

Extremal Rotating BTZ Black Holes Cannot Be Dressed in (anti-)Self-Dual Maxwell Field

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 Under the (anti-)self-dual condition for orthonormal components of the Faraday tensor, the 3D Einstein–Maxwell system with a negative cosmological constant Λ admits a solution obtained by Kamata and Koikawa and later by Cataldo and Salgado in the most general form. Actually, Clément first obtained this solution and interpreted it as a regular particle-like solution without horizon. Nevertheless, it has been erroneously stated in some literature that this Clément–Cataldo–Salgado (CCS) solution, locally characterized by a single parameter, describes a black hole even in the charged case as it reduces to the extremal rotating Bañados–Teitelboim–Zanelli (BTZ) solution in the vacuum limit and its curvature invariants are constant. In this paper, we supplement Clément’s interpretation by showing that there appears a parallelly propagated curvature singularity corresponding to an infinite affine parameter along spacelike geodesics at the location of the Killing horizon in the extremal rotating BTZ solution when the (anti-)self-dual Maxwell field is added. If the spatial coordinate θ is periodic, closed timelike curves exist near the singularity. It is also shown that the CCS solution is of Cotton type N (in contrast to charged rotating BTZ black holes which are of type I away from the horizon), and the energy-momentum tensor of the Maxwell field is of Hawking–Ellis type II. The metric solves the Einstein- Λ equations also with a massless scalar field or a null dust fluid. We explicitly demonstrate that it belongs to the Kundt shear-free, nontwisting, and nonexpanding class of geometries, whereas extremal rotating BTZ black holes have expanding principal null directions. It means that the CCS metric represents the specific null (i.e. “radiative”) Maxwell field generated by a singular source, rather than an extremal rotating BTZ black hole dressed in an (anti-)self-dual Maxwell field.

Subject Index E00, E01

1. Introduction

Three-dimensional (3D) gravity has been studied very intensively so far as a testing ground for quantum gravity due to its simplicity [1]. In particular, the Bañados–Teitelboim–Zanelli (BTZ) vacuum black-hole solution [2] in the presence of a negative cosmological constant Λ is expected to provide clues to the description of black holes in quantum gravity. Since the numbers of independent components of the Riemann tensor and Ricci tensor are the same in three dimensions, the BTZ spacetime is locally identical to the maximally symmetric anti-de Sitter (AdS) spacetime. Nevertheless, their global structures are different as the BTZ spacetime is obtained by identifications in the AdS spacetime [3].

In the 3D Einstein–Maxwell- Λ system, the charged nonrotating BTZ solution was obtained in Ref. [2] and independently in Ref. [4]. Then, the charged rotating BTZ solution was obtained by linear coordinate transformations from the nonrotating solution by Clément [5]. (See the introduction in Ref. [6] for the history of this solution.) Therefore, the rotating and nonrotating charged BTZ solutions are locally identical, but again, they are globally different if one uses a periodic coordinate. The charged rotating BTZ solution for $\Lambda < 0$, which corresponds to the Kerr–Newman-AdS solution in four dimensions, can be extended for a wider range of Λ by coordinate transformations [6]; however, the extended solution for $\Lambda \geq 0$ does not describe a black hole due to Ida’s theorem [7].

In fact, other exact solutions exist in the 3D Einstein–Maxwell- Λ system besides the charged rotating BTZ solution as summarized in Chapter 11 of the textbook [8]. Among them, there is a charged solution obtained by Kamata and Koikawa [9], which reduces to the *extremal* rotating BTZ solution in the uncharged limit. Unlike the charged BTZ solution, orthonormal components of the Faraday tensor of the Kamata–Koikawa solution satisfy the so-called *self-dual* or *anti-self-dual* condition¹ and the curvature invariants are constant. As a generalization of the Kamata–Koikawa solution, Cataldo and Salgado obtained the known most general stationary and axisymmetric solution under the (anti-)self-dual condition on the Maxwell field [10]. The Cataldo–Salgado solution is characterized by four parameters and its curvature invariants are also constant. For some reason, only the Kamata–Koikawa solution is mentioned in García-Díaz’s textbook [8], whereas the Cataldo–Salgado solution is not included.

In Ref. [5], Clément pointed out that the Kamata–Koikawa solution has previously been presented in Eq. (29) in Ref. [11]. In fact, as we will show in the present paper, the Cataldo–Salgado solution, including the Kamata–Koikawa solution as a special case, is also locally identical to the Clément solution. For this reason, we will refer to this solution as the Clément–Cataldo–Salgado (CCS) solution. As described in some literature [8,9], one might expect that the CCS solution describes a black hole since the curvature invariants are constant as the uncharged rotating BTZ solution. In Ref. [5], Clément interpreted the CCS solution as a perfectly regular particle-like solution without horizon. However, in this work, we supplement Clément’s interpretation by showing that there appears a parallelly propagated (p.p.) curvature singularity at the location of the Killing horizon in the extremal rotating BTZ solution if the (anti-)self-dual Maxwell field is added. We also clarify the specific Cotton type of the spacetime and the Hawking–Ellis type of the energy-momentum tensor.

In this paper, we first explain some mathematical tools and review the CCS solution and the charged rotating BTZ solution in Section 2. Then, we study the CCS solution in detail in Section 3. The summary of our main results and final remarks are given in the final section. Appendix A shows that the Cataldo–Salgado solution and the Clément solution are locally identical. Appendix B explains the parameters of the Cataldo–Salgado solution. Throughout this article, the signature of the Minkowski spacetime is $(-, +, +)$, and a prime denotes differentiation with respect to the argument. We adopt units such that $c = 1$, and the conventions of curvature tensors such as $[\nabla_\rho, \nabla_\sigma]V^\mu = \mathcal{R}^\mu{}_{\nu\rho\sigma}V^\nu$ and $\mathcal{R}_{\mu\nu} = \mathcal{R}^\rho{}_{\mu\rho\nu}$. We use $\kappa := 8\pi G$ instead of the gravitational constant G .

¹This terminology differs from the usual one for the Maxwell field and will be explained in Section 2.2.

2. Preliminaries

The action of the 3D Einstein–Maxwell– Λ system for the spacetime metric $g_{\mu\nu}$ and the $U(1)$ gauge field A_μ is given by

$$S[g_{\mu\nu}, A_\mu] = \int d^3x \sqrt{-g} \left(\frac{1}{2\kappa} (\mathcal{R} - 2\Lambda) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad (1)$$

where \mathcal{R} is the Ricci scalar and $F_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the Faraday tensor satisfying an identity $\nabla_{[\rho} F_{\mu\nu]} \equiv 0$. We have omitted the boundary term in the action for simplicity. Variation of the action gives the field equations

$$\begin{aligned} G_{\mu\nu} + \Lambda g_{\mu\nu} &= \kappa T_{\mu\nu}, \\ \nabla_\nu F^{\mu\nu} &= 0, \end{aligned} \quad (2)$$

where $G_{\mu\nu}$ is the Einstein tensor and the energy-momentum tensor $T_{\mu\nu}$ for the Maxwell field is given by

$$T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}. \quad (3)$$

Recently, the field equations in this system have been fully solved without imposing any space-time symmetry [12].

In this paper, we study stationary and axisymmetric solutions described in the coordinates (t, r, θ) by the following metric:

$$ds^2 = -\frac{r^2}{R^2} f dt^2 + \frac{dr^2}{f} + R^2 \left(d\theta + \frac{h}{R^2} dt \right)^2, \quad (4)$$

where $f = f(r)$, $R = R(r)$, and $h = h(r)$. In this spacetime, $\sqrt{-g} = r$ holds, and a regular null hypersurface $r = r_h$ defined by the condition $f(r_h) = 0$ is a Killing horizon. In the domain $f > 0$, natural orthonormal basis one-forms $\{E_\mu^{(0)}, E_\mu^{(1)}, E_\mu^{(2)}\}$ in the coordinate system (4) are given by

$$E_\mu^{(0)} dx^\mu = \frac{\sqrt{r^2 f}}{R} dt, \quad E_\mu^{(1)} dx^\mu = \frac{1}{\sqrt{f}} dr, \quad E_\mu^{(2)} dx^\mu = R \left(d\theta + \frac{h}{R^2} dt \right), \quad (5)$$

of which the contravariant components are

$$E^{(0)\mu} \partial_\mu = \frac{1}{\sqrt{r^2 f}} \left(-R \partial_t + \frac{h}{R} \partial_\theta \right), \quad E^{(1)\mu} \partial_\mu = \sqrt{f} \partial_r, \quad E^{(2)\mu} \partial_\mu = \frac{1}{R} \partial_\theta. \quad (6)$$

However, the coordinate system (4) does not cover the Killing horizons, as the metric and the natural basis (5) diverge for $f = 0$. To properly investigate the geometrical and physical properties, it is necessary to investigate the metric more carefully [6].

2.1. Kinoshita–Gundlach–Bourg–Davey quasi-local mass and angular momentum

For general axisymmetric spacetimes in three dimensions, Kinoshita–Gundlach–Bourg–Davey (KGBD) quasi-local mass m and angular momentum j are defined by [13,14]:

$$m := \frac{\pi}{\kappa} \left(-\Lambda \psi_\mu \psi^\mu + K_\mu K^\mu \right), \quad (7)$$

$$j := \frac{1}{\kappa} \varepsilon^{\mu\rho\sigma} \psi_\mu \nabla_\rho \psi_\sigma \quad \left(= -\frac{2\pi}{\kappa} \psi_\mu K^\mu \right), \quad (8)$$

where $\psi^\mu = (\partial/\partial\theta)^\mu$ is the Killing vector generating axisymmetry. Here K^μ is the generalized Kodama vector defined in Refs. [13,14] by

$$K^\mu := -\frac{1}{2} \varepsilon^{\mu\rho\sigma} \nabla_\rho \psi_\sigma, \quad (9)$$

where $\varepsilon_{\mu\rho\sigma}$ is the totally antisymmetric volume three-form. The vector K^μ shares the same properties as the Kodama vector in $n (\geq 3)$ dimensions [15,16]. If ψ^μ is hypersurface-orthogonal, $j = 0$ holds and then m and K^μ reduce to the Misner–Sharp quasi-local mass [17,18] and the Kodama vector [15,16] in three dimensions ($n = 3$), respectively. In the spacetime (4), the generalized Kodama vector is given by

$$K^\mu \partial_\mu = \frac{1}{2r} (2RR' \partial_t - h' \partial_\theta). \tag{10}$$

For the rotating BTZ vacuum solution [2], m and j are constants and the metric is written as [6]:

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 \left(d\theta - \frac{4Gj}{r^2} dt \right)^2, \tag{11}$$

$$f(r) = -\Lambda r^2 - 8Gm + \frac{(4Gj)^2}{r^2}.$$

Depending on the parameters m and j , the spacetime admits two Killing horizons, at most. They are located at $r = r_h$ determined by $f(r_h) = 0$, namely, $r_h = r_\pm$, where

$$r_\pm^2 := \frac{4Gm}{(-\Lambda)} \left(1 \pm \sqrt{1 + \Lambda \frac{j^2}{m^2}} \right). \tag{12}$$

The extremal rotating BTZ vacuum solution is realized for

$$|m| = \sqrt{-\Lambda} |j|. \tag{13}$$

In such a case the metric function becomes

$$f(r) = (-\Lambda) \frac{(r^2 - r_{\text{ex}}^2)^2}{r^2}, \tag{14}$$

with (assuming $j > 0$) one double-degenerate Killing horizon located at $r = r_{\text{ex}}$, where

$$r_{\text{ex}}^2 := \frac{4Gj}{\sqrt{-\Lambda}} = \frac{4Gm}{(-\Lambda)}. \tag{15}$$

Recall that, in the rotating ($j \neq 0$) BTZ vacuum solution, $r = 0$ is a coordinate singularity and the spacetime can be analytically extended using the coordinate $y := r^2$ into the region of $y < 0$. In the coordinates (t, y, θ) , the metric (11) is written as

$$ds^2 = -(-\Lambda y - M) dt^2 - 8Gj dt d\theta + \frac{dy^2}{4y\bar{f}} + y d\theta^2$$

$$= -\bar{f} dt^2 + \frac{dy^2}{4y\bar{f}} + y \left(d\theta - \frac{4Gj}{y} dt \right)^2, \tag{16}$$

$$\bar{f}(y) := -\Lambda y - M + \frac{(4Gj)^2}{y},$$

where $M = 8Gm$, which give the determinant of the metric $\det(g) = -1/4$, and the nonzero components of the inverse metric as

$$g^{tt} = -\frac{1}{\bar{f}}, \quad g^{t\theta} = -\frac{4Gj}{y\bar{f}}, \quad g^{yy} = 4y\bar{f}, \quad g^{\theta\theta} = -\frac{\Lambda y + M}{y\bar{f}}. \tag{17}$$

Because the metric and its inverse are analytic at $y (= r^2) = 0$ for $j \neq 0$, the spacetime in the region of $y > 0$ can be analytically extended beyond $y = 0$ into the region of $y < 0$. As $g^{\mu\nu} (\nabla_\mu y) (\nabla_\nu y)|_{y=0} = g^{yy}|_{y=0} = 4(4Gj)^2 > 0$, $y = 0$ is a regular timelike hypersurface.²

²In Ref. [19], the authors use the singular coordinates (t, r, θ) of Eq. (11) and claim that $r = 0$ is singular based on the analysis of the holonomy operator on a closed path (i.e. Wilson loop) around $r = 0$.

2.2. Clément–Cataldo–Salgado solution

Cataldo and Salgado obtained the most general stationary and axisymmetric solution known by assuming a *self-dual* or *anti-self-dual* condition imposed on the orthonormal basis components of the electric and magnetic fields [10]. It can be written in the form of the line element (4) in the coordinates (t, r, θ) with the metric functions given by Eqs. (40)–(42) in Ref. [10], namely

$$\begin{aligned} f(r) &= (-\Lambda) \frac{(r^2 - D)^2}{r^2}, \\ R(r) &= \sqrt{r^2 + \frac{\kappa C_0^2}{2(-\Lambda)} \ln|r^2 - D|}, \\ h(r) &= \epsilon \left(D\sqrt{-\Lambda} + \frac{\kappa C_0^2}{2\sqrt{-\Lambda}} \ln|r^2 - D| \right), \end{aligned} \tag{18}$$

while the gauge field is

$$A_\mu dx^\mu = \frac{1}{2} C_0 \ln|r^2 - D| \left(\epsilon dt + \frac{d\theta}{\sqrt{-\Lambda}} \right). \tag{19}$$

Using the useful identities

$$\begin{aligned} \sqrt{r^2 f} &= \sqrt{-\Lambda} (r^2 - D), \\ h + \epsilon \sqrt{r^2 f} &= \epsilon \sqrt{-\Lambda} R^2, \\ h - \epsilon \sqrt{r^2 f} &= \epsilon \sqrt{-\Lambda} R^2 - 2\epsilon \sqrt{r^2 f} \end{aligned} \tag{20}$$

in the region of $r^2 > D$, the metric can be rewritten in a simple form:

$$ds^2 = (-\Lambda)[R^2 - 2(r^2 - D)] dt^2 + 2\epsilon \sqrt{-\Lambda} [R^2 - (r^2 - D)] dt d\theta + R^2 d\theta^2 + \frac{dr^2}{f}, \tag{21}$$

and the inverse metric is given by

$$\begin{aligned} g^{tt} &= \frac{R^2}{\Lambda(r^2 - D)^2}, & g^{t\theta} &= \epsilon \frac{R^2 - (r^2 - D)}{\sqrt{-\Lambda}(r^2 - D)^2}, \\ g^{\theta\theta} &= -\frac{R^2 - 2(r^2 - D)}{(r^2 - D)^2}, & g^{rr} &= f. \end{aligned} \tag{22}$$

The nonzero components of the Maxwell field $F_{\mu\nu}$ and $F^{\mu\nu}$ are given by

$$\begin{aligned} F_{tr} &= -\frac{\epsilon C_0 r}{r^2 - D}, & F_{r\theta} &= \frac{C_0 r}{\sqrt{-\Lambda} (r^2 - D)}, \\ F^{tr} &= \epsilon \frac{C_0}{r}, & F^{r\theta} &= \sqrt{-\Lambda} \frac{C_0}{r}, \end{aligned} \tag{23}$$

and hence the main Maxwell electromagnetic invariant is vanishing,

$$F_{\mu\nu} F^{\mu\nu} = 0. \tag{24}$$

As shown in Appendix A, this Cataldo–Salgado solution is locally identical to Clément’s solution given by Eq. (29) in Ref. [11], namely, one solution can be obtained by coordinate transformations from the other. For this reason, we will refer to this solution as the CCS solution.

Apart from the (negative) cosmological constant Λ , the CCS solution contains *two continuous real parameters*, D and C_0 , and also a *discrete parameter* $\epsilon = \pm 1$, introduced already in Refs. [9,10]. In fact, the original form of the solution presented in Ref. [10] includes two additional parameters C and E , but we have set $C = 1$ and $E = 0$ by using the coordinate freedom and suitable redefinitions of the other parameters, as shown explicitly in Appendix B. Although

those additional parameters C and E possibly may have *global* meanings, in this paper we focus on the local properties of the solution.

Moreover, one can also set $D = 0$ in the charged case $C_0 \neq 0$ by coordinate transformations as shown in Appendix B, so that C_0 is the *only* continuous parameter characterizing the CCS solution. However, by setting $D = 0$, we miss the limit from the CCS solution to the Kamata–Koikawa solution [9] for $\kappa C_0^2 = -2\Lambda D$. For this reason, we will keep D arbitrary in the following analysis.

Since the CCS solution reduces to the extremal rotating BTZ *vacuum* solution for $C_0 = 0$, the parameter C_0 is related to the electric charge Q_E , while the parameter D corresponds to the extreme horizon r_{ex}^2 of the vacuum BTZ black hole in view of Eq. (14). In fact, $C_0 = Q_E$ holds, as can be immediately seen from the asymptotic behavior $r \rightarrow \infty$ of the electric field component F^{tr} given by Eq. (23) (provided r is the radial distance from the charge). Therefore, the constant factor which occurs in Eq. (18) is actually $\kappa C_0^2 = 8\pi G Q_E^2$.

On the other hand, the physical meaning of the discrete parameter $\epsilon = \pm 1$ is twofold. First, since the metric (21) shows that changing the sign of ϵ is equivalent to the transformation of time reversal $t \leftrightarrow -t$, the two possibilities $\epsilon = \pm 1$ represent the choices of the time orientation. Second, it denotes two possible types of the electromagnetic field, namely,

- $\epsilon = +1$: *self-dual* Maxwell field,
- $\epsilon = -1$: *anti-self-dual* Maxwell field,

which is exhibited by the frame components of the Faraday tensor $F_{(a)(b)} := F_{\mu\nu} E_{(a)}^\mu E_{(b)}^\nu$. With the orthonormal basis one-forms (5) in the domain $r^2 > D$, the nonzero components of $F_{(a)(b)}$ are

$$\begin{aligned} F_{(0)(1)} &= -F_{(1)(0)} =: \mathcal{E}, \\ F_{(2)(1)} &= -F_{(1)(2)} =: \mathcal{B}, \end{aligned} \tag{25}$$

where the electric component \mathcal{E} and the magnetic component \mathcal{B} of $F_{(a)(b)}$ are given by

$$\mathcal{E} = \epsilon \mathcal{B} = -\epsilon \frac{C_0}{R}. \tag{26}$$

Hence, the *self-dual condition* $\mathcal{E} = \mathcal{B}$ and the *anti-self-dual condition* $\mathcal{E} = -\mathcal{B}$ introduced in Ref. [9] are satisfied for $\epsilon = 1$ and $\epsilon = -1$, respectively, and a self-dual solution is the time reversal of an anti-self-dual solution.

Using $\psi^\mu \partial_\mu = \partial_\theta$ and Eq. (10), which for Eq. (18) is

$$K^\mu \partial_\mu = \partial_t + \frac{\kappa C_0^2}{2\sqrt{-\Lambda} |r^2 - D|} \left(\frac{1}{\sqrt{-\Lambda}} \partial_t - \epsilon \partial_\theta \right), \tag{27}$$

we compute the KGBD mass (7) and the KGBD angular momentum (8) to obtain

$$m = -\epsilon \sqrt{-\Lambda} j = \frac{1}{8G} \left[2D(-\Lambda) - \kappa C_0^2 (1 - \ln |r^2 - D|) \right]. \tag{28}$$

While the CCS solution (18) reduces to the extremal rotating BTZ *vacuum* solution (11) for $C_0 = 0$, it satisfies the extremality condition (13) independently of the parameters D , C_0 , and ϵ .

2.3. Charged rotating BTZ solution

In this subsection, for comparison with the CCS solution we review the charged rotating BTZ solution. In Ref. [5], Clément obtained the so-called charged rotating BTZ solution in the system (1) with $\Lambda < 0$. After suitable reparametrizations, the solution is written in the form of the

line element (4) with the metric functions given by

$$\begin{aligned} f(r) &= -\Lambda r^2 - M - \kappa Q^2 \ln r, \\ R(r) &= \sqrt{r^2 + \frac{\omega^2}{(-\Lambda)(1-\omega^2)}(M + \kappa Q^2 \ln r)}, \\ h(r) &= -\frac{\omega}{\sqrt{-\Lambda}(1-\omega^2)}(M + \kappa Q^2 \ln r), \end{aligned} \tag{29}$$

while the gauge field is

$$A_\mu dx^\mu = -\frac{Q}{\sqrt{1-\omega^2}} \ln r \left(dt - \frac{\omega}{\sqrt{-\Lambda}} d\theta \right). \tag{30}$$

We refer to the solution given by Eqs. (29) and (30) as the charged rotating BTZ solution in the *Clément form*. The nonzero components of the inverse metric are given by

$$\begin{aligned} g^{tt} &= -\frac{\omega^2 f + \Lambda r^2}{\Lambda r^2 f(1-\omega^2)}, & g^{t\theta} &= \frac{\omega(f + \Lambda r^2)}{\sqrt{-\Lambda} r^2 f(1-\omega^2)}, \\ g^{\theta\theta} &= \frac{f + \omega^2 \Lambda r^2}{r^2 f(1-\omega^2)}, & g^{rr} &= f. \end{aligned} \tag{31}$$

This solution is parameterized by three constants, namely $\omega (\neq \pm 1)$, M , and Q , and requires $\Lambda < 0$ in order for the metric to be real. (A different parametrization for a wider range of Λ has been presented in Ref. [6].) Since there is a curvature singularity at $r = 0$ for $Q \neq 0$, the domain of r is restricted to $r \in (0, \infty)$.

The nonzero components of the Maxwell field $F_{\mu\nu}$ and $F^{\mu\nu}$ are given by

$$\begin{aligned} F_{tr} &= \frac{Q}{\sqrt{1-\omega^2} r}, & F_{r\theta} &= \frac{\omega Q}{\sqrt{-\Lambda}(1-\omega^2) r}, \\ F^{tr} &= -\frac{Q}{\sqrt{1-\omega^2} r}, & F^{r\theta} &= \frac{\omega Q \sqrt{-\Lambda}}{\sqrt{1-\omega^2} r}, \end{aligned} \tag{32}$$

which give the following main Maxwell electromagnetic invariant as

$$F_{\mu\nu} F^{\mu\nu} = -\frac{2Q^2}{r^2}. \tag{33}$$

We note that, although the gauge field A_μ and the Faraday tensor $F_{\mu\nu}$ become pure imaginary if a condition $-1 < \omega < 1$ is not satisfied, the energy-momentum tensor $T_{\mu\nu}$ remains real even in such a case. With the orthonormal basis one-forms (5), the nonzero components of $F_{(a)(b)}$ are

$$\begin{aligned} \mathcal{E} &:= F_{(0)(1)} = -F_{(1)(0)} = \frac{Q}{\sqrt{1-\omega^2} R}, \\ \mathcal{B} &:= F_{(2)(1)} = -F_{(1)(2)} = -\omega \sqrt{\frac{f}{-\Lambda r^2}} \mathcal{E}, \end{aligned} \tag{34}$$

where $f(r)$ is given by Eq. (29). As they do not satisfy the relation in Eq. (26), the charged rotating BTZ solution is neither self-dual nor anti-self-dual. We note that the solution is purely electric in the nonrotating limit $\omega \rightarrow 0$. In contrast, the CCS solution reduces to the massless BTZ *vacuum* solution in the nonrotating limit given by $D \rightarrow 0$ and $C_0 \rightarrow 0$.

The KGBD mass and the KGBD angular momentum are given by

$$\begin{aligned}
 m &= \frac{1 + \omega^2}{8G(1 - \omega^2)} \left(\frac{1}{2} r f' - f \right) + \frac{\pi Q^2}{2(1 - \omega^2)} \left(1 + \frac{\omega^2}{2\Lambda r} f' \right) \\
 &= \frac{1 + \omega^2}{8G(1 - \omega^2)} (M + 8\pi G Q^2 \ln r) - \frac{\pi \omega^2 Q^2}{1 - \omega^2} \left(1 + \frac{2\pi G Q^2}{\Lambda r^2} \right), \tag{35}
 \end{aligned}$$

$$\begin{aligned}
 j &= \frac{\omega}{4G\sqrt{-\Lambda}(1 - \omega^2)} \left(\frac{1}{2} r f' - f \right) \\
 &= \frac{\omega}{4G\sqrt{-\Lambda}(1 - \omega^2)} (M + 8\pi G Q^2 \ln r - 4\pi G Q^2). \tag{36}
 \end{aligned}$$

We note that m and j are nonconstant in general for $Q \neq 0$. We also observe that $j = 0$ is possible for $\omega = 0$ with $Q \neq 0$.

In Ref. [20], the authors identified the global mass M_{global} , charge Q_{global} , and angular momentum J_{global} as conserved quantities by the Regge–Teitelboim Hamiltonian approach. In the units such that $8\pi G = \kappa = 1/2$ and $-\Lambda = 1$, our parameters M and Q are identical to \tilde{M} and \tilde{Q} in Ref. [20], respectively, and hence M_{global} , J_{global} , and Q_{global} are given by

$$\begin{aligned}
 M_{\text{global}} &= \frac{1}{1 - \omega^2} \left((1 + \omega^2)M - \frac{1}{2}\omega^2 Q^2 \right), \\
 J_{\text{global}} &= \frac{2\omega}{1 - \omega^2} \left(M - \frac{1}{4}Q^2 \right), \quad Q_{\text{global}} = \frac{Q}{\sqrt{1 - \omega^2}}. \tag{37}
 \end{aligned}$$

With $8\pi G = 1/2$ and $-\Lambda = 1$, the KGBD mass (35) and the KGBD angular momentum (36) become

$$m = 2\pi \left[\frac{1}{1 - \omega^2} \left((1 + \omega^2)M - \frac{1}{2}\omega^2 Q^2 \right) + \frac{1 + \omega^2}{2(1 - \omega^2)} Q^2 \ln r + \frac{\omega^2 Q^4}{16(1 - \omega^2)r^2} \right], \tag{38}$$

$$j = 2\pi \left[\frac{2\omega}{1 - \omega^2} \left(M - \frac{1}{4}Q^2 \right) + \frac{\omega}{1 - \omega^2} Q^2 \ln r \right]. \tag{39}$$

Interestingly, up to the normalization factor 2π , the *constant terms* in the right-hand sides are *exactly* M_{global} and J_{global} , respectively.

Clément’s charged rotating BTZ solution (29) admits an *extremal Killing horizon* if M and Q satisfy the relation

$$M = 4\pi G Q^2 \left[1 - \ln \left(\frac{4\pi G Q^2}{-\Lambda} \right) \right]. \tag{40}$$

The location of the extremal horizon $r = r_{\text{ex}}$ is determined by $f(r_{\text{ex}}) = f'(r_{\text{ex}}) = 0$ such as

$$r_{\text{ex}} = \sqrt{\frac{4\pi G Q^2}{-\Lambda}}. \tag{41}$$

Evaluating Eqs. (35) and (36) on $r = r_{\text{ex}}$, we obtain

$$m(r_{\text{ex}}) = \frac{\pi Q^2}{2(1 - \omega^2)}, \quad j(r_{\text{ex}}) = 0. \tag{42}$$

In particular, the KGBD quasi-local angular momentum j vanishes on the extremal horizon.

Unfortunately, the charged rotating BTZ solution in the Clément form (29) *cannot treat the extremal case in vacuum*. In the uncharged case $Q = 0$, the metric in Eq. (4) with Eq. (29)

becomes

$$\begin{aligned}
 ds^2 &= -F dt^2 + F^{-1} dR^2 + R^2 \left(d\theta - \frac{M\omega}{\sqrt{-\Lambda}(1-\omega^2)R^2} dt \right)^2, \\
 F(R) &:= \frac{r^2}{R^2} f = -\Lambda R^2 - \frac{1+\omega^2}{1-\omega^2} M + \frac{M^2\omega^2}{(-\Lambda)(1-\omega^2)^2 R^2} \\
 &= \frac{[M + \Lambda(1-\omega^2)R^2][M\omega^2 + \Lambda(1-\omega^2)R^2]}{(-\Lambda)(\omega^2 - 1)^2 R^2}
 \end{aligned} \tag{43}$$

in the coordinates (t, R, θ) , which is locally maximally symmetric and identical to the metric in Eq. (11) with Eq. (45) below.

The solution admits two Killing horizons (at most) in the region $y > 0$ (recall that $y := R^2$), and their locations $y = y_1$ and $y = y_2$ are the roots of $F(R) = 0$, namely

$$y_1 = \frac{M\omega^2}{(-\Lambda)(1-\omega^2)}, \quad y_2 = \frac{M}{(-\Lambda)(1-\omega^2)}. \tag{44}$$

As $\omega^2 \neq 1$ is assumed in the Clément form, the extremal case $y_1 = y_2$ cannot be treated.

Let us show this fact differently. In the uncharged case $Q = 0$, the KGBD mass (35) and angular momentum (36) for Clément’s charged rotating BTZ solution reduce to the following constants:

$$m = \frac{1+\omega^2}{8G(1-\omega^2)} M, \quad j = \frac{M\omega}{4G\sqrt{-\Lambda}(1-\omega^2)}. \tag{45}$$

The above equations are solved for M and ω to give

$$M = \mp 8G\sqrt{m^2 + \Lambda j^2}, \quad \omega = \frac{m \pm \sqrt{m^2 + \Lambda j^2}}{\sqrt{-\Lambda} j}. \tag{46}$$

Under the extremality condition (13), namely $m = \pm\sqrt{-\Lambda} j$, we obtain

$$M = 0, \quad \omega = \pm 1, \tag{47}$$

where the latter is not allowed in the Clément form. In fact, m and j are then *undetermined* in Eq. (45). Hence, the extreme case in vacuum cannot be treated properly under the parametrization (43).

This is also the case under a new parametrization of the charged rotating BTZ solution for a wider range of Λ introduced in Ref. [6]. In Ref. [6], the gauge field and the metric functions are written as

$$A_\mu dx^\mu = -Q \ln r (dt - a d\theta), \tag{48}$$

$$R(r) = \sqrt{\zeta r^2 + a^2(M + \kappa Q^2 \ln r)}, \tag{49}$$

$$f(r) = -\Lambda r^2 - M - \kappa Q^2 \ln r, \tag{50}$$

$$h(r) = -a(M + \kappa Q^2 \ln r), \tag{51}$$

which are parametrized by M , a , and Q . While M and Q are the same as in the Clément form, the rotation parameter a and the constant ζ are related to ω as

$$a = \frac{\omega}{\sqrt{-\Lambda}(1-\omega^2)}, \quad \zeta = \frac{1}{1-\omega^2}. \tag{52}$$

Those two constants satisfy

$$\zeta^2 - \zeta + a^2 \Lambda = 0, \tag{53}$$

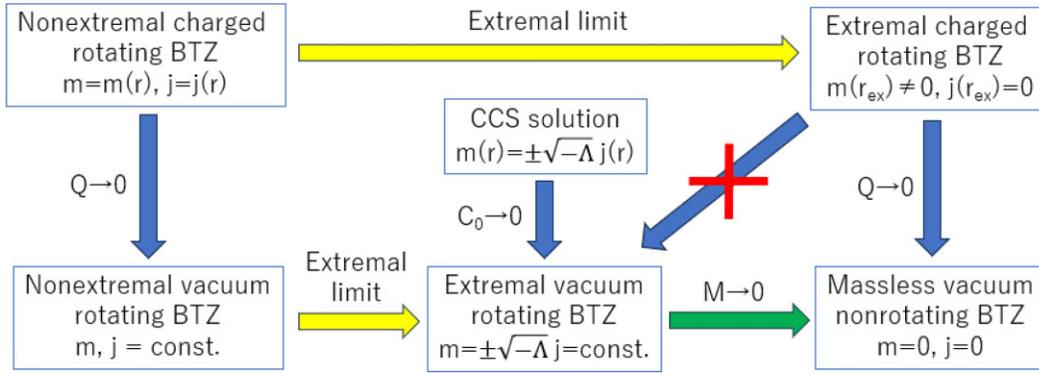


Fig. 1. Various limits from Clément’s charged rotating BTZ solution, and from the CCS solution.

which shows

$$\zeta = \frac{1}{2} \left(1 \pm \sqrt{1 - 4a^2\Lambda} \right). \tag{54}$$

In this new parametrization, there are two branches of solutions depending on the sign in Eq. (54). In the uncharged case ($Q = 0$) for $\zeta \neq 0$, one can write both branches of solutions in the same form as

$$ds^2 = -F d\bar{t}^2 + F^{-1} dR^2 + R^2 \left(d\theta - \frac{\zeta a M}{R^2} d\bar{t} \right)^2, \tag{55}$$

$$F(R) = -\Lambda R^2 - \zeta(2\zeta - 1)M + \frac{\zeta^2 a^2 M^2}{R^2},$$

where $\bar{t} := t/\zeta$. The solution admits two Killing horizons (at most) in the region of $y > 0$, and their locations $y = y_{\pm}$ are roots of $F(R) = 0$, namely

$$y_{\pm} = \frac{\zeta M(2\zeta - 1 \pm 1)}{2(-\Lambda)}, \tag{56}$$

where we have used Eq. (53). Clearly, the extremal case $y_+ = y_-$ is not possible for $\zeta M \neq 0$.

In the uncharged case $Q = 0$, the KGBD mass (35) and angular momentum (36) under the new parametrization become

$$m = \frac{\zeta(2\zeta - 1)}{8G} M, \quad j = \frac{\zeta}{4G} aM, \tag{57}$$

which give

$$m - \varepsilon\sqrt{-\Lambda} j = \frac{\zeta M}{8G} (2\zeta - 1 - 2\varepsilon a\sqrt{-\Lambda}). \tag{58}$$

As the bracket in the right-hand side cannot be zero due to Eq. (54), the extremality condition (13), namely $m = \varepsilon\sqrt{-\Lambda}j$, with $\varepsilon = \pm 1$, in the uncharged case is not satisfied for $\zeta M \neq 0$. Hence, the *extremal case in vacuum cannot be treated in the new parametrization as well*.

The results obtained in this subsection are summarized in Fig. 1. We note that, unlike the extremal charged rotating BTZ solution in three dimensions, the extremal Kerr–Newman–AdS solution in four dimensions reduces to the extremal Kerr-AdS solution in the uncharged limit $Q \rightarrow 0$.

3. Properties of the CCS solution

In this section, we investigate geometrical and physical properties of the CCS solution in detail to give its physical interpretation.

3.1. Geodesics and locally AdS infinity

In the original coordinate system (t, r, θ) with the metric in Eq. (21), we consider the domain of r given by $r \in (0, \infty)$, in which the metric component g_{rr} or g^{rr} diverges as $r \rightarrow 0$ or degenerates for any value of D . Actually, it is a coordinate singularity for $D \neq 0$ and can be removed by introducing a new radial coordinate $y := r^2$. In the new coordinates (t, y, θ) , the metric (21) and the gauge field (19) are written as

$$\begin{aligned}
 ds^2 &= (-\Lambda)[S - 2(y - D)] dt^2 + 2\epsilon\sqrt{-\Lambda} [S - (y - D)] dt d\theta \\
 &\quad + S d\theta^2 + \frac{dy^2}{4(-\Lambda)(y - D)^2}, \\
 A_\mu dx^\mu &= \frac{1}{2} C_0 \ln |y - D| \left(\epsilon dt + \frac{d\theta}{\sqrt{-\Lambda}} \right),
 \end{aligned} \tag{59}$$

where

$$S(y) := y + \frac{\kappa C_0^2}{2(-\Lambda)} \ln |y - D| \quad [= R^2(r(y))]. \tag{60}$$

The nonzero components of the inverse metric are

$$\begin{aligned}
 g^{tt} &= \frac{S}{\Lambda(y - D)^2}, & g^{t\theta} &= \epsilon \frac{S - (y - D)}{\sqrt{-\Lambda}(y - D)^2}, \\
 g^{\theta\theta} &= -\frac{S - 2(y - D)}{(y - D)^2}, & g^{yy} &= 4(-\Lambda)(y - D)^2.
 \end{aligned} \tag{61}$$

Because for $C_0 = 0$ the metric and its inverse are both analytic at $y = 0$ at $D \neq 0$, the spacetime defined in the domain $y \geq 0$ can be analytically extended beyond $y = 0$ into the domain $y < 0$ in that case. In contrast, as shown in Section 3.2 below, in the *nonvacuum case* for $C_0 \neq 0$, $y = D$ is not a coordinate singularity but a *p.p. scalar curvature singularity*. Consequently, the domain of y in the coordinate system (59) is $y \in (D, \infty)$.

For later use, we derive geodesic equations in the coordinates (t, y, θ) . Consider an affinely parametrized geodesic γ represented as $x^\mu = x^\mu(\lambda)$ with its tangent vector v^μ (given by $dx^\mu/d\lambda$), where λ is an affine parameter along γ in the coordinate system (59). As the spacetime admits two Killing vectors, namely $\xi^\mu = (\partial/\partial t)^\mu$ and $\Theta^\mu = (\partial/\partial \theta)^\mu$, there are conserved quantities $E := -\xi_\mu v^\mu$ and $L := \Theta_\mu v^\mu$ along γ . Using them and the normalization $\varepsilon = v_\mu v^\mu$, where $\varepsilon = -1, 0$, and 1 , corresponds to timelike, null, and spacelike γ , respectively, we can write down the geodesic equations as

$$\begin{aligned}
 \dot{t} &= \frac{\epsilon\sqrt{-\Lambda}L(S - y + D) + ES}{(-\Lambda)(y - D)^2}, \\
 \dot{y}^2 &= 4\varepsilon(-\Lambda)(y - D)^2 + 4(E + \epsilon\sqrt{-\Lambda}L)[(E + \epsilon\sqrt{-\Lambda}L)S - 2\epsilon\sqrt{-\Lambda}L(y - D)], \\
 \dot{\theta} &= \frac{-\epsilon\sqrt{-\Lambda}L[S - 2(y - D)] - E(S - y + D)}{\epsilon\sqrt{-\Lambda}(y - D)^2},
 \end{aligned} \tag{62}$$

where a dot denotes differentiation with respect to λ . (Notice also the difference between ε and ϵ .)

Since $\lim_{y \rightarrow \infty} R^{\mu\nu}_{\rho\sigma} = \Lambda(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)$ is satisfied, the CCS spacetime is asymptotically (at least) locally AdS near the coordinate infinity $y \rightarrow \infty$. In fact, null geodesics ($\varepsilon = 0$) with $E^2 > -\Lambda L^2$ or $E = -\epsilon\sqrt{-\Lambda}L$ and timelike geodesics ($\varepsilon = -1$) cannot reach $y \rightarrow \infty$ because the right-hand side of the radial geodesic equation (62) becomes negative in the limit $y \rightarrow \infty$.

Along other geodesics, we obtain

$$\dot{y}^2 \simeq \begin{cases} 4(E^2 + \Lambda L^2)y & (\varepsilon = 0, E^2 > -\Lambda L^2) \\ 8\kappa C_0^2 E^2 \ln|y - D|/(-\Lambda) & (\varepsilon = 0, E = \epsilon\sqrt{-\Lambda}L \neq 0) \\ 4(-\Lambda)y^2 & (\varepsilon = 1) \end{cases} \quad (63)$$

near $y \rightarrow \infty$. In the first and the third cases, Eq. (63) is integrated to give

$$y(\lambda) \simeq \begin{cases} (E^2 + \Lambda L^2)(\lambda - \lambda_0)^2 & (\varepsilon = 0, E^2 > -\Lambda L^2) \\ e^{2\sqrt{-\Lambda}(\lambda - \lambda_0)} & (\varepsilon = 1) \end{cases}, \quad (64)$$

where λ_0 is a constant. Hence, $y \rightarrow \infty$ corresponds to an *infinite* affine parameter $\lambda \rightarrow \infty$. It is also true in the second case in Eq. (63) for $\varepsilon = 0$ with $E = \epsilon\sqrt{-\Lambda}L \neq 0$ because the left-hand side blows up in the following inequality:

$$\lim_{y \rightarrow D} \int^y \frac{dy}{y} < \lim_{y \rightarrow D} \int^y \frac{dy}{\sqrt{\ln y}}. \quad (65)$$

Therefore, the asymptotically locally AdS region $y \rightarrow \infty$ corresponds to both *spacelike and null infinities*.

Lastly, we can also show that the infinity $y \rightarrow \infty$ is *causally timelike* by the conformal compactification of spacetime. The line element of the CCS spacetime $(\mathcal{M}_3, g_{\mu\nu})$ described by the metric $g_{\mu\nu}$ given by Eq. (59) can be written as $ds^2 = \Omega^{-2}d\bar{s}^2$ with a conformal factor $\Omega = S^{-1/2}$ that satisfies $\lim_{y \rightarrow \infty} \Omega = 0$. Here $d\bar{s}^2$ is the line element of the conformally compactified spacetime $(\bar{\mathcal{M}}_3, \bar{g}_{\mu\nu})$ given by

$$d\bar{s}^2 = (\epsilon d\theta + \sqrt{-\Lambda} dt) \left[\epsilon d\theta + \sqrt{-\Lambda} \left(1 - \frac{2(y(y_*) - D)}{S(y(y_*))} \right) dt \right] + dy_*^2, \quad (66)$$

$$y_* := \int^y \frac{dy}{2\sqrt{-\Lambda}(y - D)\sqrt{S}}.$$

In the asymptotically locally AdS region $y \rightarrow \infty$, $(\bar{\mathcal{M}}_3, \bar{g}_{\mu\nu})$ which shares the same light-cone structure with $(\mathcal{M}_3, g_{\mu\nu})$ is asymptotically flat,

$$d\bar{s}^2|_{y \rightarrow \infty} = -(-\Lambda)dt^2 + d\theta^2 + dy_*^2. \quad (67)$$

Such a boundary $y \rightarrow \infty$ is causally timelike because it corresponds to a finite value of y_* , as shown by

$$\lim_{y \rightarrow \infty} y_* \simeq \lim_{y \rightarrow \infty} \int^y \frac{dy}{2\sqrt{-\Lambda}y^{3/2}} = C_\infty - \frac{1}{\sqrt{-\Lambda}} \lim_{y \rightarrow \infty} y^{-1/2} = C_\infty, \quad (68)$$

where C_∞ is an integration constant.

3.2. Curvature singularity at $r^2(\equiv y) = D$

Here we investigate the properties of $y = D$. We first show that $y = D$ is *causally null* by the conformal compactification of spacetime. Since S is negative near $y = D$, we write the line element of the CCS solution (59) as $ds^2 = \Omega^{-2}d\bar{s}^2$, but now with a conformal factor $\Omega = (-S)^{-1/2}$ that also satisfies $\lim_{y \rightarrow D} \Omega = 0$. The line element $d\bar{s}^2$ of such conformally compactified spacetime is

$$d\bar{s}^2 = -(\epsilon d\theta + \sqrt{-\Lambda} dt) \left[\epsilon d\theta + \sqrt{-\Lambda} \left(1 - \frac{2(y(y_*) - D)}{S(y(y_*))} \right) dt \right] + dy_*^2, \quad (69)$$

$$y_* := \int^y \frac{dy}{2\sqrt{-\Lambda}(y - D)\sqrt{-S}}.$$

Near $y = D$, it reduces to

$$d\bar{s}^2|_{y \rightarrow D} \simeq -(\epsilon d\theta + \sqrt{-\Lambda} dt)^2 + dy_*^2, \tag{70}$$

which is 2D. The boundary $y = D$ is causally null because it corresponds to $|y_*| \rightarrow \infty$, shown by

$$\lim_{y \rightarrow D} |y_*| \simeq \lim_{y \rightarrow D} \left| \int^y \frac{dy}{\sqrt{2\kappa C_0^2(y-D)}\sqrt{-\ln|y-D|}} \right| = \lim_{y \rightarrow D} \sqrt{\frac{-2 \ln|y-D|}{\kappa C_0^2}} \rightarrow \infty. \tag{71}$$

In Ref. [5], Clément showed that only a particular class of *spacelike* geodesics can reach $y = D$, and it corresponds to an *infinite affine distance*. Because the right-hand side of Eq. (62) becomes negative as $y \rightarrow D$ unless $E = -\epsilon\sqrt{-\Lambda}L$ due to $\lim_{y \rightarrow D} S \rightarrow -\infty$, geodesics with $E \neq -\epsilon\sqrt{-\Lambda}L$ cannot reach $y = D$. With $E = -\epsilon\sqrt{-\Lambda}L$, the geodesic equations (62) for in-going γ reduce to

$$\dot{t} = \frac{E}{(-\Lambda)(y-D)}, \quad \dot{\theta} = -\frac{E}{\epsilon\sqrt{-\Lambda}(y-D)}, \quad \dot{y} = -2\sqrt{\epsilon(-\Lambda)}(y-D), \tag{72}$$

which show that only spacelike geodesics ($\epsilon = 1$) with $E = -\epsilon\sqrt{-\Lambda}L$ can arrive at $y = D$. Along it, $R_{\mu\nu}v^\mu v^\nu = 2\Lambda$ is satisfied, and the radial geodesic equation is integrated to give

$$y(\lambda) = D + e^{-2\sqrt{-\Lambda}(\lambda-\lambda_0)}, \tag{73}$$

where λ_0 is a constant. The above expression shows that $y = D$ corresponds to an infinite affine parameter $\lambda \rightarrow \infty$ along such a spacelike γ . For this reason, Clément concluded that the CCS solution is perfectly regular [5]. However, we will show that $y = D$ is not a regular spacelike infinity but a *p.p. curvature singularity*.

It is known that curvature singularities are classified into two main categories [21]. Although a scalar polynomial (s.p.) curvature singularity is usually examined, it may miss a p.p. curvature singularity. An s.p. curvature singularity is defined by the blowing up of a scalar, formed as a polynomial in the curvature tensor, such as the Ricci scalar \mathcal{R} and the Kretschmann scalar $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$. On the other hand, a p.p. curvature singularity is defined by the blowing up of a component of the Riemann tensor in a p.p. (pseudo-)orthonormal frame $\mathcal{R}_{(a)(b)(c)(d)} := \mathcal{R}_{\mu\nu\rho\sigma} E_{(a)}^\mu E_{(b)}^\nu E_{(c)}^\rho E_{(d)}^\sigma$ with basis vectors $E_{(a)}^\mu$ along a curve. An s.p. curvature singularity is a p.p. curvature singularity but the latter is not always the former. (See Section 3 in Ref. [22].) In fact, the following curvature invariants of the CCS solution (59) are constant:

$$\begin{aligned} \mathcal{R} &= 6\Lambda, & \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} &= \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} = 12\Lambda^2, \\ \mathcal{R}_\mu{}^\nu\mathcal{R}_\nu{}^\rho\mathcal{R}_\rho{}^\mu &= 24\Lambda^3, & (\nabla_\rho\mathcal{R}_{\mu\nu})(\nabla^\rho\mathcal{R}^{\mu\nu}) &= 0. \end{aligned} \tag{74}$$

Nevertheless, as shown below, $y = D$ in the CCS spacetime (59) is a p.p. curvature singularity.

We consider a spacelike geodesic given by Eq. (72) with $E = 0 = L$, of which the tangent vector is

$$v^\mu \partial_\mu = -2\sqrt{-\Lambda}(y-D) \partial_y. \tag{75}$$

We also introduce p.p. basis vectors $\{E_{(0)}^\mu, E_{(1)}^\mu, E_{(2)}^\mu\}$ with $E_{(2)}^\mu = v^\mu$ along the geodesic as

$$\begin{aligned} E_{(0)}^\mu \partial_\mu &= \frac{1}{2\sqrt{-\Lambda}(y-D)^{3/2}} \left[-(S+y-D) \partial_t + \epsilon\sqrt{-\Lambda}(S-y+D) \partial_\theta \right], \\ E_{(1)}^\mu \partial_\mu &= \frac{1}{2\sqrt{-\Lambda}(y-D)^{3/2}} \left[-(S-y+D) \partial_t + \epsilon\sqrt{-\Lambda}[S-3(y-D)] \partial_\theta \right], \\ E_{(2)}^\mu \partial_\mu &= -2\sqrt{-\Lambda}(y-D) \partial_y, \end{aligned} \tag{76}$$

which satisfy $E^{(a)\mu}E_{(b)\mu} = \text{diag}(-1, 1, 1)$ and $E_{(2)}^\nu \nabla_\nu E_{(a)}^\mu = 0$ for $a = 0, 1, 2$. Then, the following orthonormal components of the Riemann tensor diverge as $y \rightarrow D$ along η for $C_0 \neq 0$:

$$R_{(0)(2)(0)(2)} = -\Lambda + \frac{\kappa C_0^2}{y - D}, \quad R_{(1)(2)(1)(2)} = \Lambda + \frac{\kappa C_0^2}{y - D}, \quad R_{(0)(2)(1)(2)} = \frac{\kappa C_0^2}{y - D}. \quad (77)$$

Therefore, $y = D$ is not a spacelike infinity but a p.p. scalar curvature singularity corresponding to an infinite affine parameter λ .

To summarize, we have shown the following properties of $y = D$:

1. It is causally null.
2. Among all geodesics, only spacelike geodesics with $E = -\epsilon\sqrt{-\Lambda}L$ can reach $y = D$, corresponding to an infinite affine parameter.
3. Some components of $\mathcal{R}_{(a)(b)(c)(d)}$ blow up as $y \rightarrow D$ along a spacelike geodesic with $E = 0 = L$.

By property 2, $y = D$ is not a null infinity nor a timelike infinity. By property 3, $y = D$ is a spacelike infinity, and also a p.p. curvature singularity. However, such a singularity may be harmless because (i) it is not naked for any observer in the region of $y > D$, and (ii) no free-falling causal observer arrives there.

In spite of the fact that no causal geodesic reaches $y = D$, causal curves could.³ However, any causal curve $t = t(y)$ with constant θ does not reach $y = D$ with a finite value of t . For such curves, from Eq. (59) we obtain

$$\left(\frac{dt}{dy}\right)^2 \geq \frac{1}{4(-\Lambda)^2(y - D)^2[-S + 2(y - D)]} \quad (78)$$

with equality holding for null curves. For a null curve, we get

$$\lim_{y \rightarrow D} t \simeq \pm \lim_{y \rightarrow D} \sqrt{\frac{-2 \ln |y - D|}{(-\Lambda)\kappa C_0^2}} \rightarrow \pm\infty. \quad (79)$$

Since the right-hand side of Eq. (78) blows up as $y \rightarrow D$, any causal curve with constant θ does not reach $y = D$ with a finite value of t .

Similarly, it can be shown that any causal curve $\theta = \theta(y)$ with constant t does not reach $y = D$ with a finite value of θ . Indeed for such curves, we obtain

$$\left(\frac{d\theta}{dy}\right)^2 \geq \frac{1}{4(-\Lambda)(y - D)^2(-S)} \quad (80)$$

with equality holding for null curves. For a null curve, we obtain

$$\lim_{y \rightarrow D} \theta \simeq \pm \lim_{y \rightarrow D} \sqrt{\frac{-2 \ln |y - D|}{\kappa C_0^2}} \rightarrow \pm\infty. \quad (81)$$

Since the right-hand side of Eq. (80) blows up as $y \rightarrow D$, any causal curve with constant t does not reach $y = D$ with a finite value of θ . Generally, there is no null curve $y = y(t, \theta)$ that arrives at $y = D$ with finite values of t and θ because we obtain

$$\lim_{y \rightarrow D} |\epsilon \theta + \sqrt{-\Lambda} t| \simeq \lim_{y \rightarrow D} \int^y \frac{dy}{2\sqrt{-\Lambda}(y - D)\sqrt{-S}} = \lim_{y \rightarrow D} \sqrt{\frac{-2 \ln |y - D|}{\kappa C_0^2}} \rightarrow \infty \quad (82)$$

³For example, in the 5D vacuum spacetime obtained in Ref. [23], there exist timelike curves, corresponding to accelerated timelike observers, that arrive at a wormhole throat in spite of the fact that no causal geodesic arrives there.

along such curves, cf. Eqs. (70) and (71). We note that a curve satisfying $\epsilon \theta + \sqrt{-\Lambda} t = 0$ is spacelike because $ds^2 = dy^2/[4(-\Lambda)(y - D)^2] > 0$ is satisfied along it.

The above facts (do not prove but strongly) suggest that there is no causal curve reaching $y = D$, and therefore the CCS spacetime cannot be extended along such curves. We thus conclude that the CCS solution with the nonvanishing (anti-)self-dual Maxwell field does *not* admit a (regular) horizon of a black hole (at $y \equiv r^2 = D$), as opposed to the claim in Ref. [8] that the solution describes an extremal black hole with mass, angular momentum, and electric charge.

3.3. Causality violations

Here we study causality in the CCS spacetime under the assumption that θ is a periodic coordinate. First, we consider the standard identification $(t, y, \theta) = (t, y, \theta + 2\pi)$ in the coordinate system (59). Then, as $g_{\theta\theta} < 0$ holds near the curvature singularity $y = D$ due to $\lim_{y \rightarrow D} S \rightarrow -\infty$, the Killing vector $\Theta^\mu = (\partial/\partial\theta)^\mu$ becomes timelike and therefore there are closed timelike curves near $y = D$.

In fact, under a different way of identification, the CCS spacetime admits closed null geodesics *everywhere*. To show it, we introduce a null coordinate defined by

$$u := \sqrt{-\Lambda} t + \epsilon \theta \tag{83}$$

and write the CCS solution (59) in the new coordinates (u, y, θ) as

$$ds^2 = -[2(y - D) - S] du^2 + 2\epsilon(y - D) du d\theta + \frac{dy^2}{4(-\Lambda)(y - D)^2},$$

$$A_\mu dx^\mu = \frac{\epsilon C_0}{2\sqrt{-\Lambda}} \ln |y - D| du. \tag{84}$$

Since $g^{\mu\nu}(\nabla_\mu u)(\nabla_\nu u) = g^{\mu\mu} = 0$ holds, $u = u_0$ is a null hypersurface. In fact, the constant u_0 labels privileged null hypersurfaces in the Kundt family, as shown in Eqs. (92) and (105) below. Also, as $g_{\theta\theta} = 0$ holds, the Killing vector $\Phi^\mu = (\partial/\partial\theta)^\mu (\neq \Theta^\mu)$ is null everywhere.

As the new coordinate system (84) admits the Killing vectors $\zeta^\mu = (\partial/\partial u)^\mu$ and $\Phi^\mu = (\partial/\partial\theta)^\mu$, $\bar{E} := -g_{\mu\nu}\zeta^\mu v^\nu$, $\bar{L} := g_{\mu\nu}\Phi^\mu v^\nu$, and $\epsilon = g_{\mu\nu}v^\mu v^\nu$ are conserved along a geodesic γ with its tangent vector v^μ , where $\epsilon = -1, 0, 1$ corresponds to timelike, null, and spacelike γ , respectively. Hence, geodesic equations for γ are given by

$$\dot{u} = \frac{\epsilon \bar{L}}{y - D},$$

$$\dot{y}^2 = 4\epsilon(-\Lambda)(y - D)^2 - 4(-\Lambda)\bar{L}[2(\bar{L} - \epsilon\bar{E})(y - D) - \bar{L}S], \tag{85}$$

$$\dot{\theta} = \frac{-\epsilon\bar{E}(y - D) + \bar{L}[2(y - D) - S]}{(y - D)^2}.$$

For null geodesics ($\epsilon = 0$) with $\bar{L} = 0$, the above geodesic equations are easily integrated to give

$$u = u_0, \quad y = y_0, \quad \theta = -\frac{\epsilon\bar{E}}{y_0 - D} \lambda + \theta_0, \tag{86}$$

where u_0, y_0 , and θ_0 are constants. Hence, if we identify $(u, y, \theta) = (u, y, \theta + 2\pi)$ in the coordinate system (84), the CCS spacetime admits closed null geodesics *everywhere*, which are described by Eq. (86) with $\bar{E} \neq 0$.

Such causal pathology is circumvented by the reinterpretation of the CCS solution in terms of the related (Kundt-type) coordinate r_k , which is introduced below in Eq. (102) and is noncyclic.

Actually, it is an affine parameter along the privileged quadruple-degenerate Cotton-aligned null direction (CAND) k^μ , as shown in Eq. (105) below.

3.4. Cotton and Maxwell algebraic types

Next, we determine the Cotton type of the CCS spacetime (59), using the method recently developed in Refs. [24,25]. It uses five real Cotton scalars Ψ_A ($A = 0, 1, \dots, 4$) defined by

$$\begin{aligned}\Psi_0 &:= C_{\mu\nu\rho} k^\mu m^\nu k^\rho, \\ \Psi_1 &:= C_{\mu\nu\rho} k^\mu l^\nu k^\rho, \\ \Psi_2 &:= C_{\mu\nu\rho} k^\mu m^\nu l^\rho, \\ \Psi_3 &:= C_{\mu\nu\rho} l^\mu k^\nu l^\rho, \\ \Psi_4 &:= C_{\mu\nu\rho} l^\mu m^\nu l^\rho,\end{aligned}\tag{87}$$

where $C_{\mu\nu\rho}$ is the Cotton tensor [26], and a null triad $\{k^\mu, l^\mu, m^\mu\}$ is properly normalized such that

$$k_\mu k^\mu = 0 = l_\mu l^\mu, \quad k_\mu l^\mu = -1, \quad m_\mu m^\mu = 1, \quad k_\mu m^\mu = 0 = l_\mu m^\mu.\tag{88}$$

Following the convention of Ref. [27], we define the Cotton tensor as

$$C_{\mu\nu\rho} := 2(\nabla_{[\mu} R_{\nu]\rho} - \frac{1}{4}\nabla_{[\mu} R g_{\nu]\rho}),\tag{89}$$

which automatically satisfies the constraints $C_{(\mu\nu)\rho} \equiv 0$, $C_{[\mu\nu\rho]} \equiv 0$, and $C_{\mu\nu}{}^\mu \equiv 0$. The Cotton scalars Ψ_A are 3D counterparts of the Newman–Penrose complex Weyl scalars of 4D gravity [28]. Moreover, for the algebraic classification, it is then convenient to employ the scalar polynomial invariants [24,25]:

$$\begin{aligned}I &:= \Psi_0\Psi_4 - 2\Psi_1\Psi_3 - 3\Psi_2^2, \\ J &:= 2\Psi_0\Psi_2\Psi_4 + 2\Psi_1\Psi_2\Psi_3 + 2\Psi_2^3 + \Psi_0\Psi_3^2 - \Psi_4\Psi_1^2, \\ G &:= \Psi_1\Psi_4^2 - 3\Psi_2\Psi_3\Psi_4 - \Psi_3^3, \\ H &:= 2\Psi_2\Psi_4 + \Psi_3^2, \\ N &:= 3H^2 + \Psi_4^2 I.\end{aligned}\tag{90}$$

As the metric (59) can be written as

$$ds^2 = -(\sqrt{-\Lambda} dt + \epsilon d\theta)[2\sqrt{-\Lambda}(y - D) dt - S(\sqrt{-\Lambda} dt + \epsilon d\theta)] + \frac{dy^2}{4(-\Lambda)(y - D)^2},\tag{91}$$

a natural triad $\{k_\mu, l_\mu, m_\mu\}$ in the coordinate system (59) satisfying Eq. (88) is given by the one-forms

$$\begin{aligned}k_\mu dx^\mu &= -\frac{1}{\sqrt{2}}(\sqrt{-\Lambda} dt + \epsilon d\theta), \\ l_\mu dx^\mu &= -\frac{1}{\sqrt{2}}(2\sqrt{-\Lambda}(y - D) dt - S(\sqrt{-\Lambda} dt + \epsilon d\theta)), \\ m_\mu dx^\mu &= \frac{dy}{2\sqrt{-\Lambda}(y - D)},\end{aligned}\tag{92}$$

of which the contravariant components are

$$\begin{aligned} k^\mu \partial_\mu &= \frac{1}{\sqrt{2}(y-D)} \left(\frac{1}{\sqrt{-\Lambda}} \partial_t - \epsilon \partial_\theta \right), \\ l^\mu \partial_\mu &= \frac{1}{\sqrt{2}(y-D)} \left(\frac{S}{\sqrt{-\Lambda}} \partial_t - \epsilon [S - 2(y-D)] \partial_\theta \right), \\ m^\mu \partial_\mu &= 2\sqrt{-\Lambda} (y-D) \partial_y. \end{aligned} \tag{93}$$

The null vector k^μ satisfies the geodesic conditions $k^\nu \nabla_\nu k^\mu = 0$, whereas l^μ does not. Also, as $k_\mu = -(\nabla_\mu u)/\sqrt{2}$ is satisfied, where u is defined by Eq. (83), k^μ is a normal vector of a null hypersurface given by $u = \text{constant}$.

The Cotton scalars (87) with respect to such a null triad are

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \quad \Psi_4 = 16\pi G \sqrt{-\Lambda} C_0^2, \tag{94}$$

which give the identically vanishing invariants,

$$I = J = G = H = N = 0. \tag{95}$$

Therefore, according to the flow diagram in Figure 1 in Ref. [24], or equivalently in Ref. [25], the *CCS spacetime is of Cotton type N everywhere* (unless we consider the vacuum case $C_0 = 0$ which is conformally flat, i.e. of type O). This is in striking contrast to the large class of charged rotating BTZ black holes which are of Cotton type I away from the horizon and type III on the horizon. (See Corollary 1 in Ref. [6].)

As generally explained in detail in Ref. [25], by a suitable choice of the triad the Cotton scalars for any type N spacetime can be put into the canonical form in which *only the scalar Ψ_4 is nonzero*. It is the case of Eq. (94), and this explicitly demonstrates that the null vector $k^\mu \partial_\mu$ given by Eq. (93) is the *quadruple Cotton-aligned null direction (CAND)* of the type N CCS spacetime. Moreover, Eq. (93) is the *principal null triad*. (See Refs. [24,25] for more details on the definition and multiplicity of CAND.)

Analogous (Newman–Penrose) scalars for the Maxwell field are computed to give

$$\begin{aligned} \phi_0 &:= F_{\mu\nu} k^\mu m^\nu = 0, \\ \phi_1 &:= F_{\mu\nu} k^\mu l^\nu = 0, \\ \phi_2 &:= F_{\mu\nu} m^\mu l^\nu = \epsilon \sqrt{2} C_0. \end{aligned} \tag{96}$$

It proves that the *electromagnetic field is aligned* with the gravitational field (because $\phi_0 = 0$), and it is *null*, i.e. *radiative* (because $\phi_0 = 0 = \phi_1$ but $\phi_2 \neq 0$). Expressed geometrically, the double-degenerate null direction of the type N Maxwell field coincides with the quadruple-degenerate CAND of the CCS type N gravitational field. In fact, it was observed in Ref. [8] that *both* the Cotton and Maxwell tensors possess the same triple zero eigenvalues, so that their algebraic types are N (see page 188). However, these key properties of the *algebraically most special* (“null,” i.e. “radiative”) type have not been taken into account for the physical (re-)interpretations of the CCS solution.

The geometrically privileged null vector field k^μ of the principal triad (93) satisfies the geodesic conditions ($k^\nu \nabla_\nu k^\mu = 0$). Then, its *optical scalars are all zero*, in particular the expansion $\rho_k := (\nabla_\mu k_\nu) m^\mu m^\nu$,

$$\rho_k = 0. \tag{97}$$

It means that the whole CCS family of solutions, including the Kamata–Koikawa solution, belongs to the *Kundt class* of spacetimes [12].

3.5. The Kundt canonical form of the solution

We have thus shown that the CCS spacetime belongs to the Kundt class in three dimensions. Now we write the metric of this solution in the canonical Kundt form.

Using a null coordinate u defined by Eq. (83) instead of t , and renaming the spatial coordinate r as $r \mapsto x$, we write the metric (21) and the gauge field (19) in the new coordinates (u, x, θ) as

$$ds^2 = \frac{dx^2}{P^2} + 2\epsilon(x^2 - D) du d\theta + H du^2, \tag{98}$$

$$A_\mu dx^\mu = \frac{\epsilon C_0}{2\sqrt{-\Lambda}} \ln|x^2 - D| du,$$

where

$$P(x) := \sqrt{-\Lambda} \frac{x^2 - D}{x}, \tag{99}$$

$$H(x) := 2D - x^2 + \frac{\kappa C_0^2}{2(-\Lambda)} \ln|x^2 - D|. \tag{100}$$

The Faraday tensor is given by

$$\mathbf{F} = F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{\epsilon C_0}{\sqrt{-\Lambda}} \frac{x}{x^2 - D} dx \wedge du. \tag{101}$$

This is a 3D analogue of the 4D metric representing a family of *all* type-N Kundt spacetimes which are the solutions in the Einstein–Maxwell- Λ system, or in the system with a null dust fluid instead of the Maxwell field. In four dimensions, such the most general type N Kundt solution was first presented in Ref. [29], and later investigated in detail [30]. (See Chap. 18 of the textbook [31] for a review.)

Finally, by introducing a new coordinate r_κ , instead of θ , defined by

$$r_\kappa := -\epsilon(x^2 - D)\theta, \tag{102}$$

the CCS metric (98) is written in the *canonical Kundt form* in the coordinates (u, x, r_κ) as

$$ds^2 = \frac{dx^2}{P^2} + \frac{4x r_\kappa}{x^2 - D} du dx - 2 du dr_\kappa + H du^2, \tag{103}$$

while A_μ and \mathbf{F} are unchanged. It actually belongs to a special family of *degenerate Kundt metrics*. (See, e.g. Section 7.1 of the topical review [32] for the definition and more details.)

The metric (103) can be directly compared with the complete family of 3D Kundt solutions with a Maxwell field (necessarily aligned) and Λ , found recently in Ref. [12]. Comparing the metric functions in Eq. (103) with the general expressions $g_{ux} = e + f r_\kappa$ and $g_{uu} = a + b r_\kappa + c r_\kappa^2$ given in Eq. (90) in Ref. [12], we identify

$$\begin{aligned} e = 0, \quad f = \frac{2x}{x^2 - D} \neq 0, \quad a = H, \\ b = 0, \quad c = -\frac{1}{2}\kappa_0 Q^2 = 0 \quad \Rightarrow \quad Q = 0. \end{aligned} \tag{104}$$

The corresponding Maxwell field, given by Eq. (95) in Ref. [12], is thus $\mathbf{F} = \xi(x) dx \wedge du$, which is easily identified with Eq. (98). It is an aligned null Maxwell field that belongs to subcase (i) with $\phi_0 = 0 = \phi_1$. (See Eq. (99) in Ref. [12].) Moreover, by performing the transformations in

Eqs. (83) and (102), the principal null triad (93) becomes

$$k^\mu \partial_\mu = \frac{1}{\sqrt{2}} \partial_{r_k}, \quad l^\mu \partial_\mu = \frac{1}{\sqrt{2}} H \partial_{r_k} + \sqrt{2} \partial_u, \quad m^\mu \partial_\mu = P(\partial_x + g_{ux} \partial_{r_k}). \quad (105)$$

After an additional simple boost $B k^\mu \rightarrow k^\mu$ and $B^{-1} l^\mu \rightarrow l^\mu$ with $B = \sqrt{2}$, the triad (105) fully agrees with Eq. (6) in Ref. [12].

It can also be observed that by introducing a modified spatial coordinate

$$z := \frac{1}{2\sqrt{-\Lambda}} \ln |x^2 - D|, \quad (106)$$

the metric (103) of the CCS solution is put into the form

$$ds^2 = dz^2 + 4\sqrt{-\Lambda} r_k du dx - 2 du dr_k + \left(D - e^{2\sqrt{-\Lambda}z} + \frac{\kappa C_0^2}{\sqrt{-\Lambda}} z \right) du^2, \quad (107)$$

in which the p.p. curvature singularity (originally at $r^2 = D$) is located at $z \rightarrow -\infty$. In this coordinate system (u, z, r_k) , the Maxwell field (101) is *uniform*, namely

$$\mathbf{F} = \epsilon C_0 dz \wedge du = \frac{\phi_2}{\sqrt{2}} dz \wedge du. \quad (108)$$

It should finally be noted that the physical interpretation of the family of closely related 4D Kundt spacetimes of algebraic type N with any Λ was investigated in Refs. [33,34]. It elucidated the character of wave surfaces in these spacetimes, and also the related p.p. singularity. It is the *caustics* formed as an envelope of the wave surfaces (see Chap. 18 of the textbook [31] for a review).

3.6. Compatible matter fields

Lastly, we show that the CCS solution (59) can be a solution not only with the original Maxwell field but also with a *null dust fluid* or a *massless scalar field*. For this purpose, we study the Hawking–Ellis type of $T_{\mu\nu}$ for the solution. With the orthonormal basis one-forms given by

$$E_\mu^{(0)} = \frac{1}{\sqrt{2}}(l_\mu + k_\mu), \quad E_\mu^{(1)} = \frac{1}{\sqrt{2}}(l_\mu - k_\mu), \quad E_\mu^{(2)} = m_\mu \quad (109)$$

constructed from a triad $\{k_\mu, l_\mu, m_\mu\}$ in Eq. (92), the nonzero components of $T^{(a)(b)} := T^{\mu\nu} E_\mu^{(a)} E_\nu^{(b)}$ are computed to give

$$T^{(0)(0)} = T^{(1)(1)} = T^{(0)(1)} (= T^{(1)(0)}) = C_0^2, \quad (110)$$

which shows

$$(T^{(0)(0)} + T^{(1)(1)})^2 - 4(T^{(0)(1)})^2 = 0. \quad (111)$$

Then, by Lemma 1 in Ref. [35], $T_{\mu\nu}$ for the Maxwell field in the CCS solution is of Hawking–Ellis type II everywhere.

Additionally, the expression (110) of $T^{(a)(b)}$ shows that the CCS metric also solves the Einstein- Λ equations with a null dust fluid, the energy-momentum tensor of which is given by

$$T_{\mu\nu} = \Omega k_\mu k_\nu. \quad (112)$$

Here k_μ is the null vector in Eq. (92) and the energy density of the null dust Ω is a constant given by

$$\Omega = 2 C_0^2. \quad (113)$$

In three dimensions, the field equations (2) are equivalent to the ones with a massless scalar field ϕ instead of the Maxwell field, using the duality. The dual Maxwell one-form is defined by

$${}^*F_\mu := \frac{1}{2} \varepsilon_{\mu\nu\rho} F^{\nu\rho} \quad (\Leftrightarrow {}^*F_\mu \varepsilon^{\mu\alpha\beta} = -F^{\alpha\beta}), \tag{114}$$

where the totally antisymmetric volume three-form $\varepsilon_{\mu\nu\rho}$ is defined by

$$\varepsilon_{\mu\nu\rho} := \sqrt{-g} \epsilon_{\mu\nu\rho} \quad (\Leftrightarrow \varepsilon^{\mu\nu\rho} = -\epsilon^{\mu\nu\rho} / \sqrt{-g}) \tag{115}$$

with the Levi-Civita symbol $\epsilon_{\mu\nu\rho}$ satisfying $\epsilon_{012} = 1$ and $\epsilon^{012} = 1$. Identifying

$$\nabla_\mu \phi \equiv {}^*F_\mu, \tag{116}$$

and using $\varepsilon^{\mu\alpha\beta} \varepsilon_{\mu\nu\rho} = -(\delta^\alpha_\nu \delta^\beta_\rho - \delta^\alpha_\rho \delta^\beta_\nu)$, we obtain the equation of motion and the energy-momentum tensors of ϕ as

$$\nabla^2 \phi = 0, \tag{117}$$

$$\begin{aligned} T_{\mu\nu} &= F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \\ &= (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{2} g_{\mu\nu} (\nabla\phi)^2, \end{aligned} \tag{118}$$

where $\nabla^2 \phi := \nabla_\mu \nabla^\mu \phi$ and $(\nabla\phi)^2 := (\nabla_\alpha \phi)(\nabla^\alpha \phi)$.

For the CCS solution, the dual massless scalar field is given in the coordinate system (4) as

$$\phi = C_0 (\sqrt{-\Lambda} t + \epsilon \theta) + \phi_0 = C_0 u + \phi_0, \tag{119}$$

where ϕ_0 is a constant, and u is the null coordinate defined by Eq. (83). Different from the case with a Maxwell field or a null dust fluid, the coordinate θ now cannot be periodic, as a periodic boundary condition $\phi(t, \theta) = \phi(t, \theta + 2\pi)$ is not satisfied unless $C_0 = 0$.

4. Concluding remarks

In our paper, we have studied in detail the CCS solution (59) with the metric function given by Eq. (60) in the coordinates (t, y, θ) . It is the most general stationary and axisymmetric solution known in the 3D Einstein–Maxwell– Λ system under the so-called *self-dual* or *anti-self-dual* condition on the Maxwell field [10]. The solution is locally characterized by a single parameter C_0 , and it reduces to the extremal rotating BTZ vacuum solution for $C_0 = 0$. Nevertheless, we have kept an auxiliary parameter D which allows the limit to the Kamata–Koikawa solution.

Our results for the *charged* case $C_0 \neq 0$ are summarized as follows.

1. For any value of D , the domain of the radial coordinate is $y \in (D, \infty)$. The spacetime is asymptotically locally AdS near the spacelike and null infinities given by $y \rightarrow \infty$. At $y = D$, there is a p.p. curvature singularity which corresponds to an infinite affine parameter along specific spacelike geodesics.
2. If θ is a periodic coordinate such that $(t, y, \theta) = (t, y, \theta + 2\pi)$, there are closed timelike curves near the singularity $y = D$.

3. The spacetime is of Cotton type N everywhere. We have identified a uniquely chosen principal null triad and the quadruple CAND.
4. The Maxwell field is null, namely radiative. The double-degenerate null direction of the type N Maxwell field coincides with the quadruple-degenerate CAND of the type N gravitational field.
5. We have written the CCS solution in the canonical Kundt form and identified the privileged null coordinate u given by Eq. (83) such that $u = u_0 = \text{constant}$ labels privileged null hypersurfaces in the Kundt family.
6. The energy-momentum tensor of the solution is of Hawking–Ellis type II everywhere. The CCS metric also solves the 3D Einstein– Λ equations with a null dust fluid given by Eqs. (112) and (113) or a massless scalar field given by Eq. (119).

To summarize, the CCS solution is algebraically, geometrically, and physically *different* from the charged rotating BTZ solution [5] and does *not* describe a black hole, unless the Maxwell field is trivial.

In the 3D Einstein–Maxwell– Λ system, many stationary and axisymmetric symmetric solutions have been classified and described by García-Díaz [36,37]. However, to the best of the authors’ knowledge, it is still an open problem whether there is a different class of charged rotating asymptotically AdS black-hole solutions other than the charged rotating BTZ solution. This problem is left for future investigation.

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Appendix A. Clément’s solution in the 1993 paper

In this appendix, we show that Clément’s solution given by Eq. (29) in Ref. [11] with the Minkowski signature (+, −, −) is locally identical to the Cataldo–Salgado solution.⁴ The original solution can be written in the coordinates (t, ρ, θ) with the Minkowski signature (−, +, +) as

$$ds^2 = \pm \frac{4}{\ell} \rho dt d\theta + (-b\rho + 2\kappa q^2 \ln |\rho/\rho_0|) d\theta^2 + \frac{\ell^2}{4\rho^2} d\rho^2, \tag{A.1}$$

$$A_\mu dx^\mu = q \ln |\rho/\rho_0| d\theta,$$

where $\rho_0, b,$ and q are constants, and ℓ is the AdS radius defined by $\ell = 1/\sqrt{-\Lambda}$.

The Cataldo–Salgado solution in the single-null coordinates (84) is

$$ds^2 = -\left(y - 2D - \frac{\kappa C_0^2}{2(-\Lambda)} \ln |y - D|\right) du^2 + 2\epsilon (y - D) du d\theta + \frac{dy^2}{4(-\Lambda)(y - D)^2}, \tag{A.2}$$

$$A_\mu dx^\mu = \frac{\epsilon C_0}{2\sqrt{-\Lambda}} \ln |y - D| du.$$

⁴Note that Eq. (23) in Ref. [11] is the charged rotating BTZ solution after the double Wick rotation.

By coordinate transformations

$$y - D = \alpha \rho, \quad u = \beta \bar{\theta}, \quad \theta = (\alpha\beta)^{-1} \frac{2}{\ell} t \tag{A.3}$$

with constant α and β satisfying

$$\frac{\kappa C_0^2}{2(-\Lambda)} \ln |\alpha| = -D - \frac{\kappa C_0^2}{2(-\Lambda)} \ln |\rho_0|, \quad \beta^2 = \frac{b}{\alpha} \tag{A.4}$$

and using the freedom of a gauge constant of A_μ , the solution (A.2) is transformed into Clément's solution (A.1) with $q = \epsilon\beta C_0/(2\sqrt{-\Lambda})$.

Appendix B. Parameters of the Cataldo–Salgado solution

Cataldo and Salgado presented their solution as

$$ds^2 = -\frac{r^2}{R^2} f dt^2 + \frac{dr^2}{f} + R^2 \left(d\theta + \frac{h}{R^2} dt \right)^2, \tag{B.1}$$

$$A_\mu dx^\mu = \frac{C_0}{2\sqrt{-\Lambda}} \ln \left| \frac{r^2 - D}{C} \right| \left[\left(\epsilon\sqrt{-\Lambda} + \frac{E}{C} \right) dt + d\theta \right]$$

with the metric functions

$$f(r) = (-\Lambda) \frac{(r^2 - D)^2}{r^2}, \tag{B.2}$$

$$R(r) = \sqrt{r^2 + \frac{\kappa C_0^2}{2(-\Lambda)} \ln \left| \frac{r^2 - D}{C} \right|}, \tag{B.3}$$

$$h(r) = \frac{E}{C} r^2 + \epsilon D \sqrt{-\Lambda} + \frac{\kappa C_0^2}{2\sqrt{-\Lambda}} \left(\epsilon + \frac{E}{\sqrt{-\Lambda} C} \right) \ln \left| \frac{r^2 - D}{C} \right|. \tag{B.4}$$

See Eqs. (37–39) in Ref. [10], in which $C (\neq 0)$, D , C_0 , and E are four arbitrary constants of integration, and $\epsilon = \pm 1$.

However, without loss of generality we can *locally* set $C = 1$ and $E = 0$ by the following coordinate transformations:

$$t = \frac{\bar{t}}{\sqrt{C}}, \quad r = \sqrt{C} \bar{r}, \quad \theta = \frac{\bar{\theta} - \bar{E} \bar{t}}{\sqrt{C}} \tag{B.5}$$

with the reparametrizations $E = C\bar{E}$, $D = C\bar{D}$, and $C_0 = \sqrt{C}\bar{C}_0$. Nevertheless, the constants C and E may possibly have global meanings. For example, if one assumes that θ is periodic as $\theta \in [0, 2\pi)$ in the coordinates (B.1) with $C \neq 1$ and $E \neq 0$, then $\bar{\theta} = 0$ is not identified with $\bar{\theta} = 2\pi$ at different times.

With $C = 1$ and $E = 0$, using the useful identities (20) in the region of $r^2 > D$, the Cataldo–Salgado solution can be rewritten in the simple form (21),

$$ds^2 = (-\Lambda) [R^2 - 2(r^2 - D)] dt^2 + 2\epsilon\sqrt{-\Lambda} [R^2 - (r^2 - D)] dt d\theta + R^2 d\theta^2 + \frac{dr^2}{f}, \tag{B.6}$$

$$A_\mu dx^\mu = \frac{1}{2} C_0 \ln |r^2 - D| \left(\epsilon dt + \frac{d\theta}{\sqrt{-\Lambda}} \right),$$

where R^2 is given by Eq. (18). Changing the sign of ϵ is equivalent to a transformation of time reversal $t \rightarrow -t$. Then, without loss of generality we can also *locally* set $D = 0$ in the charged

case $C_0 \neq 0$ by the following coordinate transformations:

$$r^2 - D = e^{-2(-\Lambda)D/(\kappa C_0^2)} \bar{r}^2, \quad t = e^{(-\Lambda)D/(\kappa C_0^2)} \bar{t}, \quad \theta = e^{(-\Lambda)D/(\kappa C_0^2)} \bar{\theta} \quad (\text{B.7})$$

with a reparametrization $C_0 = e^{(-\Lambda)D/(\kappa C_0^2)} \bar{C}_0$, which generate a gauge constant for A_μ . However, setting $D = 0$ forces us to miss the limit from the Cataldo–Salgado solution (B.6) to the Kamata–Koikawa solution [9] for $\kappa C_0^2 = -2\Lambda D$.

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