Pinball spring model

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In general relativity it is difficult to work with the concept of a classical phenomenological spring model since it involves action at a distance. We propose a toy relativistic model exhibiting oscillatory behavior and investigate its motion in selected gravitational fields. The simple model can be used to study the behavior of extended objects falling freely in a background gravitational field. Unlike previous glider-type models, the present system only involves local interaction, avoiding thus the pitfalls of superluminal motion. Moreover, it locally satisfies energy-momentum conservation.

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I. INTRODUCTION

Geodesic motion has been a very useful tool to probe the properties of solutions to Einstein equations since the very beginning of general relativity [1], including the famous GR tests—the precession of the perihelion of Mercury and deflection of light by the Sun. Geodesics deal with point masses, which, however, are only an idealization of real bodies of finite dimensions. It is thus of interest to describe trajectories of finite objects that necessarily deform along their paths since rigid bodies are forbidden by the special theory of relativity [2]. It is well known from the classical Newtonian theory that such bodies show nontrivial effects, such as the long-term changes of the Moon's orbit due to its tidal deformations and transfer of Earth's rotational energy into Moon's orbital energy, pushing it ever farther away from the Earth [3]. The question is whether there are some analogous, purely relativistic effects not present in the Newtonian case that would perhaps appear in the strong gravitational regime near compact objects. We thus study the motion of nonpoint masses on the background of a Schwarzschild black hole, which is closely related to the so-called swimming and swinging effects [4–8] reported previously, whereby an object is able to actively change its course through spacetime by altering its shape periodically.

In a previous paper [9] we inspected the feasibility of a test "glider" consisting of two massive point particles connected by a massless rod and falling radially in Schwarzschild spacetime while the rod coordinate length varies in a predefined periodic manner. The particular model in question followed a previous work [10], which showed the presence of a "swimming" effect whereby the glider fell more slowly than a point mass starting its radial fall with the same initial conditions. We were interested in a curious apparent divergence occurring for low oscillation frequencies. Although this feature is obviously interesting from the observational point of view, previous papers did not comment on it. We explained the low-frequency "divergence" as a projection of the plotted line from a 3D space to a 2D space combined with the fact the model is no longer tenable from the point of view of physics as one of the ends of the dumbbell touches the null cone and requires an infinite amount of energy to adhere to the prescribed deformation curve. We further found a similar divergence for high frequencies which was again due to the dumbbell reaching the speed of light.

We further investigated the energy required for the glider to adhere to the prescribed deformation curve and we came to the conclusion that it becomes infinite as the glider approaches the horizon or the center in the relativistic and Newtonian cases, respectively. The unsettling fact that the work exerted by the dumbbell engine diverges, together with the upper limit on admissible frequencies due to superluminal motion imply it is arguable that one should not use the implicitly troublesome model with a predefined dumbbell deformation. Most decisively, the model predicts a non-vanishing displacement of the glider even in (anti-)de Sitter spacetime, contradicting thus the results of Dixon's theory [11–13] as pointed out in [14–16]. The intuitive argument here is that due to the high symmetry of the spacetime, there is no preferred direction in the spacetime in which the path would deviate from a geodesic. For a detailed discussion, we refer the reader to the references above. We thus need to resort to a physically more explicit system such as a spring in the Newtonian case. In such a case we control the energy of the system as a whole but its specific length at each moment is also influenced by its position relative to the gravitational field. From the point of view of physics, this seems to be a more plausible approach

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to the problem. It is however difficult to find a general relativistic analogue of a dumbbell with its endpoints following Hooke's law [17] since it necessarily involves nonlocal interaction and we thus chose to employ a discrete model capturing some of the properties of the spring.

The main advantage of the model presented here consists in the fact that unlike in the previous glider systems, all interactions taking place within the system are purely local, removing thus the "spooky action at a distance" [18], which renders unphysical all glider models with a prescribed length variation. We investigate a discrete system consisting of point masses that decay and merge, interacting via the exchange of momentum during their mutual collisions and otherwise following geodesics—the model was devised by T. Ledvinka, see [19]. The system starts as a single point particle decaying in two steps into a collection of particles that recede from one another, collide, and later come back together to ultimately merge into a single point particle again.

The central question to ask remains the same: do we still get some sort of a swimming effect even with our improved model that fixes the problems mentioned above? How does the toy model compare to a reference point particle with the same initial conditions—does it end up farther away from or closer to a Schwarzschild black hole after completing one full stroke? Let us point out that although the value of this relative radial shift depends on coordinates, its sign is gauge-invariant since in Schwarzschild geometry, the distance from the center can be defined geometrically. We further emphasize that our results do not contradict the universality of free fall since geodesic motion only applies to point test particles while our model is a system of interacting particles.

The paper is organized as follows: Sec. II summarizes the basic interactions between point particles following radial geodesics along a preferred direction in selected spacetimes. In Sec. III, we introduce a system of colliding and decaying particles that we use as a discrete model of a spring. Section IV confirms that the discrete spring model passes the test of motion in a maximally symmetric spacetime, namely in (anti) de Sitter. The final Sec. V then explores what happens when we use the discrete spring model in Schwarzschild and compares the trajectory of our non-point spring model to that of a reference particle.

II. KINEMATICS OF THE DECAY AND MERGER OF TEST PARTICLES

Our system consists of two kinds of particles: ones of a positive rest mass and ones of a negative rest mass. The latter mediate nongravitational attraction enabling the discrete "spring" to contract. These exotic particles purely serve us to mimic the springy behavior—we do not propose they be observed outside of our system. In fact, this loosely corresponds to the effective collective behavior in some complex systems, see, e.g., [20–24]. The negative-mass particles only interact with partners of positive mass. All the constituents follow a geodesic until they either decay or collide with another particle. We will only consider simple two-body decays and mergers. In this section, we will discuss the kinematics of these processes in a general $1 + 1$ spacetime with a diagonal metric such as the radial part of Schwarzschild or (anti-)de Sitter spacetime, or the symmetry axis of the Kerr solution. The symmetry of these spacetimes ensures that the system remains within the chosen subspace during its entire motion. In Schwarzschild, it is possible to derive a closed-form expression for radial geodesics and approach thus the problem analytically, which we used to check our numerical code.

Let us start by discussing a decay of a parent particle of mass M into two particles of masses m_1 and m_2. Let us denote $\mu_i \coloneqq m_i/M$. We require the decay to be special with the product particles moving along the same radial coordinate line and thus they must respect the following simple four-momentum conservation relation

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \mu_1 \frac{\mathrm{d}t_1}{\mathrm{d}\tau_1} + \mu_2 \frac{\mathrm{d}t_2}{\mathrm{d}\tau_2}, \tag{1}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \mu_1 \frac{\mathrm{d}r_1}{\mathrm{d}\tau_1} + \mu_2 \frac{\mathrm{d}r_2}{\mathrm{d}\tau_2}. \tag{2}$$

Here, the coordinates without a subscript correspond to the decaying particle and the coordinates with a subscript correspond to the two product particles, t is the timelike coordinate and r is the spacelike coordinate.

In our numerics, we typically described the decays and mergers using t as the independent variable thanks to the relation between coordinate- and four-velocity components

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}, \tag{3}$$

since this way it was easier to check whether the relevant particles indeed met. The final relation we need is the normalization of four-velocity components

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{-g_{tt} - g_{rr}(\frac{\mathrm{d}r}{\mathrm{d}t})^2}}. \tag{4}$$

We thus derive the following equation for the spatial component of the four-velocity for one of the product particles

$$4\mu_1^2 g_{rr}\left(\frac{\mathrm{d}r_1}{\mathrm{d}\tau_1}\right)^2 - 2(\mu_1^2 - \mu_2^2 + 1)\left(2\mu_1 g_{rr}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)\frac{\mathrm{d}r_1}{\mathrm{d}\tau_1} - (\mu_1^2 - \mu_2^2 + 1)^2 + 4\mu_1^2 + 4\mu_1^2 g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = 0, \tag{5}$$

while the expression for the other product just requires us to exchange the indices. This is a quadratic equation for the sought four-velocity component, which has two solutions in general because the product particle can go left or right. The temporal component of the four-velocity can then be expressed from the normalization (4) again. Subsequently, we calculate the radial four-velocity component for the second particle from (1). If necessary, one can express product coordinate velocities by inverting (3). The above equations apply to all decays occurring in the studied system.

The collision process is analogous. We now know the parameters of the two colliding particles and want to find the four-velocity and mass of the single product particle. We again use the conservation of the total four-momentum (1)—which we now read right to left—and normalization (4) which determine the three parameters of the resulting particle and the mass of the emergent particle as a bonus.

III. THE DISCRETE SPRING

We now apply the kinematic equations given above. We begin with a single, momentarily static particle that splits into two particles of equal masses moving radially in opposite directions at initial velocities u_0 corresponding to the energy of the primary decay. The pair move freely on geodesics until they decay, too. In the model, the decay occurs after the particles exist for a specific value of proper time τ_0. For this part of the process we will, therefore, use the proper time of each particle as the independent parameter.

We will integrate the radial geodesic equations for the chosen period of the proper time for both particles τ_0. This parameter corresponds to the decay rate for the particles and it determines the elastic properties of the corresponding spring but we will treat it as a free parameter. Afterwards both particles decay into a particle of a higher mass and an exotic particle with a negative mass. Note that this decay happens at different coordinate times for the lower and upper particles. We consider the masses of the new products or, more specifically, their ratios to the mass of the second decaying particle μ_1, μ_2 as free parameters. The velocity components of the products are calculated using (5) and (4). We have to use particles with negative rest masses because by colliding with the other particles, they effectively pull their nonexotic partners back together. A system involving only particles with positive rest masses would always push some of the involved particles farther apart but our goal is to fuse the particles back into a single product so that we can evaluate its position shift, $\delta r \coloneqq r_{\mathrm{spring}} - r_{\mathrm{ref}}$, which is the difference at the same coordinate time between the final positions of the now collapsed and pointlike spring and the reference particle with the same initial conditions.

After the 1st generation product particles decay, all four 2nd generation product particles follow geodesics again. With an appropriate choice of masses, the exotic particle moves very fast toward the other pair, passing on its way through the other exotic particle. As it reaches the other standard particle, they collide and recombine. This process further pulls the nonexotic particle back. The four-velocity of the product is given by the conservation of the total four-momentum again. Finally, we have the two particles, which are now moving toward one another, collide, at which point we stop the integration and read off the final shift, δr. The entire process representing one oscillation period or “stroke” of the “spring” is sketched in Fig. 1. Although the process can be repeated, we are only interested in the shift after one stroke—the qualitative results after multiple strokes are the same.

There are in total four parameters of the model: the initial velocity of the first pair of particles u_0, their decay time τ_0, and the two relative masses of the products μ_1, μ_2. One could also include the initial position and velocity of the primary decaying particle among the parameters but we will only consider one set of initial conditions for now, identical to the initial conditions of the reference particle and the original glider model [10]. The calculation of the decay is “safer” in the sense that the products are

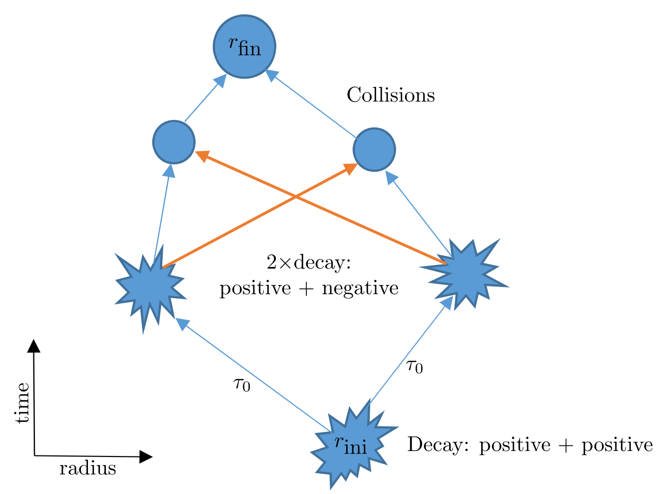

FIG. 1. A schematic diagram of the discrete spring model with time pointing upwards and the center located to the left. The original particle is dropped from rest at an initial distance r_{ini} from the center and it is shown at the bottom of the diagram as it decays into two intermediate particles flying in opposite directions for an interval of their proper time τ_0 after which they decay again, both releasing a standard particle with a positive mass and an interaction particle with a negative mass. Choosing the parameters of the system judiciously, the interaction particles fly in the direction opposite to their parent particles, hitting the other end of the “spring” after some time and bringing it back to finally collide and merge to produce the final standard particle at a distance r_{fin}. This process constitutes one full “stroke” of the toy model after which we compare it to a reference point particle, calculating the final relative shift $\delta r \coloneqq r_{\mathrm{spring}} - r_{\mathrm{ref}}$. Trajectories of positive-mass particles are shown using blue arrows, while negative-mass particles have orange arrows. Decays are depicted as stars, collisions as circles. Between their encounters, all particles move on geodesics.

automatically subluminal. The product of a merger can be superluminal for an invalid choice of the parameters and one has to check for this manually. We would like to compare the results of our model to the glider model, which attains a maximum body length $\delta l = 5 \times 10^{-3} M$ in Schwarzschild. We can approximately fix the corresponding quantity by setting $u_0 = \delta l / 2\tau_0$.

The masses μ_1, μ_2 have to be chosen appropriately because if the negative mass of the exotic particle is not high enough, the decay and recombination processes will not be able to pull the particles back together. We chose $\mu_1 = 3/2$ and $\mu_2 = -2/5$ so that the part of the motion from the exotic decay until the final merger happens very fast in comparison with the rest of the process. This allows us to easily control the total duration of the process because it is approximately equal to τ_0. This choice effectively restricts us to a single positive value of the asymmetry parameter α of the glider model from [10]. In this case the glider model predicts the largest swimming effect, i.e., a positive shift δr where in the end of the process, the glider is farther away from the center than the reference particle.

To be able to compare our results to [10], we need to plot the position shift δr as a function of ω, the characteristic frequency of the system. Since the whole process takes coordinate time Δt, which is a function of the free parameters described above, we choose $\omega \coloneqq 1/\Delta t$ and plot the position shift as a function of this parameter.

IV. PINBALL SPRING MODEL IN (ANTI-)DE SITTER UNIVERSE

As mentioned in the Introduction, an extended body cannot deviate from geodesic motion in maximally symmetric spacetimes since Dixon's theory [11–13] predicts a vanishing displacement of a suitably defined center of mass of the system. In our case, the reference geodesic coincides necessarily with the trajectory of a free test particle with the same initial conditions, because our model starts as a single particle. It is therefore a good test for any model of an extended body and for the numerical software to try to obtain this net zero position shift in (anti-)de Sitter spacetime. The main purpose of this computation is to check our numerical calculations and estimate the computational error.

We used Wolfram *Mathematica* to integrate the equations of motion and to compute the position shifts. A nonvanishing position shift is due to either a numerical error or an error in the code, which could also mean a biased integration method implemented in the software. We tried several methods and chose the one that consistently showed the smallest displacement. We verified that increasing the precision of the method indeed decreased the final value of the shift we obtained.

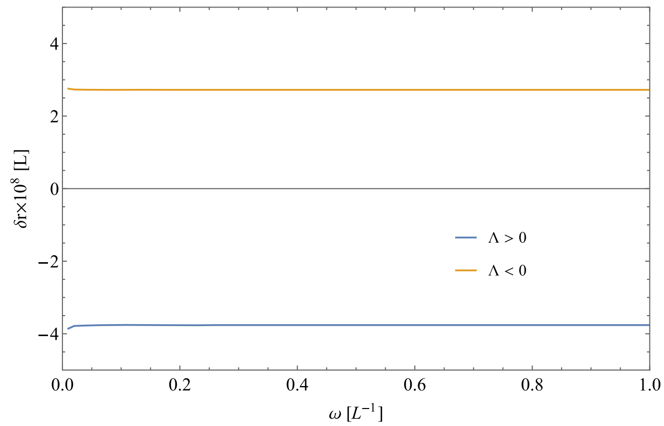

FIG. 2. Position shifts for the unphysical glider model of [10] in de Sitter (lower curve) and anti de Sitter universe (upper curve). The glider systematically falls more slowly than the reference particle in the direction of the universal expansion due to the cosmological constant, contradicting Dixon's theory.

We first chose our length scale characterizing the initial position of the spring and its maximum length: $L = 1/\sqrt{30000|\Lambda|}$. In the de Sitter case, the cosmological event horizon is then located at $r_H = 300L$. We began by calculating the shifts for the glider model to be able to compare them to the spring model, see Fig. 2, and then we proceeded to the discrete spring model. The initial position of both the spring and the reference point particle is $R_0 = 120L$ (this was the same as for the glider). They start their motion from rest and thus the first two decay products have the same initial velocities but in opposite directions. We have to make sure that none of the involved particles goes through the center of the coordinate system $r = 0$ where the description breaks down due to the nature of the coordinate system. This means that the process must not take too much coordinate time and the particles must not move away from one another too fast. With our choice of the mass ratios the process takes a very short time from the exotic decay until the end which allows us to easily control the total duration of the process through τ_0 while keeping $u_0 = \delta l/2\tau_0$ with the maximal length set to $\delta l = 5 \times 10^{-3} L$.

The resulting position shift is presented in Figs. 3 and 4. We can see that the values are many orders of magnitude smaller than those for the glider model (see Fig. 2 again) and for the Schwarzschild case (see Fig. 5 below) and they are rather random, which indicates they are due to numerical errors as is to be expected. These values can be used as a rough estimate of numerical errors in the Schwarzschild case below.

V. PINBALL SPRING MODEL IN SCHWARZSCHILD SPACETIME

We now follow the same steps for the spring model falling freely in the field of a Schwarzschild black hole.

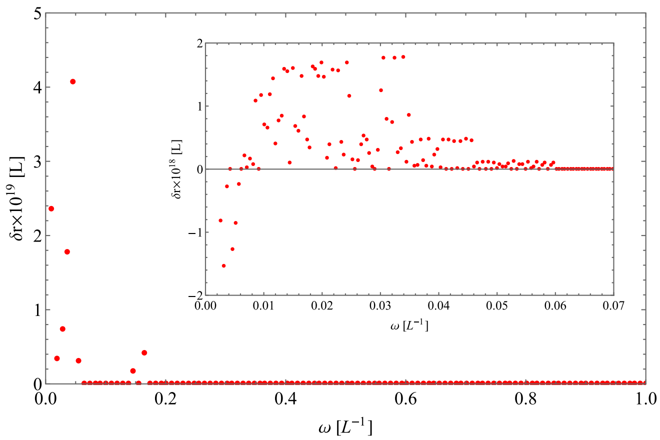

FIG. 3. Position shifts as a function of frequency for the discrete spring model in de Sitter universe, see main text for the definition of the scale L. In an exact calculation, these position shifts are to vanish according to Dixon's theory. We show the range of frequencies $\omega L \in [0, 1]$ as in Fig. 2 and, in the inset, a detail of the situation for $\omega L \in [0, 0.07]$ to be compared to the Schwarzschild case of Fig. 5 below. The largest values in the region $\omega L \in [0.2, 1]$ are three to four orders of magnitude smaller than the peaks below $\omega L \approx 0.04$. The shifts seem random across the plotted range, including also negative values for lower frequencies, and they appear to be a remnant of our set of initial conditions and the numerical calculation. The plot indicates there is indeed no systematic nonzero shift in de Sitter spacetime.

We choose its initial position to be $R_0 = 120M$, consistently with the original glider model of [9,10] to be able to compare our calculation with the previous results. With our choice of parameters, the particles move away from one another for a long time and then shrink back to a single point very rapidly so that, based on the previous glider model, we might hope for the spring to perhaps fall more

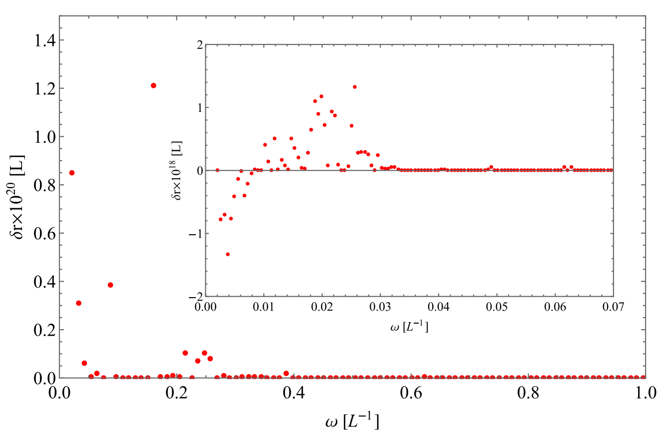

FIG. 4. Position shifts calculated for the discrete spring model in anti–de Sitter universe. We again include frequencies $\omega L \in [0, 1]$ and a blow-up of the lowest frequencies. The resulting values are of a magnitude comparable to the de Sitter case, showing a similar random pattern so that we conclude there is no overall shift as expected.

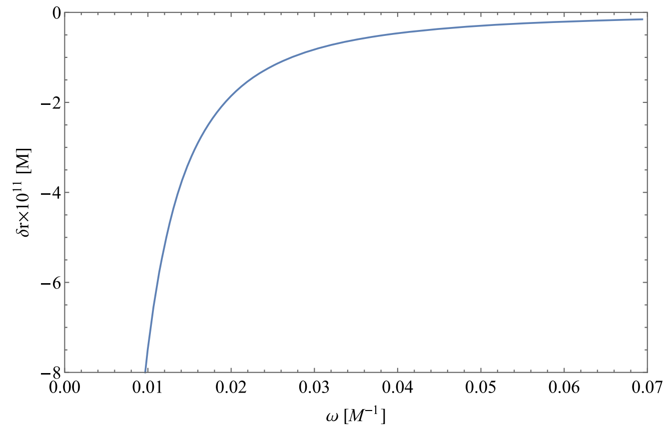

FIG. 5. Position shifts for the discrete spring model falling toward a Schwarzschild black hole. The resulting values are always negative so that the spring falls faster than the reference particle unlike the glider of [10].

slowly than the reference particle. One can indeed see in Fig. 5 that there is a non vanishing shift even in the present discrete spring model, which is allowed by Dixon's theory. However, contrary to the predictions of the glider model, the shift is always negative (the spring falls faster than the reference point particle) and has a similar frequency dependence as the glider model in the Newtonian field. Moreover, it vanishes for large frequencies, which means the alleged swimming effect does not occur. We still get the low-frequency divergence, which is due to a geometric effect explained in [9]. We do not observe any other divergence of the shifts unlike for the glider model [9] since all particles follow timelike geodesics now and they do not approach the speed of light.

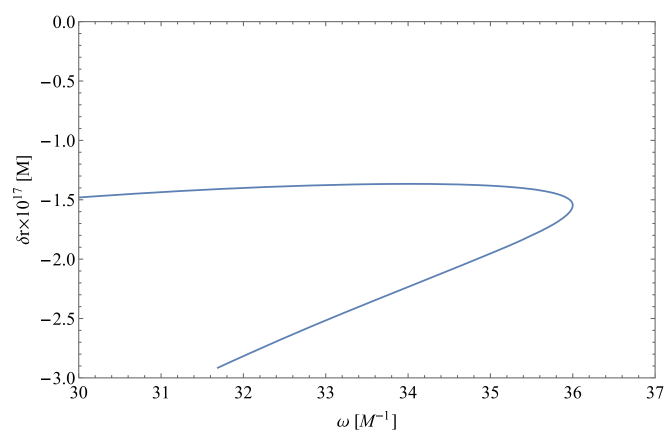

FIG. 6. The same plot of position shifts in Schwarzschild but for much higher frequencies. Since we can only change ω indirectly by varying the decay time τ_0 we can achieve just a limited range of frequencies. This is because although the entire process of stretching and shrinking of the "spring" takes ever shorter proper times, the total coordinate time is longer due to time dilation since the required velocities are higher to keep the maximum length of the spring constant.

However, the set of achievable frequencies ω is bounded from both sides. We cannot access an arbitrarily small ω because if the time of decay τ_0 is too large, the particles fall too close to the event horizon (or possibly dive under it before decaying). The tidal forces then tug on the particles so strongly apart that their decays and recombinations cannot pull them back together. On the other hand, if we choose the time decay τ_0 too small then the initial component of the four-velocity is very large since we need to keep $u_0 = \delta l/2\tau_0$ fixed. This means that the time-dilation effect becomes very significant. Although it takes the particles only a short period of the proper time to decay, the process takes much longer in terms of the coordinate time. This explains why there is a maximum ω if the other parameters of our model are fixed, see Fig. 6.

VI. CONCLUSIONS

With the model discussed in the present paper, we cannot get into trouble with causality and energy divergence described in [9] because the model simply prevents accessing unphysical frequencies. Since the position shift is always negative, our results contradict those of the glider model. Moreover, spring position shifts are smaller than the results for the glider model (these can be compared since we use the same Schwarzschild coordinate system). We should like to mention again that although our results are evaluated using coordinate distance, the main result is the sign of the displacement, which is coordinate independent. We note that in effect, we only used one value of the asymmetry parameter because controlling the frequency is rather simple in this model if we want to achieve a slow expansion and a fast contraction of the spring, which corresponded to positive position shifts in the glider model. The situation would be much more complicated if we wanted to control ω while also having the spring stretch fast and shrink slowly.

Our toy model does not require superluminal exchange of information among the different parts of the extended body. The predictions are thus more credible than the results of the glider model. The system requires some fine-tuning: all the particles need to follow radial paths to keep the model as simple as possible. Its parameters are chosen in such a way that we would be able to compare our results with the previous papers on the glider model [7,8,10]. Ultimately, the model did not reproduce the positive values of the position shifts from the glider model in the high-frequency region and we are thus led to conclude that a general relativistic swimming effect of this kind seems unlikely. In fact, the discrete spring model does deviate from a geodesic but, alas, it goes in the wrong direction so that we should rather refer to it as "diving" or "drowning". We note that if we wanted to construct a Newtonian analog of the discrete spring, we would need to add more information about the decays since energy is not conserved unlike in general relativity where local energy conservation follows from the four-momentum conservation.

In this paper we discussed the motion of a passive model that simply follows the tug of gravity on its component parts. As such, it is unlikely to be able to swim against the tide so that our results in Schwarzschild spacetime are not surprising. Let us point out that once we fix the 4 parameters defining our model, as discussed toward the end of Sec. III, we have no freedom to change the properties of the single final particle. Note also that its rest mass generally differs from that of the initial particle due to the decays and mergers involved: in our case, it was always higher than the initial mass and thus, in this respect, our model is unlike an active swimmer that we would expect to draw energy from some internal storage and use it to counter the fall. In the present paper we attempted to reproduce a situation from previous literature—but with a physically plausible model. We succeeded but the model always fell faster than the reference geodetic particle. Based on this observation, we investigated a wider range of values within the parameter space defining our model but, so far, none of these combinations produced qualitatively different results. In our future work, we plan to further modify our model in such a way as to allow a loss of rest mass and perhaps even swimming.

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