+2\Psi_3[\kappa(\epsilon-\bar\epsilon-5\rho)+\bar\kappa\sigma+\mathrm{D}\kappa]+2\Psi_4\kappa^2+\text{c.c.}],
\end{aligned}
\tag{D1}
$$

the ***kl***-projection is

$$\begin{aligned}
0 ={}& -4\mathfrak{a}[\Phi_{21}\Psi_1 + \Phi_{12}\bar{\Psi}_1 - 2\Phi_{11}(\Psi_2+\bar{\Psi}_2) + \Phi_{01}\Psi_3 + \Phi_{10}\bar{\Psi}_3] + 2\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\Phi_{11} + \frac{1}{\mathsf{k}}\left(\frac{R}{4}-\Lambda\right)\\
&+ 2\mathfrak{b}[\Delta \mathrm{D}R - \delta\bar{\delta}R - \bar{\delta}\delta R - (\gamma+\bar{\gamma}-\mu-\bar{\mu})\mathrm{D}R - (\rho+\bar{\rho})\Delta R + (\alpha-\bar{\beta}+\bar{\tau})\delta R + (\bar{\alpha}-\beta+\tau)\bar{\delta}R]\\
&- 4\mathfrak{a}[\bar{\delta}\Delta\Psi_1 - \mathrm{D}\Delta\Psi_2 - \bar{\delta}\delta\Psi_2 + \mathrm{D}\delta\Psi_3 - \lambda\Delta\Psi_0 - \nu\bar{\delta}\Psi_0 + 2\nu \mathrm{D}\Psi_1 + (2\pi-\alpha+\bar{\beta})\Delta\Psi_1 + \lambda\delta\Psi_1\\
&+ (2\mu-\bar{\mu}-2\gamma)\bar{\delta}\Psi_1 + (\bar{\mu}-3\mu)\mathrm{D}\Psi_2 + (2\rho-\epsilon-\bar{\epsilon})\Delta\Psi_2 + (\alpha-\bar{\beta}-2\pi)\delta\Psi_2 + (\bar{\pi}+3\tau)\bar{\delta}\Psi_2\\
&+ (2\beta-\bar{\pi}-2\tau)\mathrm{D}\Psi_3 - \kappa\Delta\Psi_3 + (\epsilon+\bar{\epsilon}-2\rho)\delta\Psi_3 - 2\sigma\bar{\delta}\Psi_3 + \sigma \mathrm{D}\Psi_4 + \kappa\delta\Psi_4 + \Psi_0[\lambda(4\gamma-\mu+\bar{\mu})\\
&+ \nu(\alpha-\bar{\beta}-2\pi) - \bar{\delta}\nu] + 2\Psi_1[\gamma(\alpha-\bar{\beta}-2\pi) - \lambda(\beta+\bar{\pi}+2\tau) + \mu(\bar{\beta}-\alpha+2\pi) + \bar{\mu}(\alpha-\pi)\\
&+ \nu(\epsilon+\bar{\epsilon}-2\rho) + \mathrm{D}\nu - \bar{\delta}\gamma + \bar{\delta}\mu] + 3\Psi_2[\kappa\nu + \mu(2\rho-\epsilon-\bar{\epsilon}) - \bar{\mu}\rho + \pi\bar{\pi} + \lambda\sigma + \tau(2\pi-\alpha+\bar{\beta})\\
&- \mathrm{D}\mu + \bar{\delta}\tau] + 2\Psi_3[\kappa(\bar{\mu}-2\mu-\gamma) + \epsilon(\beta-\tau-\bar{\pi}) + \bar{\epsilon}(\beta-\tau) + \rho(\bar{\pi}-2\beta+2\tau) + \sigma(\alpha-\bar{\beta}-2\pi)\\
&+ \mathrm{D}\beta - \mathrm{D}\tau - \bar{\delta}\sigma] + \Psi_4[\kappa(4\beta-\bar{\pi}-\tau) + \sigma(\epsilon+\bar{\epsilon}-2\rho) + \mathrm{D}\sigma] + \text{c.c.}],
\end{aligned} \tag{D2}$$

the ***km***-projection is

$$\begin{aligned}
0 ={}& -4\mathfrak{a}[\Phi_{21}\Psi_0 - 2\Phi_{11}\Psi_1 + \Phi_{02}\bar{\Psi}_1 + \Phi_{01}(\Psi_2 - 2\bar{\Psi}_2) + \Phi_{00}\bar{\Psi}_3] + 2\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\Phi_{01}\\
&+ 2\mathfrak{b}[\bar{\pi}\mathrm{D}R - \mathrm{D}\delta R - \kappa\Delta R + (\epsilon-\bar{\epsilon})\delta R] - 4\mathfrak{a}[\bar{\delta}\Delta\Psi_0 - \mathrm{D}\Delta\Psi_1 - \bar{\delta}\delta\Psi_1 + \mathrm{D}\delta\Psi_2 + \nu\mathrm{D}\Psi_0\\
&+ (\pi-3\alpha+\bar{\beta})\Delta\Psi_0 + (\mu-\bar{\mu}-4\gamma)\bar{\delta}\Psi_0 + (2\gamma-2\mu+\bar{\mu})\mathrm{D}\Psi_1 + (\epsilon-\bar{\epsilon}+3\rho)\Delta\Psi_1\\
&+ (3\alpha-\bar{\beta}-\pi)\delta\Psi_1 + (2\beta+\bar{\pi}+4\tau)\bar{\delta}\Psi_1 - (\bar{\pi}+3\tau)\mathrm{D}\Psi_2 - 2\kappa\Delta\Psi_2 - (\epsilon-\bar{\epsilon}+3\rho)\delta\Psi_2\\
&- 3\sigma\bar{\delta}\Psi_2 + 2\sigma\mathrm{D}\Psi_3 + 2\kappa\delta\Psi_3 + \Psi_0[(4\gamma-\mu)(3\alpha-\bar{\beta}-\pi) + \bar{\mu}(4\alpha-\pi) + \nu(\bar{\epsilon}-\epsilon-3\rho)\\
&- \lambda\bar{\pi} + \mathrm{D}\nu - 4\bar{\delta}\gamma + \bar{\delta}\mu] + 2\Psi_1[2\kappa\nu + (\mu-\gamma)(\epsilon-\bar{\epsilon}+3\rho) - \bar{\mu}(2\rho+\epsilon) + (\beta+2\tau)(\pi-3\alpha+\bar{\beta})\\
&+ \bar{\pi}(\pi-\alpha) + \mathrm{D}\gamma - \mathrm{D}\mu + \bar{\delta}\beta + 2\bar{\delta}\tau] + 3\Psi_2[\kappa(\bar{\mu}-2\mu) + \bar{\pi}\rho + \sigma(3\alpha-\bar{\beta}-\pi)\\
&+ \tau(\epsilon-\bar{\epsilon}+3\rho) - \mathrm{D}\tau - \bar{\delta}\sigma] + 2\Psi_3[\kappa(2\beta-\bar{\pi}-2\tau) + \sigma(\bar{\epsilon}-\epsilon-3\rho) + \mathrm{D}\sigma] + 2\Psi_4\kappa\sigma\\
&+ \delta\delta\bar{\Psi}_1 - \delta\mathrm{D}\bar{\Psi}_2 - \mathrm{D}\delta\bar{\Psi}_2 + \mathrm{D}\mathrm{D}\bar{\Psi}_3 - 2\bar{\lambda}\delta\bar{\Psi}_0 + 3\bar{\lambda}\mathrm{D}\bar{\Psi}_1 + \sigma\Delta\bar{\Psi}_1 + (4\bar{\pi}-3\bar{\alpha}-\beta)\delta\bar{\Psi}_1\\
&+ (\bar{\alpha}+\beta-5\bar{\pi})\mathrm{D}\bar{\Psi}_2 - \kappa\Delta\bar{\Psi}_2 + (\epsilon-\bar{\epsilon}+5\bar{\rho})\delta\bar{\Psi}_2 - \sigma\bar{\delta}\bar{\Psi}_2 + (3\bar{\epsilon}-\epsilon-4\bar{\rho})\mathrm{D}\bar{\Psi}_3 - 3\bar{\kappa}\delta\bar{\Psi}_3\\
&+ \kappa\bar{\delta}\bar{\Psi}_3 + 2\bar{\kappa}\mathrm{D}\bar{\Psi}_4 + \bar{\Psi}_0[\bar{\lambda}(5\bar{\alpha}+\beta-3\bar{\pi}) - \bar{\nu}\sigma - \delta\bar{\lambda}] + 2\bar{\Psi}_1[\kappa\bar{\nu} + \bar{\alpha}(\bar{\alpha}+\beta) + \bar{\pi}(2\bar{\pi}-3\bar{\alpha}-\beta)\\
&- \bar{\lambda}(4\bar{\rho}+\epsilon) + \sigma(\bar{\mu}-\bar{\gamma}) + \mathrm{D}\bar{\lambda} - \delta\bar{\alpha} + \delta\bar{\pi}] + 3\bar{\Psi}_2[2\bar{\kappa}\,\bar{\lambda} - \kappa\bar{\mu} + \bar{\pi}(\epsilon-\bar{\epsilon}) + \bar{\rho}(4\bar{\pi}-\bar{\alpha}-\beta)\\
&+ \sigma\bar{\tau} - \mathrm{D}\bar{\pi} + \delta\bar{\rho}] + 2\bar{\Psi}_3[\kappa(\bar{\beta}-\bar{\tau}) + \bar{\kappa}(\beta-4\bar{\pi}) - \sigma\bar{\sigma} + (\bar{\rho}-\bar{\epsilon})(\epsilon-\bar{\epsilon}+2\bar{\rho}) + \mathrm{D}\bar{\epsilon} - \mathrm{D}\bar{\rho} - \delta\bar{\kappa}]\\
&+ \bar{\Psi}_4[\bar{\kappa}(5\bar{\epsilon}-\epsilon-3\bar{\rho}) + \kappa\bar{\sigma} + \mathrm{D}\bar{\kappa}]],
\end{aligned} \tag{D3}$$

the ***ll***-projection is

$$\begin{aligned}
0 ={}& -4\mathfrak{a}[\Phi_{22}(\Psi_2+\bar{\Psi}_2) - 2\Phi_{12}\Psi_3 - 2\Phi_{21}\bar{\Psi}_3 + \Phi_{02}\Psi_4 + \Phi_{20}\bar{\Psi}_4] + 2\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\Phi_{22}\\
&+ 2\mathfrak{b}[-\Delta\Delta R - (\gamma+\bar{\gamma})\Delta R + \nu\delta R + \bar{\nu}\,\bar{\delta}\,R] - 4\mathfrak{a}[\Delta\Delta\Psi_2 - \Delta\delta\Psi_3 - \delta\Delta\Psi_3 + \delta\delta\Psi_4 - 4\nu\Delta\Psi_1\\
&+ (\gamma+\bar{\gamma}+6\mu)\Delta\Psi_2 + 5\nu\delta\Psi_2 - \bar{\nu}\,\bar{\delta}\,\Psi_2 + \bar{\nu}\mathrm{D}\Psi_3 + (3\tau-\bar{\alpha}-5\beta)\Delta\Psi_3 - (3\gamma+\bar{\gamma}+7\mu)\delta\Psi_3 + \bar{\lambda}\,\bar{\delta}\,\Psi_3\\
&- \bar{\lambda}\mathrm{D}\Psi_4 - \sigma\Delta\Psi_4 + (\bar{\alpha}+7\beta-2\tau)\delta\Psi_4 + 2\Psi_0\nu^2 + 2\Psi_1[\nu(\gamma-\bar{\gamma}-5\mu) + \lambda\bar{\nu} - \Delta\nu] + 3\Psi_2[\mu(\gamma+\bar{\gamma}+3\mu)\\
&+ \nu(\bar{\alpha}+3\beta-3\tau) - \lambda\bar{\lambda} - \bar{\nu}\pi + \Delta\mu + \delta\nu] + 2\Psi_3[\bar{\nu}(\epsilon-\rho) + \bar{\lambda}(\alpha+2\pi) + \gamma(2\tau-\bar{\alpha}-4\beta)\\
&+ \bar{\gamma}(\tau-\beta) + \mu(5\tau-2\bar{\alpha}-9\beta) + 2\nu\sigma - \Delta\beta + \Delta\tau - \delta\gamma - 2\delta\mu] + \Psi_4[\kappa\bar{\nu} + \bar{\lambda}(\rho-4\epsilon) - \sigma(\gamma+\bar{\gamma}+3\mu)\\
&+ 4\beta(3\beta+\bar{\alpha}) + \tau(\tau-\bar{\alpha}-7\beta) - \Delta\sigma + 4\delta\beta - \delta\tau] + \text{c.c.}],
\end{aligned} \tag{D4}$$

the ***lm***-projection is

$$
\begin{aligned}
0 ={}& -4\mathfrak{a}[\Phi_{22}\Psi_1 - \Phi_{12}(2\Psi_2 - \bar{\Psi}_2) + \Phi_{02}\Psi_3 - 2\Phi_{11}\bar{\Psi}_3 + \Phi_{10}\bar{\Psi}_4] + 2\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\Phi_{12} \\
& + 2\mathfrak{b}[\bar{\nu}\mathrm{D}R - \tau\Delta R - \Delta\delta R + (\gamma - \bar{\gamma})\delta R] - 4\mathfrak{a}[\Delta\Delta\Psi_1 - \Delta\delta\Psi_2 - \delta\Delta\Psi_2 + \delta\delta\Psi_3 - 2\nu\Delta\Psi_0 \\
& + (4\mu - 3\gamma + \bar{\gamma})\Delta\Psi_1 + 3\nu\delta\Psi_1 - \bar{\nu}\,\bar{\delta}\,\Psi_1 + \bar{\nu}\mathrm{D}\Psi_2 + (5\tau - \bar{\alpha} - \beta)\Delta\Psi_2 + (\gamma - \bar{\gamma} - 5\mu)\delta\Psi_2 + \bar{\lambda}\,\bar{\delta}\,\Psi_2 \\
& - \bar{\lambda}\mathrm{D}\Psi_3 - 3\sigma\Delta\Psi_3 + (\bar{\alpha} + 3\beta - 4\tau)\delta\Psi_3 + 2\sigma\delta\Psi_4 + \Psi_0[\nu(5\gamma - \bar{\gamma} - 3\mu) + \lambda\bar{\nu} - \Delta\nu] \\
& + 2\Psi_1[\nu(\bar{\alpha} - 4\tau) + \bar{\nu}(\alpha - \pi) - \lambda\bar{\lambda} + (\gamma - \mu)(\gamma - \bar{\gamma} - 2\mu) - \Delta\gamma + \Delta\mu + \delta\nu] \\
& + 3\Psi_2[\mu(4\tau - \bar{\alpha} - \beta) + \bar{\lambda}\pi - \bar{\nu}\rho + 2\nu\sigma + \tau(\bar{\gamma} - \gamma) + \Delta\tau - \delta\mu] + 2\Psi_3[\kappa\bar{\nu} - \sigma(\bar{\gamma} + 4\mu) \\
& + \tau(2\tau - \bar{\alpha} - 3\beta) + \beta(\bar{\alpha} + \beta) + \bar{\lambda}(\rho - \epsilon) - \Delta\sigma + \delta\beta - \delta\tau] - \Psi_4[\kappa\bar{\lambda} - \sigma(\bar{\alpha} + 5\beta - 3\tau) - \delta\sigma] \\
& - \Delta\mathrm{D}\bar{\Psi}_3 + \Delta\delta\bar{\Psi}_2 + \bar{\delta}\mathrm{D}\bar{\Psi}_4 - \bar{\delta}\delta\bar{\Psi}_3 - 2\bar{\lambda}\Delta\bar{\Psi}_1 - 2\bar{\nu}\delta\bar{\Psi}_1 + 2\bar{\nu}\mathrm{D}\bar{\Psi}_2 + (3\bar{\pi} + \tau)\Delta\bar{\Psi}_2 + (\bar{\gamma} - \gamma + 3\bar{\mu})\delta\bar{\Psi}_2 \\
& + 3\bar{\lambda}\,\bar{\delta}\,\bar{\Psi}_2 + (\gamma - \bar{\gamma} - 3\bar{\mu})\mathrm{D}\bar{\Psi}_3 + (2\bar{\rho} - \rho - 2\bar{\epsilon})\Delta\bar{\Psi}_3 + (\alpha - 3\bar{\beta} + \bar{\tau})\delta\bar{\Psi}_3 - (2\bar{\alpha} + 4\bar{\pi} + \tau)\bar{\delta}\bar{\Psi}_3 \\
& + (3\bar{\beta} - \alpha - \bar{\tau})\mathrm{D}\bar{\Psi}_4 - \bar{\kappa}\Delta\bar{\Psi}_4 + (4\bar{\epsilon} + \rho - \bar{\rho})\bar{\delta}\bar{\Psi}_4 + 2\bar{\Psi}_0\bar{\lambda}\,\bar{\nu} + 2\bar{\Psi}_1[\bar{\lambda}(\gamma - \bar{\gamma} - 3\bar{\mu}) + \bar{\nu}(2\bar{\alpha} - 2\bar{\pi} - \tau) - \Delta\bar{\lambda}] \\
& + 3\bar{\Psi}_2[\bar{\lambda}(3\bar{\beta} - \bar{\tau} - \alpha) + \bar{\pi}(3\bar{\mu} - \gamma + \bar{\gamma}) + \bar{\nu}(\rho - 2\bar{\rho}) + \bar{\mu}\tau + \Delta\bar{\pi} + \bar{\delta}\,\bar{\lambda}] \\
& + 2\bar{\Psi}_3[2\bar{\kappa}\,\bar{\nu} + (\bar{\epsilon} - \bar{\rho})(\gamma - \bar{\gamma} - 3\bar{\mu}) - \rho(\bar{\gamma} + 2\bar{\mu}) + \tau(\bar{\tau} - \bar{\beta}) + (\bar{\alpha} + 2\bar{\pi})(\alpha - 3\bar{\beta} + \bar{\tau}) - \Delta\bar{\epsilon} + \Delta\bar{\rho} \\
& - \bar{\delta}\,\bar{\alpha} - 2\bar{\delta}\,\bar{\pi}] + \bar{\Psi}_4[\bar{\kappa}(\gamma - \bar{\gamma} - 3\bar{\mu}) + \rho(4\bar{\beta} - \bar{\tau}) + \bar{\rho}(\alpha - 3\bar{\beta} + \bar{\tau}) + 4\bar{\epsilon}(3\bar{\beta} - \bar{\tau} - \alpha) - \bar{\sigma}\tau - \Delta\bar{\kappa} + 4\bar{\delta}\,\bar{\epsilon} - \bar{\delta}\,\bar{\rho}]], \qquad \text{(D5)}
\end{aligned}
$$

the ***mm***-projection is

$$
\begin{aligned}
0 ={}& -4\mathfrak{a}[\Phi_{22}\Psi_0 - 2\Phi_{12}\Psi_1 + \Phi_{02}(\Psi_2 + \bar{\Psi}_2) - 2\Phi_{01}\bar{\Psi}_3 + \Phi_{00}\bar{\Psi}_4] + 2\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\Phi_{02} \\
& + 2\mathfrak{b}[\bar{\lambda}\mathrm{D}R - \sigma\Delta R - (\bar{\alpha} - \beta)\delta R - \delta\delta R] - 4\mathfrak{a}[\Delta\Delta\Psi_0 - \Delta\delta\Psi_1 - \delta\Delta\Psi_1 + \delta\delta\Psi_2 + (2\mu - 7\gamma + \bar{\gamma})\Delta\Psi_0 \\
& + \nu\delta\Psi_0 - \bar{\nu}\,\bar{\delta}\,\Psi_0 + \bar{\nu}\mathrm{D}\Psi_1 + (7\tau - \bar{\alpha} + 3\beta)\Delta\Psi_1 + (5\gamma - \bar{\gamma} - 3\mu)\delta\Psi_1 + \bar{\lambda}\,\bar{\delta}\,\Psi_1 - \bar{\lambda}\mathrm{D}\Psi_2 - 5\sigma\Delta\Psi_2 \\
& + (\bar{\alpha} - \beta - 6\tau)\delta\Psi_2 + 4\sigma\delta\Psi_3 + \Psi_0[\mu(\mu - 7\gamma + \bar{\gamma}) + \nu(\bar{\alpha} - \beta - 3\tau) + \bar{\nu}(4\alpha - \pi) + 4\gamma(3\gamma - \bar{\gamma}) - \lambda\bar{\lambda} \\
& - 4\Delta\gamma + \Delta\mu + \delta\nu] + 2\Psi_1[2\nu\sigma - \bar{\nu}(\epsilon + 2\rho) + \bar{\lambda}(\pi - \alpha) + (\bar{\gamma} - 2\gamma)(\beta + 2\tau) + (\mu - \gamma)(5\tau - \bar{\alpha} + 2\beta) \\
& + \Delta\beta + 2\Delta\tau + \delta\gamma - \delta\mu] + 3\Psi_2[\kappa\bar{\nu} + \bar{\lambda}\rho + \sigma(3\gamma - \bar{\gamma} - 3\mu) + \tau(3\tau - \bar{\alpha} + \beta) - \Delta\sigma - \delta\tau] \\
& - 2\Psi_3[\kappa\bar{\lambda} - \sigma(\bar{\alpha} + \beta - 5\tau) - \delta\sigma] + 2\Psi_4\sigma^2 + \mathrm{D}\mathrm{D}\bar{\Psi}_4 - \mathrm{D}\delta\bar{\Psi}_3 - \delta\mathrm{D}\bar{\Psi}_3 + \delta\delta\bar{\Psi}_2 - 4\bar{\lambda}\delta\bar{\Psi}_1 + 5\bar{\lambda}\mathrm{D}\bar{\Psi}_2 \\
& + \sigma\Delta\bar{\Psi}_2 + (\bar{\alpha} - \beta + 6\bar{\pi})\delta\bar{\Psi}_2 + (\beta - 3\bar{\alpha} - 7\bar{\pi})\mathrm{D}\bar{\Psi}_3 - \kappa\Delta\bar{\Psi}_3 + (\epsilon - 5\bar{\epsilon} + 3\bar{\rho})\delta\bar{\Psi}_3 - \sigma\bar{\delta}\bar{\Psi}_3 \\
& + (7\bar{\epsilon} - \epsilon - 2\bar{\rho})\mathrm{D}\bar{\Psi}_4 - \bar{\kappa}\delta\bar{\Psi}_4 + \kappa\bar{\delta}\bar{\Psi}_4 + 2\bar{\Psi}_0\bar{\lambda}^2 + 2\bar{\Psi}_1[\bar{\lambda}(\bar{\alpha} + \beta - 5\bar{\pi}) - \bar{\nu}\sigma - \delta\bar{\lambda}] \\
& + 3\bar{\Psi}_2[\kappa\bar{\nu} + \bar{\lambda}(3\bar{\epsilon} - \epsilon - 3\bar{\rho}) + \bar{\mu}\sigma + \bar{\pi}(\bar{\alpha} - \beta + 3\bar{\pi}) + \mathrm{D}\bar{\lambda} + \delta\bar{\pi}] + 2\bar{\Psi}_3[2\bar{\kappa}\,\bar{\lambda} - \kappa(2\bar{\mu} + \bar{\gamma}) \\
& + \sigma(\bar{\tau} - \bar{\beta}) + (\bar{\rho} - \bar{\epsilon})(2\bar{\alpha} - \beta + 5\bar{\pi}) + (\epsilon - 2\bar{\epsilon})(2\bar{\pi} + \bar{\alpha}) - \mathrm{D}\bar{\alpha} - 2\mathrm{D}\bar{\pi} - \delta\bar{\epsilon} + \delta\bar{\rho}] \\
& + \bar{\Psi}_4[\kappa(4\bar{\beta} - \bar{\tau}) + \bar{\kappa}(\beta - \bar{\alpha} - 3\bar{\pi}) + (\bar{\rho} - 4\bar{\epsilon})(\epsilon - 3\bar{\epsilon} + \bar{\rho}) - \sigma\bar{\sigma} + 4\mathrm{D}\bar{\epsilon} - \mathrm{D}\bar{\rho} - \delta\bar{\kappa}]], \qquad \text{(D6)}
\end{aligned}
$$

the ***m$\bar{m}$***-projection is

$$
\begin{aligned}
0 ={}& -4\mathfrak{a}[\Phi_{21}\Psi_1 + \Phi_{12}\bar{\Psi}_1 - 2\Phi_{11}(\Psi_2 + \bar{\Psi}_2) + \Phi_{01}\Psi_3 + \Phi_{10}\bar{\Psi}_3] + 2\left(\frac{1}{\mathsf{k}} + 2\mathfrak{b}R\right)\Phi_{11} + \frac{1}{\mathsf{k}}\left(\Lambda - \frac{R}{4}\right) \\
& + 2\mathfrak{b}[(\gamma + \bar{\gamma} - \bar{\mu})\mathrm{D}R - \mathrm{D}\Delta R - \Delta\mathrm{D}R + (\rho - \epsilon - \bar{\epsilon})\Delta R - (\alpha - \bar{\beta} - \pi + \bar{\tau})\delta R + (\bar{\pi} - \tau)\bar{\delta}R + \bar{\delta}\delta R] \\
& - 4\mathfrak{a}[\bar{\delta}\Delta\Psi_1 - \mathrm{D}\Delta\Psi_2 - \bar{\delta}\delta\Psi_2 + \mathrm{D}\delta\Psi_3 - \lambda\Delta\Psi_0 - \nu\bar{\delta}\Psi_0 + 2\nu\mathrm{D}\Psi_1 + (2\pi - \alpha + \bar{\beta})\Delta\Psi_1 + \lambda\delta\Psi_1 \\
& + (2\mu - \bar{\mu} - 2\gamma)\bar{\delta}\Psi_1 + (\bar{\mu} - 3\mu)\mathrm{D}\Psi_2 + (2\rho - \epsilon - \bar{\epsilon})\Delta\Psi_2 + (\alpha - \bar{\beta} - 2\pi)\delta\Psi_2 + (\bar{\pi} + 3\tau)\bar{\delta}\Psi_2
\end{aligned}
$$

$$+ (2\beta - \bar{\pi} - 2\tau)\mathrm{D}\Psi_3 - \kappa\Delta\Psi_3 + (\epsilon + \bar{\epsilon} - 2\rho)\delta\Psi_3 - 2\sigma\bar{\delta}\Psi_3 + \sigma\mathrm{D}\Psi_4 + \kappa\delta\Psi_4 + \Psi_0[\lambda(4\gamma - \mu + \bar{\mu})$$
$$+ \nu(\alpha - \bar{\beta} - 2\pi) - \bar{\delta}\nu] + 2\Psi_1[\gamma(\alpha - \bar{\beta} - 2\pi) - \lambda(\beta + \bar{\pi} + 2\tau) + \mu(\bar{\beta} - \alpha + 2\pi) + \bar{\mu}(\alpha - \pi) + \nu(\epsilon + \bar{\epsilon} - 2\rho)$$
$$+ \mathrm{D}\nu - \bar{\delta}\gamma + \bar{\delta}\mu] + 3\Psi_2[\kappa\nu + \mu(2\rho - \epsilon - \bar{\epsilon}) - \bar{\mu}\rho + \pi\bar{\pi} + \lambda\sigma + \tau(2\pi - \alpha + \bar{\beta}) - \mathrm{D}\mu + \bar{\delta}\tau]$$
$$+ 2\Psi_3[\kappa(\bar{\mu} - 2\mu - \gamma) + \epsilon(\beta - \tau - \bar{\pi}) + \bar{\epsilon}(\beta - \tau) + \rho(\bar{\pi} - 2\beta + 2\tau) + \sigma(\alpha - \bar{\beta} - 2\pi) + \mathrm{D}\beta - \mathrm{D}\tau - \bar{\delta}\sigma]$$
$$+ \Psi_4[\kappa(4\beta - \bar{\pi} - \tau) + \sigma(\epsilon + \bar{\epsilon} - 2\rho) + \mathrm{D}\sigma] + \mathrm{c.c.}]. \tag{D7}$$

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