Regularized conformal electrodynamics: Novel C metric in $2+1$ dimensions

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Conformal electrodynamics is a particularly interesting example of power Maxwell nonlinear electrodynamics, designed to possess conformal symmetry in all dimensions. In this paper, we propose a regularized version of conformal electrodynamics, minimally regularizing the field of a point charge at the origin by breaking the conformal invariance of the theory with a dimensionful "Born-Infeld-like" parameter. In four dimensions the new theory reduces to the recently studied regularized Maxwell electrodynamics, distinguished by its "Maxwell-like" solutions for accelerated and slowly rotating black hole spacetimes. Focusing on three dimensions, we show that the new theory shares many of the properties of its four-dimensional cousin, including the existence of the charged C-metric solution (currently unknown in the Maxwell theory).

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I. INTRODUCTION

Nonlinear electrodynamics (NLE) arose out of attempts to deal with the singular nature of the classical Maxwell's (linear) theory of electrodynamics when applied to point charges. One of the first and certainly the most famous nonlinear model was proposed by Born and Infeld almost a hundred years ago [1]—it is distinguished by the absence of birefringence and other unique properties [2]. Subsequently, other models were proposed to achieve better regularization [3], to embody quantum corrections to Maxwell's theory coming from QED [4], and (much later) string theory [5], or to offer a maximally symmetric alternative to Maxwell's theory [6] (see also [7] for the "democratic formulation" of this theory).

NLE was also used as a "physical" source of regular black holes, increasing its significance for physics of spacetime, see [8,9] for original references and [10–12] for more recent works employing double copy. In this regard, one may formulate new criterion for the importance of a given NLE model by demanding its compatibility with essential spacetime geometries (going beyond spherical symmetry) [13–16], thus mimicking the success of Maxwell's linear theory in this regard.

While predominantly studied in four dimensions, theories of nonlinear electrodynamics are also interesting in lower-/higher-dimensional settings. Among these, conformal electrodynamics [17] is of particular interest. It is a special example of power Maxwell electrodynamics [18], designed in a way to preserve Weyl symmetry in any number of dimensions, such that in four dimensions it reduces to the Maxwell theory and yields dimension-independent (four-dimensional) Coulomb law for a point charge.

In this paper, we propose a "regularized" version of conformal electrodynamics. Namely, we design a one-parametric generalization of conformal electrodynamics characterized by a dimensionful Born-Infeld-like parameter α, which yields a finite (minimally regularized) field of a point charge in the origin. While the new regularized conformal electrodynamics naturally breaks the Weyl symmetry of the original theory, it possesses a number of interesting properties. Namely, in four dimensions it reduces to the recently studied regularized Maxwell (RegMax) electrodynamics, which is a unique NLE (constructed from a single field invariant $F_{\mu\nu}F^{\mu\nu}$ that admits Maxwell-like Robinson-Trautman [13,16], C metric [15], and slowly rotating [14] spacetimes; see also [19] for a recent discussion of optical properties of the corresponding RegMax black holes). As we shall show in this paper, in three dimensions the regularized theory admits a well-behaved generalized charged Bañados-Teitelboim-Zanelli (BTZ) black hole with improved thermodynamic charges that are not plagued by at infinity logarithmically divergent vector potential. Perhaps most importantly, it also admits a novel charged C-metric solution (at the moment unknown to exist in three-dimensional Einstein-Maxwell theory). We shall argue that the last property is very exceptional among all three-dimensional theories of NLE.

Our paper is organized as follows. The basic properties of NLE theories are reviewed in the next section. Conformal electrodynamics together with an overview of its spherical solutions are gathered in Sec. III. The novel regularized

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conformal electrodynamics is proposed in Sec. IV. Focusing on three dimensions, the corresponding generalized charged BTZ black holes solutions are studied in Sec. V. The novel charged C metric in $2+1$ dimensions is constructed in Sec. VI. We conclude in Sec. VII. Appendix A overviews spherical charged black holes in Maxwell's theory, Appendix B is devoted to construction of rotating charged BTZ black holes, and Appendix C discusses an alternative "lift" of RegMax electrodynamics to other dimensions.

II. THEORIES OF NONLINEAR ELECTRODYNAMICS

In this paper, we consider Einstein gravity coupled to nonlinear electrodynamics, described by the following d-dimensional action:

$$I = \frac{1}{16\pi}\int_M d^d x\sqrt{-g}(R + 4\mathcal{L} - 2\Lambda), \tag{1}$$

allowing for a possibility of (negative) cosmological constant Λ, which we parametrize in terms of the corresponding anti–de Sitter (AdS) radius ℓ as follows:

$$\Lambda = -\frac{(d-1)(d-2)}{2\ell^2}, \tag{2}$$

and relate it to the thermodynamic pressure according to, e.g., [20]

$$P = -\frac{\Lambda}{8\pi}. \tag{3}$$

Here, $\mathcal{L}$ is the electromagnetic Lagrangian, which is taken to be a function of electromagnetic field strength invariants of the Maxwell tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (not considering its covariant derivatives). In d number of spacetime dimensions, there are up to $[d/2]$ such invariants, related to the $[d/2]$ nontrivial eigenvalues of $F_{\mu\nu}$. One convenient way for extracting such eigenvalues is, for example, to consider the traces of the even powers of the Maxwell tensor, namely,

$$\mathrm{Tr}(F^2), \qquad \mathrm{Tr}(F^4), \quad \dots \quad \mathrm{Tr}(F^{2[d/2]}), \tag{4}$$

see, e.g., [21] for a construction of quasitopological electromagnetism in terms of powers of such traces.

A canonical example of nonlinear electrodynamics is the Born-Infeld (BI) theory [1], whose Lagrangian in all dimensions is naturally written as (see, e.g., [22] for examples of solutions in higher dimensions)

$$\mathcal{L}_{\mathrm{BI}} = -\frac{b^2}{\sqrt{-g}}\sqrt{-\det\left(g_{\mu\nu} + \frac{F_{\mu\nu}}{b}\right)} + b^2, \tag{5}$$

where b is the Born-Infeld dimensionful parameter (with dimensions $1/L$), which regularizes the field of a point charge and determines the maximal field strength allowed in the theory. Theories considered in this paper will possess similar parameters.

In this paper, we focus on a simple class of nonlinear theories that are characterized by a single electromagnetic invariant[1]

$$\mathcal{L} = \mathcal{L}(\mathcal{S}), \qquad \mathcal{S} = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}. \tag{6}$$

To further restrict the possibilities, one might require that a given theory of nonlinear electrodynamics should approach that of Maxwell in the weak field approximation

$$\lim_{\mathcal{S}\to 0}\mathcal{L} = \mathcal{L}^{(\mathrm{M})} + o(\mathcal{S}), \qquad \mathcal{L}^{(\mathrm{M})} = -\frac{1}{2}\mathcal{S}, \tag{7}$$

a condition known as the "principle of correspondence." However, while such a condition is important in four dimensions, there is no reason *a priori* to consider it in other dimensions as well. In particular, theories studied in this paper will obey the principle of correspondence in four dimensions but will not approach Maxwell's theory in other dimensions.[2]

Introducing the following notation:

$$\mathcal{L}_{,\mathcal{S}} = \frac{\partial\mathcal{L}}{\partial\mathcal{S}}, \qquad \mathcal{L}_{,\mathcal{S}\mathcal{S}} = \frac{\partial^2\mathcal{L}}{\partial\mathcal{S}^2}, \tag{8}$$

the "generalized Maxwell equations" read

$$d * D = 0, \qquad dF = 0, \tag{9}$$

where

$$D_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial F^{\mu\nu}} = 2\mathcal{L}_{,\mathcal{S}}F_{\mu\nu}. \tag{10}$$

We also obtain the following Einstein equation:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{11}$$

where the generalized electromagnetic energy-momentum tensor reads

$$T^{\mu\nu} = -\frac{1}{4\pi}\left(2F^{\mu\sigma}F^{\nu}{}_{\sigma}\mathcal{L}_{,\mathcal{S}} - \mathcal{L}g^{\mu\nu}\right). \tag{12}$$

[1] In $d = 3$ spacetime dimensions (of main interest in this paper) this is really no restriction, as any NLE therein is characterized by a single field invariant.

[2] Another criterion for restricting possible nonlinear theories of electrodynamics is related to the birefringence phenomena, causality, and energy conditions. In this work, we shall not deal with these issues and refer the interested reader to recent papers on this topic [23–25].

We shall discuss various examples of nonlinear electrodynamics below.

III. CONFORMAL ELECTRODYNAMICS

The conformal electrodynamics [17] is described by the following Lagrangian:

$$\mathcal{L}^{(\mathrm{C})} = \frac{2}{d}\beta^{4-d}(-\mathcal{S})^{d/4}, \tag{13}$$

where β is a dimensionful coupling constant, with dimensions $1/\sqrt{L}$. It plays a slightly different role compared to parameter b introduced for the Born-Infeld model (5) since it does not facilitate the Maxwell limit or determine maximal field strength. Rather, it controls the strength of coupling between electromagnetic and gravitational fields. Obviously, in $d=4$ this coupling constant disappears and (13) reduces to the Maxwell electrodynamics (7). Moreover, upon the Weyl scaling $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and $A_\mu \to A_\mu$, we find that $\sqrt{-g}\mathcal{L}^{(\mathrm{C})}$ remains in any number of dimensions invariant. We also have

$$\mathcal{L}^{(\mathrm{C})}_{,\mathcal{S}} = -\frac{1}{2}\beta^{4-d}(-\mathcal{S})^{\frac{d}{4}-1} = \frac{d}{4\mathcal{S}}\mathcal{L}^{(\mathrm{C})}. \tag{14}$$

With this it is easy to check that the corresponding energy-momentum tensor (12) is traceless, $T = T^{\mu}{}_{\mu} = 0$.

A. Spherical solutions

In any number of spacetime dimensions, the field of a point charge in conformal electrodynamics is given by the (four-dimensional) Coulomb law,

$$A_{\mathrm{C}} = -\frac{e}{r}dt, \tag{15}$$

where e is a charge parameter of dimensions of length; in what follows (and to simplify our notations) we restrict ourselves to positive charges, $e > 0$. Then, e is related to the electric charge according to the following formula:

$$Q = \frac{1}{4\pi}\int_{S^{d-2}} *D = \frac{\omega_{d-2}}{4\pi} e^{\frac{d-2}{2}}\beta^{4-d}, \tag{16}$$

where ω_d is the volume of the d-dimensional sphere, namely,

$$\omega_d = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}. \tag{17}$$

The corresponding spherically symmetric solution is then given by [26,27] (see Appendix A for comparison to solutions in standard Einstein-Maxwell theory)

$$ds^2 = -f_{\mathrm{C}}dt^2 + \frac{dr^2}{f_{\mathrm{C}}} + r^2 d\Omega^2_{d-2},$$
$$A_{\mathrm{C}} = -\frac{e}{r}dt, \tag{18}$$

where $d\Omega_d^2$ stands for the standard element on S^d, and the metric function f_{C} reads

$$f_{\mathrm{C}} = 1 - \frac{m}{r^{d-3}} + \frac{4}{d}\frac{e^{d/2}\beta^{4-d}}{r^{d-2}} + \frac{r^2}{\ell^2}. \tag{19}$$

One can show that, when the above solution describes a black hole, the corresponding thermodynamic quantities are given by (see also [28,29])

$$M = \frac{d-2}{16\pi}\omega_{d-2}m, \qquad T = \frac{f_{\mathrm{C}}'(r_+)}{4\pi}, \qquad S = \frac{\omega_{d-2}r_+^{d-2}}{4},$$
$$\phi = \frac{e}{r_+}, \qquad V = \frac{\omega_{d-2}r_+^{d-1}}{d-1}, \qquad P = \frac{(d-1)(d-2)}{16\pi\ell^2},$$
$$\Pi_\beta = \frac{(d-4)\omega_{d-2}e^{d/2}\beta^{3-d}}{2\pi d r_+}. \tag{20}$$

With these at hand, it is easy to verify that the following extended first law holds:

$$\delta M = T\delta S + \phi\delta Q + V\delta P + \Pi_\beta\delta\beta, \tag{21}$$

which reduces to the standard first law upon fixing the cosmological constant Λ and the coupling constant β. The above first law is accompanied by the corresponding extended Smarr relation, which reads

$$(d-3)M = (d-2)TS + (d-3)\phi Q - 2PV - \frac{1}{2}\Pi_\beta\beta, \tag{22}$$

with the two related by Euler's scaling argument. Note that a conjugate quantity to the coupling constant β, Π_β, was introduced in these relations. Such a quantity is necessary for the validity of the Smarr relation, as is the case with other theories characterized by dimensionful couplings, the Born-Infeld theory, for example [30]. The physical meaning of Π_β, which has dimensions of $L^{d-5/2}$, remains to be explored.

B. Conformally charged BTZ black hole

Contrary to Maxwell's case (see Appendix A), many of the above formulas remain also valid in $d=3$ dimensions. Let us state these explicitly for future reference. Namely, in $d=3$, the conformal electrodynamics reduces to

$$\mathcal{L}^{(\mathrm{C})}_3 = \frac{2}{3}\beta(-\mathcal{S})^{3/4}, \tag{23}$$

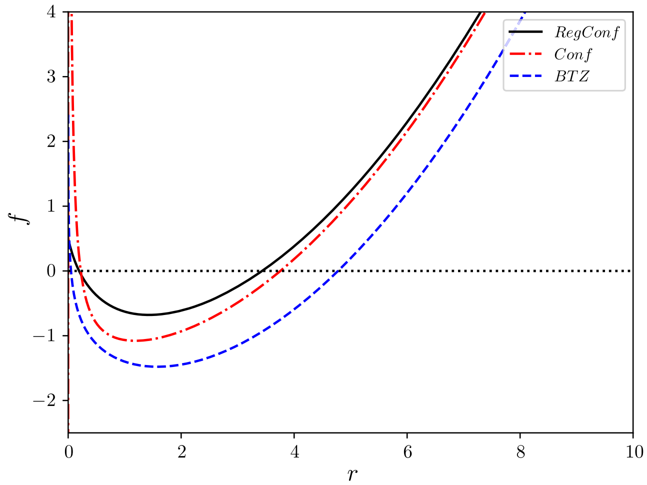

FIG. 1. Charged BTZ-like black holes. Here we display the behavior of the metric function f (determining the horizons) for charged BTZ-like black holes in various theories. Namely, the solid black curve corresponds to the Reissner-Nordström-like RegConf black hole, the red dash-dotted curve to a charged black hole in conformal electrodynamics, and the blue dashed curve to the charged BTZ black hole in Maxwell's theory. The figure is displayed for $e = 1$, $\beta = 1$, $\Lambda = -0.1$, and $\alpha = 1$.

and admits the following charged BTZ black hole solution [31–33] (see also [34] for its holographic studies):

$$\begin{aligned} ds^2 &= -f_{\mathrm{C}}dt^2 + \frac{dr^2}{f_{\mathrm{C}}} + r^2 d\varphi^2, \\ A_{\mathrm{C}} &= -\frac{e}{r}dt, \end{aligned} \tag{24}$$

where the metric function reads

$$f_{\mathrm{C}} = -m + \frac{4\beta e^{3/2}}{3r} + \frac{r^2}{\ell^2}. \tag{25}$$

It demonstrates a typical Reissner-Nordström-AdS-like behavior with two, one extremal, or no black hole horizons. In particular, in Fig. 1 we display an example of a black hole with two horizons and compare it to other charged BTZ black holes studied in this paper.

The solution is characterized by the following thermodynamic charges:

$$\begin{aligned} &Q = \frac{\sqrt{e}\beta}{2}, \qquad \phi = \frac{e}{r_+}, \qquad M = \frac{m}{8}, \\ &S = \frac{\pi r_+}{2}, \qquad T = \frac{f_{\mathrm{C}}'(r_+)}{4\pi} = \frac{r_+}{2\pi\ell^2} - \frac{\beta e^{3/2}}{3\pi r_+^2}, \\ &P = \frac{1}{8\pi\ell^2}, \qquad V = \pi r_+^2, \qquad \Pi_\beta = -\frac{e^{3/2}}{3r_+}. \end{aligned} \tag{26}$$

Note that, contrary to what happens in Maxwell's electrodynamics, cf. (A9), thermodynamic volume V here is the "standard" 2D geometric volume. One can then easily verify that the above thermodynamic quantities obey the generalized first law (21) and the Smarr formula (22), which now reduces to a simple relation,

$$TS = 2PV + \frac{1}{2}\Pi_\beta \beta, \tag{27}$$

without explicit M and ϕQ terms.

IV. REGULARIZED CONFORMAL ELECTRODYNAMICS

A. Constructing the theory

Let us now construct a theory that minimally regularizes the conformal electrodynamics. More precisely, we seek a theory whose vector potential of a pointlike charge in flat space (written in spherical coordinates) takes the following minimally regularized form in any number of dimensions:

$$A_{\mathrm{RC}} = -\frac{e}{r + r_0}dt, \qquad r_0 = \frac{\sqrt{e}}{\alpha}. \tag{28}$$

Here we have introduced a dimensionful Born-Infeld-like parameter α, which plays the role of a maximum field strength and has dimensions $1/\sqrt{L}$; the conformal electrodynamics is recovered upon setting

$$\alpha \to \infty. \tag{29}$$

Calculating the field invariant for the above field, we find

$$\mathcal{S} = -\frac{\alpha^4 e^2}{(\alpha r + \sqrt{e})^4}. \tag{30}$$

The generalized Maxwell equation (9) in d dimensions then reads

$$\left(\mathcal{L}_{\mathcal{S}} \frac{\alpha^2 e}{(\alpha r + \sqrt{e})^2} r^{d-2}\right)_{,r} = 0. \tag{31}$$

Upon integrating this equation and expressing r in terms of $\mathcal{S}$ via (30), we recover

$$\mathcal{L}_{\mathcal{S}} = \frac{c s^{d-4}}{(1-s)^{d-2}}, \tag{32}$$

where c is some (rescaled) integration constant, and we introduced a shorthand

$$s = \left(-\frac{\mathcal{S}}{\alpha^4}\right)^{\frac{1}{4}} \in (0, 1). \tag{33}$$

Expanding (32) for large α and comparing it to (14) fixes the integration constant to $c = -\frac{1}{2}\beta^{4-d}\alpha^{d-4}$, giving

$$\mathcal{L}_{\mathcal{S}} = -\frac{1}{2}\beta^{4-d}\alpha^{d-4}\frac{s^{d-4}}{(1-s)^{d-2}}. \tag{34}$$

The full Lagrangian is then obtained by integration

$$\mathcal{L} = -4\alpha^4 \int s^3 \mathcal{L}_{\mathcal{S}} ds. \tag{35}$$

This yields the following regularized conformal (RegConf) theory,

$$\mathcal{L} = \frac{2\alpha^d \beta^{4-d} s^d}{d}\,{}_2F_1[d, d-2; d+1; s] + \text{const}, \tag{36}$$

where ${}_2F_1$ is a hypergeometric function and the integration constant needs to be fixed so that we recover the conformal electrodynamics in the large α limit.

In particular, in the lowest dimensions $d = 3, 4, 5, 6$, we recover

$$\begin{aligned}
\mathcal{L}_3 &= -\alpha^3\beta\left(s^2 + 2s + 2\log(1-s)\right),\\
\mathcal{L}_4 &= -\alpha^4\left(\frac{s^3 + 3s^2 - 6s}{1-s} - 6\log(1-s)\right),\\
\mathcal{L}_5 &= -\frac{\alpha^5}{\beta}\left(\frac{s^4 + 4s^3 - 18s^2 + 12s}{(1-s)^2} + 12\log(1-s)\right),\\
\mathcal{L}_6 &= -\frac{\alpha^6}{\beta^2}\left(\frac{s^5 + 5s^4 - 110s^3/3 + 50s^2 - 20s}{(1-s)^3}\right.\\
&\qquad \left. - 20\log(1-s)\right).
\end{aligned} \tag{37}$$

Obviously, $\mathcal{L}_4$ is nothing else than the regularized Maxwell Lagrangian studied in [13–16,19].

B. Three dimensions

In what follows we shall focus on the regularized conformal electrodynamics in $d = 3$ dimensions. Let us summarize here the corresponding formulas. The theory is described by the following Lagrangian:

$$\begin{aligned}
\mathcal{L} &= -2\beta\alpha^3\left(s + \frac{s^2}{2} + \ln(1-s)\right),\\
s &\equiv \left(-\frac{\mathcal{S}}{\alpha^4}\right)^{\frac{1}{4}} \in (0,1).
\end{aligned} \tag{38}$$

In addition to the dimensionful parameter β of the conformal electrodynamics, the theory is characterized by a new dimensionful parameter $\alpha > 0$, $[\alpha^2] = (\text{length})^{-1}$, and reduces to the conformal electrodynamics in $2+1$ dimensions upon setting

$$\alpha \to \infty. \tag{39}$$

Namely, we have

$$\mathcal{L} = \frac{2}{3}\beta(-\mathcal{S})^{\frac{3}{4}} - \frac{1}{2}\beta\frac{\mathcal{S}}{\alpha} + \frac{2\beta(-\mathcal{S})^{5/4}}{5\alpha^2} + O(\alpha^{-3}); \tag{40}$$

the limit $\beta \to 0$ yields the vacuum case. So the two parameters α, β play two different roles: β is a coupling constant controlling the strength of interaction of electromagnetic and gravitational fields, while α controls the deviation of the Lagrangian from the conformal case. The first and second derivatives of $\mathcal{L}$ with respect to $\mathcal{S}$, which are important for the field equations and the optical metric, are given by

$$\mathcal{L}_{\mathcal{S}} = -\frac{1}{2}\frac{\beta}{\alpha}\frac{1}{s(1-s)}, \qquad \mathcal{L}_{\mathcal{S}\mathcal{S}} = \frac{\beta}{8\alpha^5}\frac{(2s-1)}{s^5(s-1)^2}. \tag{41}$$

It is interesting to compute the trace of energy-momentum tensor (12) using (38) and (41) in the $\alpha \to \infty$ expansion

$$T = T^{\mu}{}_{\mu} = \frac{1}{8\pi}\frac{\beta}{\alpha}S + O(\alpha^{-2}). \tag{42}$$

Evidently, the parameter α is controlling the breaking of conformal invariance for the model. In the limit of $\alpha = \infty$ we have $T = 0$ which naturally corresponds to recovering the original conformal model. Vanishing trace can also be achieved by $\beta = 0$, but that corresponds to a vacuum case, which makes it a trivial statement.

We shall now turn to constructing simple (black hole) solutions in this theory.

V. GENERALIZED CHARGED BTZ BLACK HOLE

Let us first show that the regularized conformal electrodynamics in $d = 3$ dimensions admits a charged BTZ-like black hole solution, generalizing (24). It takes the following simple form:

$$ds^2 = -f_{\text{RC}}dt^2 + \frac{dr^2}{f_{\text{RC}}} + r^2 d\varphi^2, \tag{43}$$

$$A_{\text{RC}} = -\frac{\alpha e}{\alpha r + \sqrt{e}}dt, \tag{44}$$

where the metric function f_{RC} reads

$$\begin{aligned}
f_{\text{RC}} &= 2\alpha\beta e - m - 4\sqrt{e}\alpha^2\beta r + \frac{r^2}{\ell^2}\\
&\quad + 4\alpha^3\beta r^2 \log\left(\frac{\alpha r + \sqrt{e}}{r\alpha}\right).
\end{aligned} \tag{45}$$

The corresponding field strength

$$F = dA_{\mathrm{RC}} = Edr \wedge dt, \qquad E = \frac{e\alpha^2}{(\alpha r + \sqrt{|e|})^2}, \tag{46}$$

approaches a finite value in the origin, $E_0 = E(r=0) = \alpha^2$. For large α (or alternatively large r) we recover the conformal electrodynamics metric function

$$f_{\mathrm{RC}} = f_{\mathrm{C}} - \frac{\beta e^2}{r^2\alpha} + O(1/\alpha^2). \tag{47}$$

On the other hand, near the origin, $r \to 0$, we find[3]

$$f_{\mathrm{RC}} = 2\alpha\beta e - m - 4\sqrt{e}\alpha^2\beta r + O(r^2). \tag{48}$$

Although the metric function remains finite at the origin, the black hole solution possesses a singularity at $r = 0$, as can be seen, for example, by expanding the Ricci scalar. In particular, setting $\Lambda = 0$ for the moment, we find the following expansions for the Ricci and Kretschmann scalars:

$$R = -\frac{8\sqrt{e}\alpha^2\beta}{r} + O(\log r),$$
$$\mathcal{K} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{32e\alpha^4\beta}{r^2} + O\left(\frac{\log r}{r}\right). \tag{49}$$

This is to be compared to the vanishing Ricci scalar of the conformal BTZ black hole (24), as well as to its (significantly more divergent) Kretschmann scalar, $\mathcal{K}_{\mathrm{C}} = \frac{32e^3\beta^2}{3r^6}$.

In order to have a black hole, we have to have $\Lambda < 0$. Dependent on the choice of parameters, we then obtain three types of black holes, see Fig. 2. Namely, since at $r = 0$ the metric function f_{RC} remains finite,

$$f_{\mathrm{RC}} = m_c - m, \qquad m_c = 2\alpha\beta e > 0, \tag{50}$$

if $m < m_c$, we have the Reissner-Nordström branch, with timelike singularity and two, one extremal, or no black hole horizons. If $m > m_c$, we have the Schwarzschild branch with spacelike singularity and one black hole horizon. Finally, $m = m_c$ is the marginal case, characterized by $f_{\mathrm{RC}}(r=0) = 0$. Formally, the origin becomes a "horizon," though the curvature scalars still diverge there. At the same time, this "place" is pointlike since we use the area gauge for the coordinate r. This behavior is similar to what happens in the scalar field spacetimes in four-dimensional general relativity [35], where a notion of the so-called "black point" is used for its description. This type of null pointlike singularity prevails also for scalar field spacetimes in the presence of nonlinear electrodynamics [36].

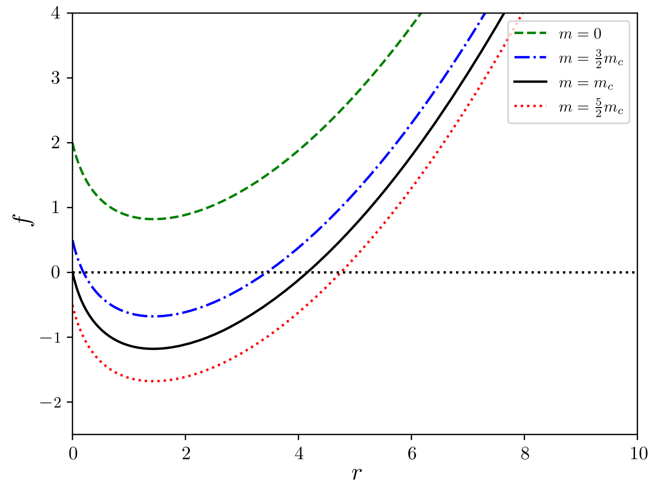

FIG. 2. Three types of RegConf black holes. Based on the behavior of the metric function f_{RC} near the origin, we distinguish three types of black holes: (i) Reissner-Nordström branch (dot-dashed blue), (ii) Schwarzschild branch (dotted red), and (iii) marginal case $m = m_c$ (solid black). This is also compared to a (naked singularity) $m = 0$ solution in the Reissner-Nordström branch (green dashed). The figure is displayed for $e = 1$, $\beta = 1$, $\Lambda = -0.1$, and $\alpha = 1$.

The above generalized BTZ black hole can be assigned the following thermodynamic quantities:

$$M = \frac{m}{8}, \qquad S = \frac{\pi r_+}{2}, \qquad V = \pi r_+^2,$$
$$T = \frac{f_{\mathrm{RC}}'(r_+)}{4\pi}, \qquad \phi = \frac{\alpha e}{\alpha r_+ + \sqrt{e}}, \qquad Q = \frac{\beta\sqrt{e}}{2}, \tag{51}$$

together with

$$\Pi_\beta = \frac{r_+^2\alpha^3}{2}\log\left(1 + \frac{\sqrt{e}}{\alpha r_+}\right) - \frac{e\alpha(2\alpha^2 r_+^2 + \alpha r_+\sqrt{e} + e)}{4(\alpha r_+\sqrt{e} + e)},$$
$$\Pi_\alpha = \frac{3}{2}r_+^2\alpha^2\beta\log\left(1 + \frac{\sqrt{e}}{\alpha r_+}\right) + \frac{\beta(e^2 - 9e\alpha^2 r_+^2 - 2e^{3/2}\alpha r_+ - 6\sqrt{e}\alpha^3 r_+^3)}{4(\alpha^2 r_+^2 + 2\alpha r_+\sqrt{e} + e)}. \tag{52}$$

It is easy to verify that these obey the following extended first law:

$$\delta M = T\delta S + \phi\delta Q + V\delta P + \Pi_\alpha\delta\alpha + \Pi_\beta\delta\beta, \tag{53}$$

together with the corresponding Smarr relation

[3]Note that even though the Lagrangian (38) possesses a logarithmically divergent term (for $s \to 1$, when maximum field strength is approached) both electromagnetic field [considering (46) or the invariant (30)] and metric function f_{RC} remain finite. This behavior corresponds to the four-dimensional regularized Maxwell theory analyzed in [15].

$$TS = 2PV + \frac{1}{2}\Pi_\alpha \delta\alpha + \frac{1}{2}\Pi_\beta \delta\beta. \tag{54}$$

Unfortunately, contrary to their four-dimensional cousins, the three-dimensional RegConf black holes do not seem to admit any remarkable thermodynamic behavior.

So far we have focused on static RegConf black holes. However, the rotating ones can easily be obtained by a "boost trick" *à* la [37–39]—we present such solutions in Appendix B. Here, we focus on a more interesting solution: a solution describing RegConf accelerated black holes.

VI. NOVEL CHARGED C METRIC IN 2+1 DIMENSIONS

Recently there has been a lot of interest in the three-dimensional C metric [40,41], see, e.g., [42,43] for analysis of the solution and [44–46] for attempts at its thermodynamic interpretation. Since its first discovery, the vacuum solution [40,41] has been generalized to include the scalar field [47] and most recently also to conformal electrodynamics [48]. While the solution may not exist in Maxwell's theory (see below), in this section we generalize it to the regularized conformal electrodynamics.

A. Accelerated BTZ black hole

The $(2+1)$-dimensional vacuum C metric is most easily written in the so-called $x-y$ coordinates [40–42] and reads

$$ds^2 = \frac{1}{\Omega^2(x,y)}\left(-F(y)dt^2 + \frac{dy^2}{F(y)} + \frac{dx^2}{G(x)}\right), \tag{55}$$

where the conformal factor is

$$\Omega = g(x+y), \tag{56}$$

with g an acceleration parameter (of dimensions of $1/L$) and the metric functions F and G that take the following form:

$$F = \frac{1}{2}c_1 y^2 + c_2 y + c_3 + \frac{1}{g^2\ell^2},$$
$$G = -\frac{1}{2}c_1 x^2 + c_2 x - c_3. \tag{57}$$

Such a metric can describe accelerated particlelike solutions or black holes. Focusing on accelerated black holes that are smoothly connected to a BTZ black hole, we set

$$c_1 = -2, \qquad c_2 = 0, \qquad c_3 = 1, \tag{58}$$

upon which

$$G = x^2 - 1, \qquad F = 1 - y^2 + \frac{1}{g^2\ell^2}, \tag{59}$$

and (to preserve the signature of the spacetime) we have to have $x > 1$ (or alternatively $x < -1$). More concretely, let $x \in (x_{\min}, x_{\max})$ be the appropriate range of coordinate x, with $x_{\min} > 1$. The zeros of the metric function F determine the position of the black hole (y_+) and Rindler (y_R) horizons; explicitly, these are given by

$$y_+ = -\frac{\sqrt{1+g^2\ell^2}}{g\ell}, \qquad y_R = \frac{\sqrt{1+g^2\ell^2}}{g\ell}. \tag{60}$$

We can now distinguish two cases: (i) the case of rapid acceleration, which happens for $x_{\max} > y_R$, and in which case both horizons are present and (ii) the case of slowly accelerating black holes with no Rindler horizon, for which $x_{\max} < y_R$.

To make connections with the four-dimensional C metric, e.g., [49,50], let us perform the following change of coordinates [42]:

$$r = -\frac{1}{\mathcal{A}y}, \quad x = \cosh(m\phi), \quad t = \frac{m^2\mathcal{A}\tau}{\omega}, \quad \mathcal{A} = \frac{g}{m}, \tag{61}$$

upon which the metric takes a more familiar form,

$$ds^2 = \frac{1}{\Omega^2}\left(-f\frac{d\tau^2}{\omega^2} + \frac{dr^2}{f} + r^2 d\phi^2\right), \tag{62}$$

where

$$f = \frac{r^2}{\ell^2} - m^2(1 - \mathcal{A}^2 r^2),$$
$$\Omega = 1 + \mathcal{A}r\cosh(m\phi). \tag{63}$$

Here, $m = \operatorname{arccosh}(x_{\max})/\pi$ regulates the tension of the wall pulling the black hole and ensures that $\phi \in (-\pi, \pi)$, and ω is not an independent parameter, but rather a combination of other parameters, ensuring the proper normalization of the proper time of an asymptotic observer.

We shall not attempt to review more properties of the above solution here and refer the interested reader to the original literature above. Instead, we proceed directly to finding the corresponding charged generalization in regularized conformal electrodynamics.

B. Regularized charged C metric

To find the charged generalization of the above vacuum solution in regularized conformal electrodynamics, we employ the *Ansätze* (55) and (56), and accompany it with the following *Ansatz* for the vector potential:

$$A = \psi(y)dt. \tag{64}$$

The Einstein equations together with the generalized Maxwell equation then yield the following solution for the metric functions appearing in the general form of the C metric (55):

$$F = \frac{4\beta\alpha^3}{g^2}\log\left(y + \frac{\alpha}{g\sqrt{e}}\right) + \frac{c_1}{2}y^2 + c_2 y + c_3 + \frac{1}{\ell^2 g^2},$$
$$G = -\frac{4\beta\alpha^3}{g^2}\log\left(\frac{\alpha}{g\sqrt{e}} - x\right) - \frac{c_1}{2}x^2 + c_2 x - c_3, \qquad (65)$$

and

$$A = \frac{\alpha e y}{\alpha + \sqrt{e} g y} dt. \qquad (66)$$

The solution looks remarkably similar to its four-dimensional cousin in this theory, cf. Sec. VA in [15]. In particular, note that, when the cosmological constant vanishes, the functions $F(y)$ and $G(x)$ have the following property: $F(w) = -G(-w)$. Similar to the vacuum case, to maintain a Lorentzian signature of the metric (55), it is necessary that $G > 0$, which implies restrictions on the domain of coordinate x. We postpone the detailed discussion of this solution to a future study. Here we only make two remarks.

First, in the large α limit we recover the charged C metric in conformal electrodynamics studied in [48], namely,

$$F = \frac{4}{3}e^{3/2}g\beta y^3 + \frac{c_1}{2}y^2 + c_2 y + c_3 + \frac{1}{g^2\ell^2},$$
$$G = \frac{4}{3}e^{3/2}g\beta x^3 - \frac{c_1}{2}x^2 + c_2 x - c_3, \qquad (67)$$

together with

$$A = e y dt. \qquad (68)$$

Second, one can easily check that, while we were able to construct the charged C metric for the conformal electrodynamics and its regularized generalization, the *Ansatz* (55) together with (64) are incompatible with many other theories of NLE, including the Maxwell theory. Namely, starting with any NLE, and using the *Ansatz* (64), the time component of the modified Maxwell equation, $(\nabla \cdot D)_t = 0$, can be once integrated, to yield

$$\psi_{,y} = \frac{c(x)}{\Omega(x,y)\mathcal{L}_S(y,x)}, \qquad (69)$$

where $c = c(x)$ is an integration constant, a function of the x coordinate only. However, since the corresponding $T_{\mu\nu}$ obeys $T_{xy} = 0$, we must also have $G_{xy} \propto \Omega_{,xy} = 0$, that is, Ω has to be separable,

$$\Omega = \Omega_x(x) + \Omega_y(y). \qquad (70)$$

Moreover, for the spacetime to describe the C metric, as we know it, both such parts have to be nontrivial. Thus, the Eq. (69) imposes a very strict restriction on the form of $\mathcal{L}_S$ for a given theory, namely,

$$\mathcal{L}_S(x,y) = \frac{c(x)h(y)}{\Omega(x,y)}, \qquad (71)$$

for some function $h = h(y)$. Obviously, for Maxwell, $\mathcal{L}_S = -1/2$ and the previous equation cannot be satisfied. Remarkably, for the regularized C-metric solution above, we find

$$\mathcal{L}_S = \frac{\beta}{2g\alpha\sqrt{e}}\frac{(yg\sqrt{e}+\alpha)^2}{(x+y)(xg\sqrt{e}-\alpha)}, \qquad (72)$$

which is precisely of the form above. It remains to be seen whether the regularized conformal electrodynamics is the most general theory for which Eq. (71) can be satisfied.

VII. SUMMARY

Conformal electrodynamics is a very interesting example of a power Maxwell theory characterized by preserving the Weyl symmetry in any number of dimensions. In four dimensions it coincides with the Maxwell theory, while it breaks the principle of correspondence in any other dimension.

In this paper, we have generalized the recently studied four-dimensional RegMax electrodynamics to any number of dimensions. A foundational feature of the new theory (inherited from its four-dimensional cousin) is that it minimally regularizes the field of a point charge—it is characterized by a dimensionful Born-Infeld-like parameter, which imposes a maximal bound on the field strength at the position of the charge. In addition, we have designed our theory so that in any number of dimensions it reduces to the conformal electrodynamics (and, in particular, to the Maxwell theory in four dimensions) in the weak field limit—thence the name regularized conformal electrodynamics. The newly introduced parameter α facilitates the regularization of the point charge field as can be seen from (46) and at the same time directly controls the deviation of this new theory from exact conformality as can be seen from (42).

Moreover, focusing on three dimensions, we have shown that the new theory admits charged BTZ-like black holes with vanishing at infinity vector potential, giving rise to a much simpler thermodynamic interpretation than is the case for the Maxwell charged BTZ black holes whose potential logarithmically diverges at infinity. Even more remarkably, the theory admits a three-dimensional generalization of a charged C metric, thus providing a nontrivial example of charged accelerating black holes in three dimensions, a property it shares with its four-dimensional RegMax cousin. We suspect that regularized conformal electrodynamics may be the most general theory for which the accelerated charged black holes can be found in the above studied form.

The extension of the RegMax theory beyond four dimensions is, of course, not unique. For example, instead of demanding that the theory in the weak field limit approaches that of the conformal electrodynamics, we might have required it to approach the Maxwell electrodynamics instead, giving rise to genuine RegMax electrodynamics in all dimensions. Such a generalization is presented in Appendix C. We shall return to it in our future investigations.

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APPENDIX A: CHARGED BLACK HOLE IN MAXWELL THEORY

The charged AdS black holes in Maxwell theory in $d > 3$ number of spacetime dimensions take the following standard form, e.g., [30]:

$$ds^2 = -f_{\mathrm{M}} dt^2 + \frac{dr^2}{f_{\mathrm{M}}} + r^2 d\Omega^2_{d-2},$$
$$A_{\mathrm{M}} = -\sqrt{\frac{d-2}{2(d-3)}} \frac{q}{r^{d-3}} dt, \tag{A1}$$

where $d\Omega^2_d$ stands for the standard element on S^d, and the metric function f_{M} reads

$$f_{\mathrm{M}} = 1 - \frac{m}{r^{d-3}} + \frac{q^2}{r^{2(d-3)}} + \frac{r^2}{\ell^2}. \tag{A2}$$

Here, the electric charge Q is given by

$$Q = \frac{\sqrt{2(d-2)(d-3)}}{8\pi} \omega_{d-2} q, \tag{A3}$$

and the mass M reads

$$M = \frac{d-2}{16\pi} \omega_{d-2} m. \tag{A4}$$

The remaining thermodynamic quantities are

$$T = \frac{f'_{\mathrm{M}}(r_+)}{4\pi}, \qquad S = \frac{\omega_{d-2} r_+^{d-2}}{4},$$
$$\phi = \sqrt{\frac{d-2}{2(d-3)}} \frac{q}{r_+^{d-3}}, \qquad V = \frac{\omega_{d-2} r_+^{d-1}}{d-1}. \tag{A5}$$

Together, they obey the standard first law of thermodynamics,

$$\delta M = T\delta S + \phi \delta Q + V\delta P. \tag{A6}$$

The above solution is valid in $d > 3$ dimensions. For $d = 3$, one has instead the charged BTZ black hole [51,52]. It reads as follows:

$$ds^2 = -f_{\mathrm{M}} dt^2 + \frac{dr^2}{f_{\mathrm{M}}} + r^2 d\varphi^2,$$
$$A_{\mathrm{M}} = -Q \log(r/r_0) dt, \tag{A7}$$

where

$$f_{\mathrm{M}} = -m - \frac{Q^2}{2} \log(r/r_0) + \frac{r^2}{\ell^2}. \tag{A8}$$

Here, r_0 is a dimensionful constant with dimensions of length; often in the literature r_0 is simply associated with the AdS length scale, $r_0 = \ell$, e.g., [53]. The logarithmic divergence renders calculation of the asymptotic mass a bit problematic, see, e.g., [54] for a possible renormalization procedure. In any case, the solution can be assigned the following thermodynamic quantities [53]:

$$M = \frac{m}{8}, \qquad T = \frac{f'_{\mathrm{M}}(r_+)}{4\pi}, \qquad S = \frac{\pi r_+}{2}, \qquad P = \frac{1}{8\pi\ell^2},$$
$$\phi = -\frac{Q}{8} \log(r/\ell), \qquad V = \pi r_+^2 - \frac{Q^2 \pi \ell^2}{4}; \tag{A9}$$

note the departure of V from the standard geometric volume. In any case, the above thermodynamic quantities obey the above standard first law (A6).

APPENDIX B: ROTATING CHARGED BTZ BLACK HOLES

Rotating charged BTZ-like black holes (in any NLE) can be obtained from nonrotating ones by the following trick [38,39]. Start from a static solution

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\varphi^2,$$
$$A = \psi dt, \tag{B1}$$

and apply the following boost:

$$t = \frac{T - \omega\phi}{\sqrt{1-\omega^2}}, \qquad \varphi = \frac{\phi - \omega T}{\sqrt{1-\omega^2}}. \tag{B2}$$

This, upon the right identification of the new coordinate ϕ, yields the rotating and charged BTZ-like black hole solution

$$ds^2 = -N^2 F dT^2 + \frac{dR^2}{F} + R^2(d\phi + N^\phi dT)^2,$$
$$A = \frac{\psi}{\sqrt{1-\omega^2}}(dT - \omega d\phi), \tag{B3}$$

where

$$R^2 = \frac{r^2 - \omega^2 f}{1-\omega^2}, \qquad F = \left(\frac{dR}{dr}\right)^2 f,$$
$$N^\phi = -\frac{\omega(r^2 - f)}{R^2(1-\omega^2)}, \qquad N = \frac{r}{R}\left(\frac{dr}{dR}\right). \tag{B4}$$

APPENDIX C: REGULARIZED MAXWELL'S THEORY IN ALL DIMENSIONS

Regularization of Maxwell theory in d dimensions can be achieved using the following prescription for the electric field (thus avoiding logarithm in potential for three dimensions) of a point charge:

$$E(r) = \frac{e}{(r + r_0)^{(d-2)}}, \qquad r_0 = \frac{e^{\frac{1}{d-2}}}{\alpha}, \tag{C1}$$

which yields the following expression for derivative of the Lagrangian:

$$\mathcal{L}_S = -\frac{1}{2}\frac{1}{(1-s)^{d-2}}, \qquad s = \frac{(-S)^{\frac{1}{2(d-2)}}}{\alpha}. \tag{C2}$$

Upon integration we arrive at

$$\mathcal{L} = -\frac{S}{2}{}_2F_1[d-2, 2(d-2); 2(d-2)+1; s], \tag{C3}$$

where ${}_2F_1$ is a hypergeometric function. Limit of the above expression for $\alpha \to \infty$ gives the same (Maxwell) Lagrangian irrespective of the dimension

$$\lim_{\alpha\to\infty} \mathcal{L} = -\frac{1}{2}S, \tag{C4}$$

as desired. In few lower dimensions, we obtain the following explicit results from (C3):

$$\mathcal{L}_3 = -\alpha^2(s + \log(1-s)), \tag{C5}$$

with $s = \frac{\sqrt{-S}}{\alpha}$,

$$\mathcal{L}_4 = -\alpha^4\left(\frac{s^3 + 3s^2 - 6s}{1-s} - 6\log(1-s)\right), \tag{C6}$$

with $s = \frac{\sqrt[4]{-S}}{\alpha}$ [notice that it coincides with the corresponding case in (37)], and

$$\mathcal{L}_5 = -\frac{1}{2}\alpha^6\left(\frac{2s^5 + 5s^4 + 20s^3 - 90s^2 + 60s}{(1-s)^2} + 60\log(1-s)\right), \tag{C7}$$

with $s = \frac{\sqrt[6]{-S}}{\alpha}$.

We have checked that contrary to RegConf presented in the main text, the theory (C5) does not seem to give rise to the (standard) C-metric solution in three dimensions.

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