Basics of calculus

- derivative of a function

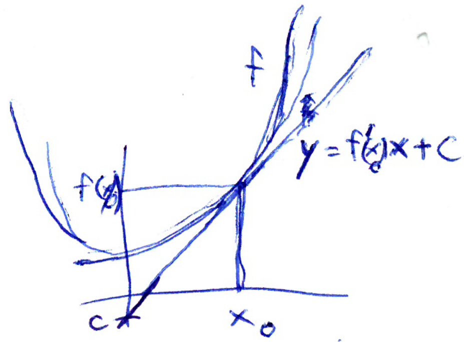
one variable $\left.\frac{df(x)}{dx}\right|_{x=x_0} = f'(x_0) \equiv \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$

more variables – partial derivatives

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

keep y fixed

similarly for y



constant C : $f(x_0) = f'(x_0)x_0 + C$

$C = f(x_0) - f'(x_0)x_0$

$\Rightarrow y = f(x_0) + f'(x_0)(x-x_0)$

$\left(+\frac{1}{2}f''(x_0)(x-x_0)^2 + \dots\right)$

first two terms of Taylor series

- rules

$(\alpha f(x) + \beta g(x))' = \alpha f'(x) + \beta g'(x)$ linearity

$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$ Leibnitz rule

$$\lim_{h\to 0}\frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0)}{h} = \lim_{h\to 0}\left(\frac{(f(x_0+h)-f(x_0))g(x_0+h)}{h} + \frac{f(x_0)(g(x_0+h)-g(x_0))}{h}\right)$$

$$= f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

$(f(g(x)))' = f'(g(x))\,g'(x)$ composition rule

$(f^{-1})'(y) = \dfrac{1}{f'(\underbrace{f^{-1}(y)}_{x})}$ inverse function $\leftarrow$ from $\left.\frac{d}{dy}\right|$ $f(f^{-1}(y)) = y$

examples

$(x^n)' = \lim_{h\to 0}\dfrac{\overbrace{(x+h)^n}^{x^n + nhx^{n-1} + \dots h^2\ h^3} - x^n}{h} = nx^{n-1}$

$(e^x)' = e^x$ $\leftarrow$ we need $\lim_{h\to 0}\frac{e^h-1}{h} = 1$

$(\sin x)' = \cos x$ $\leftarrow$ we need $\lim_{h\to 0}\frac{\sin h}{h} = 1$, $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$

- fundamental theorem of calculus

$f \in C[a,b]$ and let $F(x) = \int_0^x f(t)\,dt$ on $[a,b]$

then $F'(x) = f(x)$ (F is antiderivative of f)

$$\Rightarrow \boxed{\int_a^b f(t)\,dt = F(b) - F(a)}$$

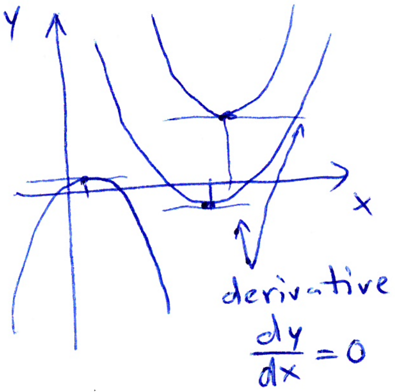
→ so we need to know the function $F(x)$ which has as its derivative the function $f(x)$ (not always in the closed form)

Examples on derivatives

1) Maxima ~~and~~ minima of functions

a) Find a derivative of $y(x) = ax^2 + bx + C$

and its maximum or minimum depending on parameters



$$\boxed{\frac{dy(x)}{dx} = 2ax + b \cdot 1 + 0}$$

$$\frac{d(x^2)}{dx} = 2x \qquad \frac{dx}{dx} = 1 \qquad \frac{dc}{dx} = 0$$

maxima or minima are at x_m

satisfying $\left.\frac{dy(x)}{dx}\right|_{x_m} = 2a\,x_m + b = 0$

or $\boxed{x_m = -\frac{b}{2a}}$

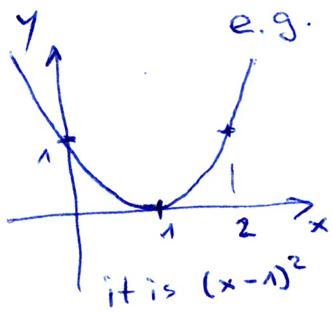
for $a \neq 0$ (otherwise it is a line!)

- how to determine whether it is a maximum or minimum?

From the sign of the second derivative!

If the second derivative is positive, then the first derivative grows and we have a minimum!

e.g. for $y(x) = x^2 - 2x + 1$ we get

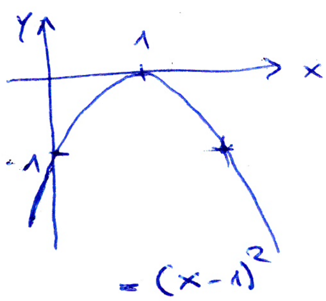


$$\frac{dy(x)}{dx} = 2x - 2 \Rightarrow 2x_m = 2, \quad x_m = 1 = -\frac{\overset{-2}{b}}{\underset{1}{2a}}$$

$$\frac{d^2y(x)}{dx} = 2 > 0 \Rightarrow \text{minimum (in general it is } = 2a)$$

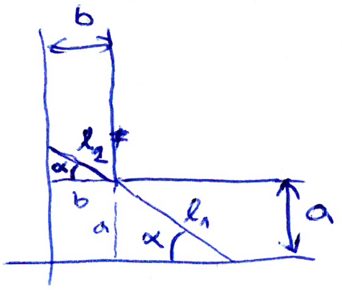
but for $y(x) = -x^2 + 2x - 1$ we get

$$\frac{dy(x)}{dx} = -2x + 2 \Rightarrow -2x_m = -2, \quad x_m = 1 = -\frac{\overset{2}{b}}{\underset{-1}{2a}}$$



$$\frac{d^2y(x)}{dx} = -2 < 0 \Rightarrow \text{maximum}$$

b) Find the maximal length of a ladder you can carry horizontally in a corridor of L-shape



- we express the length as a function of the angle α:

$$\frac{a}{l_1} = \sin\alpha \qquad \frac{b}{l_2} = \cos\alpha$$

$$L(\alpha) = l_1(\alpha) + l_2(\alpha) = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$$

- the minimum of this function corresponds to the maximal length of the ladder

- we calculate the derivative using

$$\frac{d}{d\alpha}\left(\frac{1}{f(\alpha)}\right) = -\frac{f'(\alpha)}{f^2(\alpha)}$$

and we get

$$\frac{dL(\alpha)}{d\alpha} = a\left(-\frac{\cos\alpha}{\sin^2\alpha}\right) + b\left(-\frac{(-\sin\alpha)}{\cos^2\alpha}\right) =$$

$$= \frac{b\sin\alpha}{\cos^2\alpha} - \frac{a\cos\alpha}{\sin^2\alpha} = \frac{b\sin^3\alpha - a\cos^3\alpha}{\cos^2\alpha\sin^2\alpha}$$

thus α_m (where minimum is) is given by

$$b\sin^3\alpha_m - a\cos^3\alpha_m = 0$$

or

$$\boxed{\tan\alpha_m = \sqrt[3]{\frac{a}{b}}}$$

- as one can expect, for $a = b$ we get

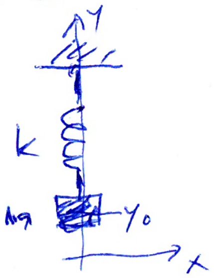
$$\alpha_m = \arctan 1 = \frac{\pi}{4} = 45^\circ$$

c)

Examples on derivatives

2) Checking that a certain „equation of motion" (description of a trajectory) is a solution of Newton's second law

a) linear harmonic oscillator



- you can find that oscillations of a mass m on a spring with the spring constant k are given by

$$y(t) = y_0 + A \sin \omega (t - t_0) \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

- check that it is a general solution of Newton's second law (in y-direction only)

$$m \cdot a_y = m \frac{d^2 y(t)}{dt^2} = F_y = -k(y(t) - y_0)$$

$$\frac{dy(t)}{dt} = 0 + A[\cos \omega (t - t_0)] \cdot \omega \qquad \frac{d\, \omega(t-t_0)}{dt} = \omega$$

$$\frac{d^2 y(t)}{dt^2} = -A \omega^2 \sin \omega (t - t_0) \qquad \text{thus}$$

$$m[-A\omega^2 \sin \omega (t - t_0)] = -k A \sin \omega (t - t_0) \Rightarrow \omega^2 = \frac{k}{m} \checkmark$$

b) damped harmonic oscillator for the frictional force

$$F_f = -c\, v$$

viscous damping coefficient

$$F = -k(y(t) - y_0) - c \frac{dy(t)}{dt}$$

- now the solution is