Projectile motion

Trajectory from Newton’s second law of motion

Force

In[1]:=
```
F = {0, -m g};
```

Initial conditions

In[2]:=
```
r0 = {x0, y0};
v0 = {w Cos[ϕ], w Sin[ϕ]};
```

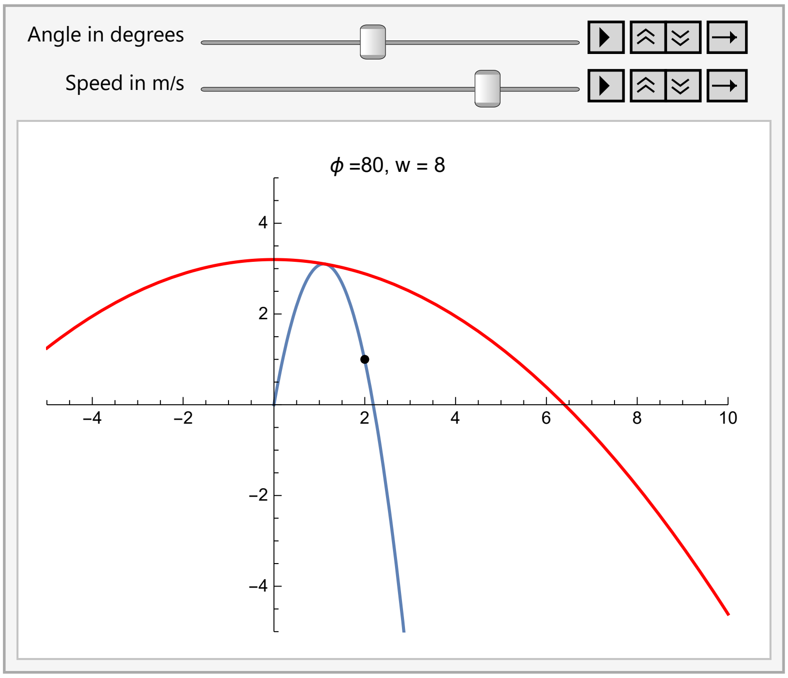
Solution of Newton’s second law

In[4]:=
```
rt = DSolve[{
    m D[x[t], t, t] == F[[1]],
    m D[y[t], t, t] == F[[2]],
    x[0] == r0[[1]], y[0] == r0[[2]],
    x'[0] == v0[[1]], y'[0] == v0[[2]]
   },
   {x[t], y[t]}, t
  ]
```

Out[4]= $\left\{\left\{x[t] \to x0 + t\, w \,\mathrm{Cos}[\phi],\ y[t] \to \frac{1}{2}\left(-g\, t^2 + 2\, y0 + 2\, t\, w\, \mathrm{Sin}[\phi]\right)\right\}\right\}$

In[5]:=
```
fixedValues = {g → 10, x0 → 0, y0 → 0};
envelope = y0 + w^2 / (2 g) + g w^2 (x^2 - 2 x0 x) / (2 (wg^2 x0^2 - w^4));
{xx1, yy1} = {2, 1};
Animate[
 Show[
  ParametricPlot[{x[t], y[t]} /. rt /. fixedValues /. {ϕ → phi * π / 180, w → ww},
   {t, 0, 10}, PlotRange → {{-5, 10}, {-5, 5}}],
  Plot[envelope /. fixedValues /. {ϕ → phi * π / 180, w → ww}, {x, -5, 10},
   PlotStyle → Red], ListPlot[{{xx1, yy1}}, PlotStyle -> Black],
  PlotLabel → Row[{"ϕ =", phi, ", w = ", ww}]],
 {{phi, 30, "Angle in degrees"}, 0, 180, 1},
 {{ww, 8, "Speed in m/s"}, 0.1, 10, 0.1},
 AnimationRunning → False
]
```

Out[8]=



Trajectory as a function y = f(x)

In[9]:=
```
traj = Simplify[Solve[
    X == (x[t] /. rt[[1, 1]]) && Y == (y[t] /. rt[[1, 2]]),
    {t, Y}]]
```

Out[9]= $\left\{\left\{t \to \frac{(X - x0)\,\text{Sec}[\phi]}{w},\ Y \to y0 - \frac{g\,(X - x0)^2\,\text{Sec}[\phi]^2}{2\,w^2} + (X - x0)\,\text{Tan}[\phi]\right\}\right\}$

Determining the angle to hit an object at (x1,y1)

Mathematica solution

Insert the point (x1 , y1) into the trajectory equation and solve it for ϕ

In[10]:=
```
eqx = (y1 - (Y /. traj[[1, 2]]) /. {X → x1})
eqd = Simplify[Cos[ϕ]^2 eqx /. {x1 -> x0 + d Cos[α], y1 -> y0 + d Sin[α]}]
```

Out[10]= $-y0 + y1 + \frac{g\,(-x0 + x1)^2\,\mathrm{Sec}[\phi]^2}{2\,w^2} - (-x0 + x1)\,\mathrm{Tan}[\phi]$

Out[11]= $\frac{1}{2}\,d\left(\frac{d\,g\,\mathrm{Cos}[\alpha]^2}{w^2} + 2\,\mathrm{Cos}[\phi]^2\,\mathrm{Sin}[\alpha] - 2\,\mathrm{Cos}[\alpha]\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\phi]\right)$

In[12]:=
```
sol = Simplify[Assuming[w > 0 && g > 0 && d > 0 && α > 0 && α < π/2, Solve[eqd == 0, ϕ]]]
```

Out[12]=

$$\Big\{\Big\{\phi \to \text{ConditionalExpression}\Big[\text{ArcTan}\Big[-\sqrt{\frac{-\sqrt{w^4\,\text{Cos}[\alpha]^4\left(w^4-d^2 g^2\,\text{Cos}[\alpha]^2-2\,d\,g\,w^2\,\text{Sin}[\alpha]\right)}+\text{Cos}[\alpha]^2\left(w^4-d\,g\,w^2\,\text{Sin}[\alpha]\right)}{w^4}},\; -\frac{1}{2\,d\,g\,w^2}\text{Sec}[\alpha]^3\left(w^4+w^4\,\text{Cos}[2\,\alpha]+2\sqrt{w^4\,\text{Cos}[\alpha]^4\left(w^4-d^2 g^2\,\text{Cos}[\alpha]^2-2\,d\,g\,w^2\,\text{Sin}[\alpha]\right)}\right)\sqrt{\frac{-\sqrt{w^4\,\text{Cos}[\alpha]^4\left(w^4-d^2 g^2\,\text{Cos}[\alpha]^2-2\,d\,g\,w^2\,\text{Sin}[\alpha]\right)}+\text{Cos}[\alpha]^2\left(w^4-d\,g\,w^2\,\text{Sin}[\alpha]\right)}{w^4}}\Big]+2\,\pi\,c_1,\; c_1\in\mathbb{Z}\Big]\Big\},$$

$$\Big\{\phi \to \text{ConditionalExpression}\Big[\text{ArcTan}\Big[\sqrt{\frac{-\sqrt{w^4\,\text{Cos}[\alpha]^4\left(w^4-d^2 g^2\,\text{Cos}[\alpha]^2-2\,d\,g\,w^2\,\text{Sin}[\alpha]\right)}+\text{Cos}[\alpha]^2\left(w^4-d\,g\,w^2\,\text{Sin}[\alpha]\right)}{w^4}},\; \frac{1}{2\,d\,g\,w^2}\text{Sec}[\alpha]^3\left(w^4+w^4\,\text{Cos}[2\,\alpha]+2\sqrt{w^4\,\text{Cos}[\alpha]^4\left(w^4-d^2 g^2\,\text{Cos}[\alpha]^2-2\,d\,g\,w^2\,\text{Sin}[\alpha]\right)}\right)\sqrt{\frac{-\sqrt{w^4\,\text{Cos}[\alpha]^4\left(w^4-d^2 g^2\,\text{Cos}[\alpha]^2-2\,d\,g\,w^2\,\text{Sin}[\alpha]\right)}+\text{Cos}[\alpha]^2\left(w^4-d\,g\,w^2\,\text{Sin}[\alpha]\right)}{w^4}}\Big]+2\,\pi\,c_1,\; c_1\in\mathbb{Z}\Big]\Big\},$$

$$\Big\{\phi \to \text{ConditionalExpression}\Big[\text{ArcTan}\Big[-\sqrt{\frac{\sqrt{w^4\,\text{Cos}[\alpha]^4\left(w^4-d^2 g^2\,\text{Cos}[\alpha]^2-2\,d\,g\,w^2\,\text{Sin}[\alpha]\right)}+\text{Cos}[\alpha]^2\left(w^4-d\,g\,w^2\,\text{Sin}[\alpha]\right)}{w^4}},\; -\frac{1}{2\,d\,g\,w^2}\text{Sec}[\alpha]^3\left(w^4+w^4\,\text{Cos}[2\,\alpha]-2\sqrt{w^4\,\text{Cos}[\alpha]^4\left(w^4-d^2 g^2\,\text{Cos}[\alpha]^2-2\,d\,g\,w^2\,\text{Sin}[\alpha]\right)}\right)\sqrt{\frac{\sqrt{w^4\,\text{Cos}[\alpha]^4\left(w^4-d^2 g^2\,\text{Cos}[\alpha]^2-2\,d\,g\,w^2\,\text{Sin}[\alpha]\right)}+\text{Cos}[\alpha]^2\left(w^4-d\,g\,w^2\,\text{Sin}[\alpha]\right)}{w^4}}\Big]+2\,\pi\,c_1,\; c_1\in\mathbb{Z}\Big]\Big\},$$

$$\Big\{\phi \to \text{ConditionalExpression}\Big[\text{ArcTan}\Big[\sqrt{\frac{\sqrt{w^4\,\text{Cos}[\alpha]^4\left(w^4-d^2 g^2\,\text{Cos}[\alpha]^2-2\,d\,g\,w^2\,\text{Sin}[\alpha]\right)}+\text{Cos}[\alpha]^2\left(w^4-d\,g\,w^2\,\text{Sin}[\alpha]\right)}{w^4}},\; \frac{1}{2\,d\,g\,w^2}\text{Sec}[\alpha]^3\left(w^4+w^4\,\text{Cos}[2\,\alpha]-2\sqrt{w^4\,\text{Cos}[\alpha]^4\left(w^4-d^2 g^2\,\text{Cos}[\alpha]^2-2\,d\,g\,w^2\,\text{Sin}[\alpha]\right)}\right)\sqrt{\frac{\sqrt{w^4\,\text{Cos}[\alpha]^4\left(w^4-d^2 g^2\,\text{Cos}[\alpha]^2-2\,d\,g\,w^2\,\text{Sin}[\alpha]\right)}+\text{Cos}[\alpha]^2\left(w^4-d\,g\,w^2\,\text{Sin}[\alpha]\right)}{w^4}}\Big]+2\,\pi\,c_1,\; c_1\in\mathbb{Z}\Big]\Big\}\Big\}$$

Only solutions number 2 and 4 give reasonable results

```
In[13]:= angle1 = Normal[ϕ /. sol[[4, 1]] /. {ℭ1 -> 0}]
         angle2 = Normal[ϕ /. sol[[2, 1]] /. {ℭ1 -> 0}]
```

Out[13]=
$$\text{ArcTan}\left[\sqrt{\frac{\sqrt{w^4 \text{Cos}[\alpha]^4 \left(w^4 - d^2 g^2 \text{Cos}[\alpha]^2 - 2 d g w^2 \text{Sin}[\alpha]\right)} + \text{Cos}[\alpha]^2 \left(w^4 - d g w^2 \text{Sin}[\alpha]\right)}{w^4}},\right.$$
$$\left.\frac{1}{2 d g w^2} \text{Sec}[\alpha]^3 \left(w^4 + w^4 \text{Cos}[2 \alpha] - 2 \sqrt{w^4 \text{Cos}[\alpha]^4 \left(w^4 - d^2 g^2 \text{Cos}[\alpha]^2 - 2 d g w^2 \text{Sin}[\alpha]\right)}\right) \sqrt{\frac{\sqrt{w^4 \text{Cos}[\alpha]^4 \left(w^4 - d^2 g^2 \text{Cos}[\alpha]^2 - 2 d g w^2 \text{Sin}[\alpha]\right)} + \text{Cos}[\alpha]^2 \left(w^4 - d g w^2 \text{Sin}[\alpha]\right)}{w^4}}\right]$$

Out[14]=
$$\text{ArcTan}\left[\sqrt{\frac{-\sqrt{w^4 \text{Cos}[\alpha]^4 \left(w^4 - d^2 g^2 \text{Cos}[\alpha]^2 - 2 d g w^2 \text{Sin}[\alpha]\right)} + \text{Cos}[\alpha]^2 \left(w^4 - d g w^2 \text{Sin}[\alpha]\right)}{w^4}},\right.$$
$$\left.\frac{1}{2 d g w^2} \text{Sec}[\alpha]^3 \left(w^4 + w^4 \text{Cos}[2 \alpha] + 2 \sqrt{w^4 \text{Cos}[\alpha]^4 \left(w^4 - d^2 g^2 \text{Cos}[\alpha]^2 - 2 d g w^2 \text{Sin}[\alpha]\right)}\right) \sqrt{\frac{-\sqrt{w^4 \text{Cos}[\alpha]^4 \left(w^4 - d^2 g^2 \text{Cos}[\alpha]^2 - 2 d g w^2 \text{Sin}[\alpha]\right)} + \text{Cos}[\alpha]^2 \left(w^4 - d g w^2 \text{Sin}[\alpha]\right)}{w^4}}\right]$$

Numerically for a given large enough speed we get two angles

```
In[15]:= ww = 8;
         {xx1, yy1} = {2, 1};
         N[{angle1 / π * 180, angle2 / π * 180} /. fixedValues /.
           {w -> ww, α -> ArcTan[yy1 / xx1], d -> Sqrt[xx1^2 + yy1^2]}]

Out[17]= {36.5887, 79.9764}
```

Solution by hand

By substituting for Sin as a function of Cos

in

$$\frac{d g \text{Cos}[\alpha]^2}{w^2} + 2 \text{Cos}[\phi]^2 \text{Sin}[\alpha] - 2 \text{Cos}[\alpha] \text{Cos}[\phi] \text{Sin}[\phi] = 0$$

and squaring it we get a quadratic equation for Cos[ϕ]^2 with the following coefficients and the discriminant Δ

```
In[18]:= β = g d / w^2;
a = 1; b = (β Sin[α] - 1) Cos[α]^2; c = β^2 Cos[α]^4 / 4;
Δ = b^2 - 4 a c
myAngle1 = ArcCos[Sqrt[(-b + Sqrt[Δ]) / 2 a]]
myAngle2 = ArcCos[Sqrt[(-b - Sqrt[Δ]) / 2 a]]
```

Out[20]= $-\frac{d^2 g^2 \text{Cos}[\alpha]^4}{w^4} + \text{Cos}[\alpha]^4 \left(-1 + \frac{d\,g\,\text{Sin}[\alpha]}{w^2}\right)^2$

Out[21]= $\text{ArcCos}\left[\frac{\sqrt{-\text{Cos}[\alpha]^2 \left(-1 + \frac{d\,g\,\text{Sin}[\alpha]}{w^2}\right) + \sqrt{-\frac{d^2 g^2 \text{Cos}[\alpha]^4}{w^4} + \text{Cos}[\alpha]^4 \left(-1 + \frac{d\,g\,\text{Sin}[\alpha]}{w^2}\right)^2}}}{\sqrt{2}}\right]$

Out[22]= $\text{ArcCos}\left[\frac{\sqrt{-\text{Cos}[\alpha]^2 \left(-1 + \frac{d\,g\,\text{Sin}[\alpha]}{w^2}\right) - \sqrt{-\frac{d^2 g^2 \text{Cos}[\alpha]^4}{w^4} + \text{Cos}[\alpha]^4 \left(-1 + \frac{d\,g\,\text{Sin}[\alpha]}{w^2}\right)^2}}}{\sqrt{2}}\right]$

Numerically we get the same result as above

```
In[23]:= ww = 8;
{xx1, yy1} = {2, 1};
N[{Δ, myAngle1 / π * 180, myAngle2 / π * 180} /. fixedValues /.
   {w -> ww, α -> ArcTan[yy1 / xx1], d -> Sqrt[xx1^2 + yy1^2]}]
Out[25]= {0.3775, 36.5887, 79.9764}
```

Minimal speed to reach the given point

The minimal speed can be determined from condition $\Delta = 0$.

Here only one of four solutions has a physical meaning

```
In[26]:= minw = Simplify[Assuming[g > 0 && d > 0 && α > 0 && α < π / 2, Solve[Δ == 0, w]]]
{xx1, yy1} = {2, 1};
N[minw /. fixedValues /. {α -> ArcTan[yy1 / xx1], d -> Sqrt[xx1^2 + yy1^2]}]
```

Out[26]= $\{\{w \to -\sqrt{d}\,\sqrt{g}\,\sqrt{-1 + \text{Sin}[\alpha]}\}, \{w \to \sqrt{d}\,\sqrt{g}\,\sqrt{-1 + \text{Sin}[\alpha]}\}, \{w \to -\sqrt{d}\,\sqrt{g}\,\sqrt{1 + \text{Sin}[\alpha]}\}, \{w \to \sqrt{d}\,\sqrt{g}\,\sqrt{1 + \text{Sin}[\alpha]}\}\}$

Out[28]= {{w → 0. - 3.51578 i}, {w → 0. + 3.51578 i}, {w → -5.68864}, {w → 5.68864}}