Obecné souřadnice tenzorů

souřadnice: x^m

báze v prostoru vektorů: $\dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^m}$ báze v prostoru 1-forem: $\mathbf{d}x^m$ platí: $\dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^m} \cdot \mathbf{d}x^n = \delta^n_m$

Souřadnice tenzorů:

souřadnice x^k

$$\boldsymbol{a} = a^k \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} \qquad \boldsymbol{\alpha} = \alpha_k\, \mathbf{d}x^k$$

$$\boldsymbol{A} = A^{k\dots}_{l\dots} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} \cdots\ \mathbf{d}x^l \cdots$$

$$a^k = a^{l'} \frac{\partial x^k}{\partial y^{l'}} \qquad \alpha_k = \alpha_{l'} \frac{\partial y^{l'}}{\partial x^k}$$

$$A^{m\dots}_{n\dots} = A^{k'}_{l'} \frac{\partial x^m}{\partial y^{k'}} \cdots \frac{\partial y^{l'}}{\partial x^n} \cdots$$

souřadnice $y^{k'}$

$$\boldsymbol{a} = a^{k'} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y^{k'}} \qquad \boldsymbol{\alpha} = \alpha_{k'}\, \mathbf{d}y^{k'}$$

$$\boldsymbol{A} = A^{k'\dots}_{l'\dots} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y^{k'}} \cdots\ \mathbf{d}y^{l'} \cdots$$

$$a^{k'} = a^{l} \frac{\partial y^{k'}}{\partial x^{l}} \qquad \alpha_{k'} = \alpha_{l} \frac{\partial x^{l}}{\partial y^{k'}}$$

$$A^{m'\dots}_{n'\dots} = A^{k}_{l} \frac{\partial y^{m'}}{\partial x^{k}} \cdots \frac{\partial x^{l}}{\partial y^{n'}} \cdots$$

Metrika, snižování a zvyšování indexů

metrika definuje skalární součin mezi vektory: $(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{q} \cdot \boldsymbol{b}$

$$\boldsymbol{q} = q_{mn}\, \mathbf{d}x^m\, \mathbf{d}x^n \qquad q_{kn}\, q^{ln} = \delta^l_k \qquad q_{kl} = q_{lk} \qquad q^{kl} = q^{lk}$$

metrika definuje korespondenci mezi vektory a 1-formami

$$a_k = q_{kl}\, a^l \qquad a^k = q^{kl}\, a_l$$

Levi-Civitův tenzor

Levi-Civitův tenzor $\boldsymbol{\epsilon}$ je totálně antisymetrický tenzor se zvolenou orientací, normalizovaný na metriku

$\boldsymbol{\epsilon} \in V^0_d$ kde $d = \dim V$

$\boldsymbol{\epsilon} = \mathcal{A}(\boldsymbol{\epsilon})$ tj. $\epsilon_{k_1\dots k_d} = \epsilon_{[k_1\dots k_d]}$ antisymetrie

$\epsilon_{k_1\dots k_d}\, \epsilon_{k_1\dots k_d}\, q^{k_1 l_1} \cdots q^{k_d l_d} = \pm d!$ normalizace

$\epsilon_{1\dots d} > 0$ v positivně orientované bázi orientace

$\Rightarrow \quad \epsilon_{1\dots d} = \pm\sqrt{|\det q_{kl}|}$

∇-operátor – kovariantní derivace

pro lineární souřadnice $\bar{x}^{\bar{k}}$ v afinním prostoru platí: $\nabla \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} \bar{x}^{\bar{k}}} = 0 \qquad \nabla\, \mathbf{d}\bar{x}^{\bar{k}} = 0$

v obecných souřadnicích x^k definujeme Γ^n_{kl}: $\nabla \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} = \Gamma^{\,n}_{mk} \dfrac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n}\, \mathbf{d}x^m \qquad \nabla\, \mathbf{d}x^k = -\Gamma^{\,k}_{mn}\, \mathbf{d}x^m\, \mathbf{d}x^n$

metrika je konstatní tenzor: $\nabla \boldsymbol{q} = 0$

souřadnice kovariantní derivace

$A^k_{l;m}$ označuje souřadnice tenzoru $\nabla \boldsymbol{A}$, $A^k_{l,m}$ označuje parciální derivace podle x^m

$$f_{;k} = f_{,k} \qquad a^k{}_{;m} = a^k{}_{,m} + \Gamma^{\,k}_{mn} a^n \qquad a_{n;m} = a_{n,m} - \Gamma^{\,k}_{mn} a_k$$

$$A^{k\dots}_{l\dots;m} = A^{k\dots}_{l\dots,m} + \Gamma^{\,k}_{mn} A^{n\dots}_{l\dots} + \dots - \Gamma^{\,n}_{ml} A^{k\dots}_{n\dots} - \cdots$$

$$\Gamma^{\,a}_{kl} = \frac{1}{2}\, q^{an}\big(q_{nk,l} + q_{nl,k} - q_{kl,n}\big) \qquad \Gamma^{\,a}_{kl} = \frac{\partial^2 \bar{x}^{\bar{n}}}{\partial x^k \partial x^l} \frac{\partial x^a}{\partial \bar{x}^{\bar{n}}} \qquad \Gamma^{\,a}_{kl} = -\frac{\partial \bar{x}^{\bar{b}}}{\partial x^k} \frac{\partial^2 x^a}{\partial \bar{x}^{\bar{b}} \partial \bar{x}^{\bar{c}}} \frac{\partial \bar{x}^{\bar{c}}}{\partial x^l}$$

Ortogonální souřadnice ($d = 3$)

ortogonální souřadnice x^m mají na sebe kolmé souřadnicové čáry, tj. q_{kl} je diagonální
v takovém případě zavádíme Lamého koeficienty: $h_k = \sqrt{q_{kk}}$
ve výrazech s Lamého koeficienty se nepoužívá sčítací konvence!

Normalizovaná báze vektorů a 1-forem

$$\boldsymbol{e}_{\hat{k}} = \frac{1}{h_k}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k} \qquad |\boldsymbol{e}_{\hat{k}}| = 1 \qquad \left|\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^k}\right| = h_k$$

$$\boldsymbol{e}^{\hat{k}} = h_k\,\mathbf{d}x^k \qquad |\boldsymbol{e}^{\hat{k}}| = 1 \qquad |\mathbf{d}x^k| = \frac{1}{h_k}$$

Metrika a objemový element

$$\boldsymbol{q} = h_1^2\boldsymbol{e}^{\hat{1}}\boldsymbol{e}^{\hat{1}} + h_2^2\boldsymbol{e}^{\hat{2}}\boldsymbol{e}^{\hat{2}} + h_3^2\boldsymbol{e}^{\hat{3}}\boldsymbol{e}^{\hat{3}} \qquad dV = h_1 h_2 h_3\, dx^1\, dx^2\, dx^3$$

Vztah souřadnic s různým typem indexů

$$\boldsymbol{a} = a^n\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^n} = a^{\hat{n}}\,\boldsymbol{e}_{\hat{n}} \qquad \boldsymbol{a} = a_n\,\mathbf{d}x^n = a_{\hat{n}}\,\boldsymbol{e}^{\hat{n}}$$

$$h_n\,a^n = a^{\hat{n}} = a_{\hat{n}} = h_n^{-1}\,a_n \qquad \text{nesčítá se!}$$

Levi-Civitův tenzor a vektorové násobení

$$\boldsymbol{\epsilon} = h_1 h_2 h_3\;\mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3 = \boldsymbol{e}^{\hat{1}} \wedge \boldsymbol{e}^{\hat{2}} \wedge \boldsymbol{e}^{\hat{3}}$$

$$\epsilon_{123} = h_1 h_2 h_3 \qquad \epsilon_{\hat{1}\hat{2}\hat{3}} = 1$$

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} \quad \Rightarrow$$

$$c^1 = \frac{h_2 h_3}{h_1}\,(a^2 b^3 - a^3 b^2) \qquad c^{\hat{1}} = (a^{\hat{2}} b^{\hat{3}} - a^{\hat{3}} b^{\hat{2}}) \qquad \text{a cyklické záměny}$$

Γ - koeficienty

$$\Gamma^{\,k}_{kn} = \Gamma^{\,k}_{nk} = (\log h_k)_{,n} = h_k^{-1}\,h_{k,n}$$

$$\Gamma^{\,k}_{nn} = -\frac{1}{2}\,h_k^{-2}\,(h_n^2)_{,k} \qquad k \neq n$$

$$\Gamma^{\,k}_{mn} = 0 \qquad k,\ m,\ n \text{ různé}$$

Operátory v ortogonálních souřadnicích

gradient skaláru: $\boldsymbol{a} = \mathbf{d}f = \nabla f$

$$a_k = f_{,k} \qquad a^{\hat{k}} = a_{\hat{k}} = \frac{1}{h_k}\, f_{,k} \qquad a^k = \frac{1}{h_k^2}\, f_{,k}$$

divergence: $f = \nabla \cdot \boldsymbol{a}$

$$\begin{aligned} f &= \frac{1}{h_1 h_2 h_3}\Big(\big(h_1 h_2 h_3\; a^1\big)_{,1} + \big(h_1 h_2 h_3\; a^2\big)_{,2} + \big(h_1 h_2 h_3\; a^3\big)_{,3}\Big) \\ &= \frac{1}{h_1 h_2 h_3}\left(\big(h_2 h_3\; a^{\hat{1}}\big)_{,1} + \big(h_1 h_3\; a^{\hat{2}}\big)_{,2} + \big(h_1 h_2\; a^{\hat{3}}\big)_{,3}\right) \\ &= \frac{1}{h_1 h_2 h_3}\left(\left(\frac{h_2 h_3}{h_1}\, a_1\right)_{,1} + \left(\frac{h_1 h_3}{h_2}\, a_2\right)_{,2} + \left(\frac{h_1 h_2}{h_3}\, a_3\right)_{,3}\right) \end{aligned}$$

rotace: $\boldsymbol{b} = \nabla \times \boldsymbol{a}$

$$b^1 = \frac{1}{h_1 h_2 h_3}\Big(\big(h_3^2\, a^3\big)_{,2} - \big(h_2^2\, a^2\big)_{,3}\Big) \qquad \text{a cyklické záměny}$$

$$b^{\hat{1}} = \frac{1}{h_2 h_3}\Big(\big(h_3\, a^{\hat{3}}\big)_{,2} - \big(h_2\, a^{\hat{2}}\big)_{,3}\Big) \qquad \text{a cyklické záměny}$$

$$b_1 = \frac{h_1}{h_2 h_3}\big(a_{3,2} - a_{2,3}\big) \qquad \text{a cyklické záměny}$$

laplace:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left(\left(\frac{h_2 h_3}{h_1}\, f_{,1}\right)_{,1} + \left(\frac{h_1 h_3}{h_2}\, f_{,2}\right)_{,2} + \left(\frac{h_1 h_2}{h_3}\, f_{,3}\right)_{,3}\right)$$

Cylindrické souřadnice $(\rho,\, \varphi,\, z)$

Vztah ke kartézským souřadnicím:

$$x = \rho\cos\varphi \qquad \rho = \sqrt{x^2 + y^2}$$
$$y = \rho\sin\varphi \qquad \varphi = \arctan\frac{y}{x}$$
$$z = z \qquad z = z$$

Metrika a objemový element:

$$\boldsymbol{g} = \mathbf{d}\rho\,\mathbf{d}\rho + \rho^2\,\mathbf{d}\varphi\,\mathbf{d}\varphi + \mathbf{d}z\,\mathbf{d}z \qquad dV = \rho\,d\rho\,d\varphi\,dz$$

Laméovy koeficienty:

$$h_\rho = 1 \quad , \quad h_\varphi = \rho \quad , \quad h_z = 1$$

Triáda (normovaná báze):

$$\boldsymbol{e}_{\hat\rho} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\rho} \qquad \boldsymbol{e}_{\hat\varphi} = \frac{1}{\rho}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} \qquad \boldsymbol{e}_{\hat z} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}z}$$
$$\boldsymbol{e}^{\hat\rho} = \mathbf{d}\rho \qquad \boldsymbol{e}^{\hat\varphi} = \rho\,\mathbf{d}\varphi \qquad \boldsymbol{e}^{\hat z} = \mathbf{d}z$$

Vztah souřadnic tenzorů (různé druhy indexů):

$$a^\rho = a^{\hat\rho} = a_{\hat\rho} = a_\rho$$
$$\rho\, a^\varphi = a^{\hat\varphi} = a_{\hat\varphi} = \frac{1}{\rho} a_\varphi$$
$$a^z = a^{\hat z} = a_{\hat z} = a_z$$

Vektorové násobení a antisymetrický tenzor:

$$\boldsymbol{\epsilon} = \rho\;\mathbf{d}\rho \wedge \mathbf{d}\varphi \wedge \mathbf{d}z$$

$$\epsilon_{\rho\varphi z} = \rho \qquad \epsilon_{\hat\rho\hat\varphi\hat z} = 1 \qquad \epsilon^{\rho\varphi z} = \frac{1}{\rho}$$
$$\epsilon^{\rho}{}_{\varphi z} = \rho \qquad \epsilon^{\varphi}{}_{z\rho} = \frac{1}{\rho} \qquad \epsilon^{z}{}_{\rho\varphi} = \rho$$

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$
$$c^\rho = \rho\,(a^\varphi b^z - a^z b^\varphi) \qquad c^\varphi = \frac{1}{\rho}\,(a^z b^\rho - a^\rho b^z) \qquad c^z = \rho\,(a^\rho b^\varphi - a^\varphi b^\rho)$$

Vztah cylindrických a kartézských bází vektorů a forem

$$\boldsymbol{e}_{\hat{x}} = \cos\varphi\,\boldsymbol{e}_{\hat{\rho}} - \sin\varphi\,\boldsymbol{e}_{\hat{\varphi}} = \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{e}_{\hat{\rho}} - \frac{y}{\sqrt{x^2+y^2}}\boldsymbol{e}_{\hat{\varphi}}$$
$$\boldsymbol{e}_{\hat{y}} = \sin\varphi\,\boldsymbol{e}_{\hat{\rho}} + \cos\varphi\,\boldsymbol{e}_{\hat{\varphi}} = \frac{y}{\sqrt{x^2+y^2}}\boldsymbol{e}_{\hat{\rho}} + \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{e}_{\hat{\varphi}}$$
$$\boldsymbol{e}_{\hat{z}} = \boldsymbol{e}_{\hat{z}}$$

$$\boldsymbol{e}_{\hat{\rho}} = \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{e}_{\hat{x}} + \frac{y}{\sqrt{x^2+y^2}}\boldsymbol{e}_{\hat{y}} = \cos\varphi\,\boldsymbol{e}_{\hat{x}} + \sin\varphi\,\boldsymbol{e}_{\hat{y}}$$
$$\boldsymbol{e}_{\hat{\varphi}} = -\frac{y}{\sqrt{x^2+y^2}}\boldsymbol{e}_{\hat{x}} + \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{e}_{\hat{y}} = -\sin\varphi\,\boldsymbol{e}_{\hat{x}} + \cos\varphi\,\boldsymbol{e}_{\hat{y}}$$
$$\boldsymbol{e}_{\hat{z}} = \boldsymbol{e}_{\hat{z}}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x} = \cos\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\rho} - \frac{\sin\varphi}{\rho}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = \frac{x}{\sqrt{x^2+y^2}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\rho} - \frac{y}{x^2+y^2}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi}$$
$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y} = \sin\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\rho} + \frac{\cos\varphi}{\rho}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = \frac{y}{\sqrt{x^2+y^2}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\rho} + \frac{x}{x^2+y^2}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi}$$
$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} z} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} z}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\rho} = \frac{x}{\sqrt{x^2+y^2}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y} = \cos\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x} + \sin\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y}$$
$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = -y\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x} + x\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y} = -\rho\sin\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x} + \rho\cos\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y}$$
$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} z} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} z}$$

$$\mathbf{d}x = \cos\varphi\,\mathbf{d}\rho - \rho\sin\varphi\,\mathbf{d}\varphi = \frac{x}{\sqrt{x^2+y^2}}\,\mathbf{d}\rho - y\,\mathbf{d}\varphi$$
$$\mathbf{d}y = \sin\varphi\,\mathbf{d}\rho + \rho\cos\varphi\,\mathbf{d}\varphi = \frac{y}{\sqrt{x^2+y^2}}\,\mathbf{d}\rho + x\,\mathbf{d}\varphi$$
$$\mathbf{d}z = \mathbf{d}z$$

$$\mathbf{d}\rho = \frac{1}{\sqrt{x^2+y^2}}(x\,\mathbf{d}x + y\,\mathbf{d}y) = \cos\varphi\,\mathbf{d}x + \sin\varphi\,\mathbf{d}y$$
$$\mathbf{d}\varphi = \frac{1}{x^2+y^2}(x\,\mathbf{d}y - y\,\mathbf{d}x) = \frac{\cos\varphi}{\rho}\,\mathbf{d}y - \frac{\sin\varphi}{\rho}\,\mathbf{d}x$$
$$\mathbf{d}z = \mathbf{d}z$$

Derivace bázových vektorů a forem

$$\nabla \boldsymbol{e}_{\hat\rho} = \frac{1}{\rho}\,\boldsymbol{e}^{\hat\varphi}\,\boldsymbol{e}_{\hat\varphi} \qquad \nabla\cdot\boldsymbol{e}_{\hat\rho} = \frac{1}{\rho} \qquad \nabla\times\boldsymbol{e}_{\hat\rho} = 0$$

$$\nabla \boldsymbol{e}_{\hat\varphi} = -\frac{1}{\rho}\,\boldsymbol{e}^{\hat\varphi}\,\boldsymbol{e}_{\hat\rho} \qquad \nabla\cdot\boldsymbol{e}_{\hat\varphi} = 0 \qquad \nabla\times\boldsymbol{e}_{\hat\varphi} = \frac{1}{\rho}\,\boldsymbol{e}_{\hat z}$$

$$\nabla \boldsymbol{e}_{\hat z} = 0 \qquad \nabla\cdot\boldsymbol{e}_{\hat z} = 0 \qquad \nabla\times\boldsymbol{e}_{\hat z} = 0$$

$$\nabla\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\rho} = \frac{1}{\rho}\,\mathbf{d}\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} \qquad \nabla\cdot\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\rho} = \frac{1}{\rho} \qquad \nabla\times\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\rho} = 0$$

$$\nabla\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = \frac{1}{\rho}\,\mathbf{d}\rho\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} - \rho\,\mathbf{d}\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\rho} \qquad \nabla\cdot\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = 0 \qquad \nabla\times\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = 2\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}z}$$

$$\nabla\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}z} = 0 \qquad \nabla\cdot\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}z} = 0 \qquad \nabla\times\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}z} = 0$$

$$\nabla\,\mathbf{d}\rho = \rho\;\mathbf{d}\varphi\,\mathbf{d}\varphi \qquad \nabla^2\rho = \frac{1}{\rho}$$

$$\nabla\,\mathbf{d}\varphi = -\frac{1}{\rho}(\,\mathbf{d}\rho\,\mathbf{d}\varphi + \,\mathbf{d}\varphi\,\mathbf{d}\rho) \qquad \nabla^2\varphi = 0$$

$$\nabla\,\mathbf{d}z = 0 \qquad \nabla^2 z = 0$$

Souřadnice kovariantní derivace

$$a^k{}_{;m} = a^k{}_{,m} + \Gamma^{\,k}_{mn}a^n \quad , \quad a_{n;m} = a_{n,m} - \Gamma^{\,k}_{mn}a_k$$

$$\Gamma^{\,\rho}_{\varphi\varphi} = -\rho \quad , \quad \Gamma^{\,\varphi}_{\rho\varphi} = \Gamma^{\,\varphi}_{\varphi\rho} = \frac{1}{\rho} \quad , \quad \text{ostatní } \Gamma\text{-koeficienty jsou nulové}$$

Operátory v cylindrických souřadnicích

gradient skaláru: $\boldsymbol{a} = \,\mathbf{d}f = \nabla f$

$$a_\rho = f_{,\rho} \qquad a_{\hat\rho} = f_{,\rho} \qquad a^\rho = f_{,\rho}$$

$$a_\varphi = f_{,\varphi} \qquad a_{\hat\varphi} = \frac{1}{\rho}\,f_{,\varphi} \qquad a^\varphi = \frac{1}{\rho^2}\,f_{,\varphi}$$

$$a_z = f_{,z} \qquad a_{\hat z} = f_{,z} \qquad a^z = f_{,z}$$

divergence: $f = \nabla\cdot\boldsymbol{a}$

$$f = \frac{1}{\rho}(\rho\,a^\rho)_{,\rho} + a^\varphi{}_{,\varphi} + a^z{}_{,z} = \frac{1}{\rho}(\rho\,a^{\hat\rho})_{,\rho} + \frac{1}{\rho}\,a^{\hat\varphi}{}_{,\varphi} + a^{\hat z}{}_{,z} = \frac{1}{\rho}(\rho\,a_\rho)_{,\rho} + \frac{1}{\rho^2}\,a_{\varphi,\varphi} + a_{z,z}$$

rotace: $\boldsymbol{b} = \nabla\times\boldsymbol{a}$

$$b_\rho = \frac{1}{\rho}\,(a_{z,\varphi} - a_{\varphi,z}) \qquad b_{\hat\rho} = \frac{1}{\rho}\,a_{\hat z,\varphi} - a_{\hat\varphi,z} \qquad b^\rho = \frac{1}{\rho}\,a^z{}_{,\varphi} - \rho\,a^\varphi{}_{,z}$$

$$b_\varphi = \rho\,(a_{\rho,z} - a_{z,\rho}) \qquad b_{\hat\varphi} = a_{\hat\rho,z} - a_{\hat z,\rho} \qquad b^\varphi = \frac{1}{\rho}\,(a^\rho{}_{,z} - a^z{}_{,\rho})$$

$$b_z = \frac{1}{\rho}\,(a_{\varphi,\rho} - a_{\rho,\varphi}) \qquad b_{\hat z} = \frac{1}{\rho}\,((\rho\,a_{\hat\varphi})_{,\rho} - a_{\hat\rho,\varphi}) \qquad b^z = \frac{1}{\rho}\,\big((\rho^2\,a^\varphi)_{,\rho} - a^\rho{}_{,\varphi}\big)$$

laplace:

$$\nabla^2 f = \frac{1}{\rho}(\rho\,f_{,\rho})_{,\rho} + \frac{1}{\rho^2}f_{,\varphi\varphi} + f_{,zz} = f_{,\rho\rho} + \frac{1}{\rho^2}f_{,\varphi\varphi} + f_{,zz} + \frac{1}{\rho}f_{,\rho}$$

Sférické souřadnice $(r,\ \vartheta,\ \varphi)$

Vztah ke kartézským souřadnicím:

$$x = r\sin\vartheta\,\cos\varphi \qquad r = \sqrt{x^2+y^2+z^2}$$

$$y = r\sin\vartheta\,\sin\varphi \qquad \vartheta = \arctan\frac{\sqrt{x^2+y^2}}{z}$$

$$z = r\cos\vartheta \qquad \varphi = \arctan\frac{y}{x}$$

Metrika a objemový element:

$$\boldsymbol{q} = \mathbf{d}r\,\mathbf{d}r + r^2\,\mathbf{d}\vartheta\,\mathbf{d}\vartheta + r^2\sin^2\vartheta\ \mathbf{d}\varphi\,\mathbf{d}\varphi \qquad dV = r^2\sin\vartheta\ dr\,d\vartheta\,d\varphi$$

Laméovy koeficienty:

$$h_r = 1 \quad , \quad h_\vartheta = r \quad , \quad h_\varphi = r\sin\vartheta$$

Triáda (normovaná báze):

$$\boldsymbol{e}_{\hat r} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} \qquad \boldsymbol{e}_{\hat\vartheta} = \frac{1}{r}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} \qquad \boldsymbol{e}_{\hat\varphi} = \frac{1}{r\sin\vartheta}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi}$$

$$\boldsymbol{e}^{\hat r} = \mathbf{d}r \qquad \boldsymbol{e}^{\hat\vartheta} = r\,\mathbf{d}\vartheta \qquad \boldsymbol{e}^{\hat\varphi} = r\sin\vartheta\ \mathbf{d}\varphi$$

Vztah souřadnic tenzorů (různé druhy indexů):

$$a^r = a^{\hat r} = a_{\hat r} = a_r$$

$$r\,a^\vartheta = a^{\hat\vartheta} = a_{\hat\vartheta} = \frac{1}{r}\,a_\vartheta$$

$$r\sin\vartheta\ a^\varphi = a^{\hat\varphi} = a_{\hat\varphi} = \frac{1}{r\sin\vartheta}\,a_\varphi$$

Vektorové násobení a antisymetrický tenzor:

$$\boldsymbol{\epsilon} = r^2\sin\vartheta\ \mathbf{d}r\wedge\mathbf{d}\vartheta\wedge\mathbf{d}\varphi$$

$$\epsilon_{r\vartheta\varphi} = r^2\sin\vartheta \qquad \epsilon_{\hat r\hat\vartheta\hat\varphi} = 1 \qquad \epsilon^{r\vartheta\varphi} = \frac{1}{r^2\sin\vartheta}$$

$$\epsilon^r{}_{\vartheta\varphi} = r^2\sin\vartheta \qquad \epsilon^\vartheta{}_{\varphi r} = \sin\vartheta \qquad \epsilon^\varphi{}_{r\vartheta} = \frac{1}{\sin\vartheta}$$

$$\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{b}$$

$$c^r = r^2\sin\vartheta\,(a^\vartheta b^\varphi - a^\varphi b^\vartheta) \qquad c^\vartheta = \sin\vartheta\,(a^\varphi b^r - a^r b^\varphi) \qquad c^\varphi = \frac{1}{\sin\vartheta}\,(a^r b^\vartheta - a^\vartheta b^r)$$

Vztah sférických a kartézských bází vektorů a forem

$$\boldsymbol{e}_{\hat{x}} = \sin\vartheta\cos\varphi\,\boldsymbol{e}_{\hat{r}} + \cos\vartheta\cos\varphi\,\boldsymbol{e}_{\hat{\vartheta}} - \sin\varphi\,\boldsymbol{e}_{\hat{\varphi}} = \frac{x}{r}\,\boldsymbol{e}_{\hat{r}} + \frac{z}{r}\frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{\vartheta}} - \frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{\varphi}}$$

$$\boldsymbol{e}_{\hat{y}} = \sin\vartheta\sin\varphi\,\boldsymbol{e}_{\hat{r}} + \cos\vartheta\sin\varphi\,\boldsymbol{e}_{\hat{\vartheta}} + \cos\varphi\,\boldsymbol{e}_{\hat{\varphi}} = \frac{y}{r}\,\boldsymbol{e}_{\hat{r}} + \frac{z}{r}\frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{\vartheta}} + \frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{\varphi}}$$

$$\boldsymbol{e}_{\hat{z}} = \cos\vartheta\,\boldsymbol{e}_{\hat{r}} - \sin\vartheta\,\boldsymbol{e}_{\hat{\vartheta}} = \frac{z}{r}\,\boldsymbol{e}_{\hat{r}} - \frac{\sqrt{x^2+y^2}}{r}\,\boldsymbol{e}_{\hat{\vartheta}}$$

$$\boldsymbol{e}_{\hat{r}} = \frac{1}{\sqrt{x^2+y^2+z^2}}\,(x\,\boldsymbol{e}_{\hat{x}} + y\,\boldsymbol{e}_{\hat{y}} + z\,\boldsymbol{e}_{\hat{z}}) = \sin\vartheta\cos\varphi\,\boldsymbol{e}_{\hat{x}} + \sin\vartheta\sin\varphi\,\boldsymbol{e}_{\hat{y}} + \cos\vartheta\,\boldsymbol{e}_{\hat{z}}$$

$$\boldsymbol{e}_{\hat{\vartheta}} = \frac{z}{\sqrt{x^2+y^2+z^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{x}} + \frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{y}} - \frac{\sqrt{x^2+y^2}}{z}\,\boldsymbol{e}_{\hat{z}}\right)$$

$$= \cos\vartheta\cos\varphi\,\boldsymbol{e}_{\hat{x}} + \cos\vartheta\sin\varphi\,\boldsymbol{e}_{\hat{y}} - \sin\vartheta\,\boldsymbol{e}_{\hat{z}}$$

$$\boldsymbol{e}_{\hat{\varphi}} = -\frac{y}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{x}} + \frac{x}{\sqrt{x^2+y^2}}\,\boldsymbol{e}_{\hat{y}} = -\sin\varphi\,\boldsymbol{e}_{\hat{x}} + \cos\varphi\,\boldsymbol{e}_{\hat{y}}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x} = \sin\vartheta\cos\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} + \frac{\cos\vartheta\cos\varphi}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} - \frac{\sin\varphi}{r\sin\vartheta}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = \frac{x}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} + \frac{z}{r^2}\frac{x}{\sqrt{x^2+y^2}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} - \frac{y}{x^2+y^2}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y} = \sin\vartheta\sin\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} + \frac{\cos\vartheta\sin\varphi}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} + \frac{\cos\varphi}{r\sin\vartheta}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = \frac{y}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} + \frac{z}{r^2}\frac{y}{\sqrt{x^2+y^2}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} + \frac{x}{x^2+y^2}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} z} = \cos\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} - \frac{\sin\vartheta}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} = \frac{z}{r}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} - \frac{\sqrt{x^2+y^2}}{r^2}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x} + y\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y} + z\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} z}\right) = \sin\vartheta\cos\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x} + \sin\vartheta\sin\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y} + \cos\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} z}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} = \frac{xz}{\sqrt{x^2+y^2}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x} + \frac{yz}{\sqrt{x^2+y^2}}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y} - \sqrt{x^2+y^2}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} z} = r\cos\vartheta\cos\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x} + r\cos\vartheta\sin\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y} - r\sin\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} z}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = -y\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x} + x\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y} = -r\sin\vartheta\sin\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x} + r\sin\vartheta\cos\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y}$$

$$\mathbf{d}x = \sin\vartheta\cos\varphi\,\mathbf{d}r + r\cos\vartheta\cos\varphi\,\mathbf{d}\vartheta - r\sin\vartheta\sin\varphi\,\mathbf{d}\varphi = \frac{x}{r}\,\mathbf{d}r + \frac{xz}{\sqrt{x^2+y^2}}\,\mathbf{d}\vartheta - y\,\mathbf{d}\varphi$$

$$\mathbf{d}y = \sin\vartheta\sin\varphi\,\mathbf{d}r + r\cos\vartheta\sin\varphi\,\mathbf{d}\vartheta + r\sin\vartheta\cos\varphi\,\mathbf{d}\varphi = \frac{y}{r}\,\mathbf{d}r + \frac{yz}{\sqrt{x^2+y^2}}\,\mathbf{d}\vartheta + x\,\mathbf{d}\varphi$$

$$\mathbf{d}z = \cos\vartheta\,\mathbf{d}r - r\sin\vartheta\,\mathbf{d}\vartheta = \frac{z}{r}\,\mathbf{d}r - \sqrt{x^2+y^2}\,\mathbf{d}\vartheta$$

$$\mathbf{d}r = \frac{1}{\sqrt{x^2+y^2+z^2}}\,(x\,\mathbf{d}x + y\,\mathbf{d}y + z\,\mathbf{d}z) = \sin\vartheta\cos\varphi\,\mathbf{d}x + \sin\vartheta\sin\varphi\,\mathbf{d}y + \cos\vartheta\,\mathbf{d}z$$

$$\mathbf{d}\vartheta = \frac{1}{x^2+y^2+z^2}\left(\frac{xz}{\sqrt{x^2+y^2}}\,\mathbf{d}x + \frac{yz}{\sqrt{x^2+y^2}}\,\mathbf{d}y - \sqrt{x^2+y^2}\,\mathbf{d}z\right)$$

$$= \frac{1}{r}\,(\cos\vartheta\cos\varphi\,\mathbf{d}x + \cos\vartheta\sin\varphi\,\mathbf{d}y - \sin\vartheta\,\mathbf{d}z)$$

$$\mathbf{d}\varphi = \frac{1}{x^2+y^2}(-y\,\mathbf{d}x + x\,\mathbf{d}y) = \frac{1}{r\sin\vartheta}(-\sin\varphi\,\mathbf{d}x + \cos\varphi\,\mathbf{d}y)$$

Derivace bázových vektorů a forem

$$\nabla \boldsymbol{e}_{\hat r} = \frac{1}{r}\,\boldsymbol{e}^{\hat\vartheta}\,\boldsymbol{e}_{\hat\vartheta} + \frac{1}{r}\,\boldsymbol{e}^{\hat\varphi}\,\boldsymbol{e}_{\hat\varphi} \qquad \nabla\cdot\boldsymbol{e}_{\hat r} = \frac{2}{r} \qquad \nabla\times\boldsymbol{e}_{\hat r} = 0$$

$$\nabla \boldsymbol{e}_{\hat\vartheta} = -\frac{1}{r}\,\boldsymbol{e}^{\hat\vartheta}\,\boldsymbol{e}_{\hat r} + \frac{1}{r}\cot\vartheta\,\boldsymbol{e}^{\hat\varphi}\,\boldsymbol{e}_{\hat\varphi} \qquad \nabla\cdot\boldsymbol{e}_{\hat\vartheta} = \frac{1}{r}\cot\vartheta \qquad \nabla\times\boldsymbol{e}_{\hat\vartheta} = \frac{1}{r}\,\boldsymbol{e}_{\hat\varphi}$$

$$\nabla \boldsymbol{e}_{\hat\varphi} = -\frac{1}{r}\,\boldsymbol{e}^{\hat\varphi}\,\boldsymbol{e}_{\hat r} - \frac{1}{r}\cot\vartheta\,\boldsymbol{e}^{\hat\varphi}\,\boldsymbol{e}_{\hat\vartheta} \qquad \nabla\cdot\boldsymbol{e}_{\hat\varphi} = 0 \qquad \nabla\times\boldsymbol{e}_{\hat\varphi} = -\frac{1}{r}\,\boldsymbol{e}_{\hat\vartheta} + \frac{1}{r}\cot\vartheta\,\boldsymbol{e}_{\hat r}$$

$$\nabla\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} = \frac{1}{r}\,\mathbf{d}\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} + \frac{1}{r}\,\mathbf{d}\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} \qquad \nabla\cdot\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} = \frac{2}{r} \qquad \nabla\times\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} = 0$$

$$\nabla\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} = -r\,\mathbf{d}\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} + \frac{1}{r}\,\mathbf{d}r\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} + \cot\vartheta\,\mathbf{d}\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} \qquad \nabla\cdot\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} = \cot\vartheta \qquad \nabla\times\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} = \frac{2}{r\sin\vartheta}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi}$$

$$\nabla\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = \frac{1}{r}\,\mathbf{d}r\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} + \cot\vartheta\,\mathbf{d}\vartheta\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} - r\sin^2\vartheta\,\mathbf{d}\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r} - \sin\vartheta\cos\vartheta\,\mathbf{d}\varphi\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta}$$

$$\nabla\cdot\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = 0 \qquad \nabla\times\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\varphi} = -\frac{2}{r}\sin\vartheta\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\vartheta} + 2\cos\vartheta\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} r}$$

$$\nabla\,\mathbf{d}r = r\;\mathbf{d}\vartheta\,\mathbf{d}\vartheta + r\sin^2\vartheta\;\mathbf{d}\varphi\,\mathbf{d}\varphi \qquad \nabla^2 r = \frac{2}{r}$$

$$\nabla\,\mathbf{d}\vartheta = -\frac{1}{r}(\,\mathbf{d}r\,\mathbf{d}\vartheta + \,\mathbf{d}\vartheta\,\mathbf{d}r) + \sin\vartheta\cos\vartheta\,\mathbf{d}\varphi\,\mathbf{d}\varphi \qquad \nabla^2\vartheta = \frac{1}{r^2}\cot\vartheta$$

$$\nabla\,\mathbf{d}\varphi = -\frac{1}{r}(\,\mathbf{d}r\,\mathbf{d}\varphi + \,\mathbf{d}\varphi\,\mathbf{d}r) - \cot\vartheta(\,\mathbf{d}\vartheta\,\mathbf{d}\varphi + \,\mathbf{d}\varphi\,\mathbf{d}\vartheta) \qquad \nabla^2\varphi = 0$$

Souřadnice kovariantní derivace

$$a^k{}_{;m} = a^k{}_{,m} + \Gamma^{\,k}_{mn} a^n \qquad , \qquad a_{n;m} = a_{n,m} - \Gamma^{\,k}_{mn} a_k$$

$$\Gamma^{\,r}_{\vartheta\vartheta} = -r \qquad \Gamma^{\,\vartheta}_{r\vartheta} = \Gamma^{\,\vartheta}_{\vartheta r} = \frac{1}{r} \qquad \Gamma^{\,\varphi}_{r\varphi} = \Gamma^{\,\varphi}_{\varphi r} = \frac{1}{r}$$

$$\Gamma^{\,r}_{\varphi\varphi} = -r\sin^2\vartheta \qquad \Gamma^{\,\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta \qquad \Gamma^{\,\varphi}_{\vartheta\varphi} = \Gamma^{\,\varphi}_{\varphi\vartheta} = \cot\vartheta$$

ostatní Γ-koeficienty jsou nulové

Operátory ve sférických souřadnicích

gradient skaláru: $\boldsymbol{a} = \mathbf{d}f = \nabla f$

$$a_r = f_{,r} \qquad a_{\hat{r}} = f_{,r} \qquad a^r = f_{,r}$$

$$a_\vartheta = f_{,\vartheta} \qquad a_{\hat{\vartheta}} = \frac{1}{r}\, f_{,\vartheta} \qquad a^\vartheta = \frac{1}{r^2}\, f_{,\vartheta}$$

$$a_\varphi = f_{,\varphi} \qquad a_{\hat{\varphi}} = \frac{1}{r\sin\vartheta}\, f_{,\varphi} \qquad a^\varphi = \frac{1}{r^2\sin^2\vartheta}\, f_{,\varphi}$$

divergence: $f = \nabla \cdot \boldsymbol{a}$

$$\begin{aligned} f &= \frac{1}{r^2}(r^2\, a^r)_{,r} + \frac{1}{\sin\vartheta}(\sin\vartheta\, a^\vartheta)_{,\vartheta} + a^\varphi{}_{,\varphi} \\ &= \frac{1}{r^2}(r^2\, a^{\hat{r}})_{,r} + \frac{1}{r\sin\vartheta}\,(\sin\vartheta\, a^{\hat{\vartheta}})_{,\vartheta} + \frac{1}{r\sin\vartheta}\, a^{\hat{\varphi}}{}_{,\varphi} \\ &= \frac{1}{r^2}(r^2\, a_r)_{,r} + \frac{1}{r^2\sin\vartheta}\,(\sin\vartheta\, a_\vartheta)_{,\vartheta} + \frac{1}{r^2\sin^2\vartheta}\, a_{\varphi,\varphi} \end{aligned}$$

rotace: $\boldsymbol{b} = \nabla \times \boldsymbol{a}$

$$b_r = \frac{1}{r^2\sin\vartheta}\,(a_{\varphi,\vartheta} - a_{\vartheta,\varphi}) \qquad b_{\hat{r}} = \frac{1}{r\sin\vartheta}\,(\sin\vartheta\, a_{\hat{\varphi}})_{,\vartheta} - a_{\hat{\vartheta},\varphi} \qquad b^r = \frac{1}{\sin\vartheta}\,((\sin^2\vartheta\, a^\varphi)_{,\vartheta} - a^\vartheta{}_{,\varphi})$$

$$b_\vartheta = \frac{1}{\sin\vartheta}\,(a_{r,\varphi} - a_{\varphi,r}) \qquad b_{\hat{\vartheta}} = \frac{1}{r}\left(\frac{1}{\sin\vartheta}\, a_{\hat{r},\varphi} - (r\, a_{\hat{\varphi}})_{,r}\right) \qquad b^\vartheta = \frac{1}{r^2\sin\vartheta}\, a^r{}_{,\varphi} - \frac{1}{r^2}(r^2\, a^\varphi)_{,r})$$

$$b_\varphi = \sin\vartheta\,(a_{\vartheta,r} - a_{r,\vartheta}) \qquad b_{\hat{\varphi}} = \frac{1}{r}\,((r\, a_{\hat{\vartheta}})_{,r} - a_{\hat{r},\vartheta}) \qquad b^\varphi = \frac{1}{r^2\sin\vartheta}\,\big((r^2\, a^\vartheta)_{,r} - a^r{}_{,\vartheta}\big)$$

laplace:

$$\begin{aligned} \nabla^2 f &= \frac{1}{r^2}\,(r^2\, f_{,r})_{,r} + \frac{1}{r^2\sin\vartheta}\,(\sin\vartheta\, f_{,\vartheta})_{,\vartheta} + \frac{1}{r^2\sin^2\vartheta}\, f_{,\varphi\varphi} \\ &= f_{,rr} + \frac{1}{r^2} f_{,\vartheta\vartheta} + \frac{1}{r^2\sin^2\vartheta} f_{,\varphi\varphi} + \frac{2}{r} f_{,r} + \frac{\cot\vartheta}{r^2} f_{,\vartheta} \end{aligned}$$