Introductory Course in Mathematics
Examples of Problems

Below are examples of problems to be solved during the Introductory Course in Mathematics intended for new students of the bachelor program Science. If you are not sure how to solve a particular problem, consider coming to the specific lecture of the course covering the topic. But, of course, you can attend the whole course just to refresh your knowledge of high school mathematics.

Equations and inequations in real numbers

1. Find the solution of the following system of linear equations dependent on the parameter p

$$3x + 2y = p\,,$$
$$px + 4y = 2p\,.$$

2. Find the solution of the quadratic inequation dependent on the parameter p

$$px^2 - 2x + 2 > 0\,.$$

3. Find the solution of the absolute value equation

$$|4 - x| - |2x + 3| = 7\,.$$

4. Find the solution of the absolute value inequation

$$x^2 - 3|x| + 2 > 0\,.$$

5. Find all $x \in \mathbb{R}$ satisfying the equation

$$\sqrt{\frac{2x+1}{x-3}} + \sqrt{\frac{x-3}{2x+1}} = \frac{34}{15}\,.$$

   Hint: Use the substitution $y = \sqrt{\frac{2x+1}{x-3}}$.

6. Find all $x \in \mathbb{R}$ satisfying

$$\sqrt{2 + x + 2\sqrt{x+1}} + \sqrt{x+1} < \frac{34}{15}\,.$$

   Hint: Use the substitution $y = \sqrt{x+1}$.

Analytic geometry

1. Write down different forms of the equation of a straight line passing through the points $A[-2, 2]$ and $B[4, -1]$.

2. Determine the intersection of and the angle between two straight lines. The first line is given in the parametric form

$$x = 5 - 3t, \quad y = 2 + 2t$$

   and the second one is determined by the points $A[1, 2]$ and $B[4, 8]$.

3. Write down the equation of a straight line passing through the point $A[2, 3]$ and making the angle $45°$ with the line given by the equation $2x + 5y - 5 = 0$.

4. Determine the distance of the point $A[7, 9, 7]$ in the space from the line given in the parametric form

$$x = 2 + 4t, \quad y = 1 + 3t, \quad z = 2t, \quad t \in \mathbb{R}.$$

5. Determine the equation of a tangent t to the circle k given by the equation

$$(x - 3)^2 + (y + 2)^2 = 100$$

   at the point $A[9, 6]$.

Trigonometry

1. Find all $x \in \mathbb{R}$ satisfying the equation

$$2 \sin(3x + \pi) = -1\,.$$

2. Find all $x \in \mathbb{R}$ satisfying the equation

$$2 \cos^2 x - 7 \cos x + 3 = 0\,.$$

3. Find all $x \in \mathbb{R}$ satisfying the equation

$$\sin x + \sqrt{3} \cos x = \sqrt{2}\,.$$

4. Find all $x \in \mathbb{R}$ satisfying the equation

$$\sin x + \cos x + \sin x \cos x = 1\,.$$

   Hint: Use the substitution $y = \sin x + \cos x$.

Complex numbers

1. Evaluate the expressions

$$(i-3)(i^6+5), \quad 2\frac{2+i}{3-i}, \quad \left|\frac{2+3i}{3-3i}\right|, \quad (1+i)^{12}\,.$$

2. Write the following complex numbers

$$i+\sqrt{3}, \quad \sqrt{6}-i\sqrt{6}, \quad -1-i, \quad 31+27i\,.$$

in the polar form $z = r(\cos\varphi + i\sin\varphi)$.

3. Find both roots of the quadratic equation

$$x^2-2x+2=0\,.$$

4. Find all solutions of the equation

$$x^4 = 64\,.$$

5. Find all solutions of the equation

$$x^3 = i\,.$$

Sequences of real numbers

1. Let $(a_k)_{k=1}^n$ be an arithmetic progression with the common difference d, i.e.

$$a_k = a_1 + (k-1)d\,.$$

(a) Show that the sum of all terms is given by

$$s_n = \sum_{k=1}^{n} a_k = \frac{n}{2}[2a_1+(n-1)d] = n\frac{a_1+a_n}{2}\,.$$

(b) Find d, a_1, a_8 and s_{11} if $a_4 = 6$ and $a_{11} = 34$.

(c) Find d and a_1 if $s_5 = s_6 = 60$.

2. Let $(a_k)_{k=1}^n$ be a geometric progression with the common ratio q, i.e.

$$a_k = a_1 q^{k-1}\,.$$

(a) Show that the sum of all terms is given by

$$s_n = \sum_{k=1}^{n} a_k = a_1\frac{1-q^n}{1-q}\,.$$

(b) Find q, a_1 and s_6 if $a_2 = 48$ and $a_5 = 162$.

(c) Find a_6 and s_8 if $a_3 = 1$ and $q = \frac{1}{3}$.

3. Let $(a_k)_{k=1}^{\infty}$ be the Fibonacci sequence given by the recursive relation

$$a_{n+2} = a_{n+1} + a_n \qquad \text{for all } n \in \mathbb{N}$$

with the initial values $a_1 = a_2 = 1$. Show that the n-th term can be expressed as

$$a_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the *golden ratio*.

4. Let $(a_k)_{k=1}^{\infty}$ be a sequence given by the recursive relation

$$a_{n+2} = 2(a_{n+1} - a_n) \qquad \text{for all } n \in \mathbb{N}$$

with the initial values $a_1 = 1, a_2 = 0$. Show that the n-th term can be expressed as

$$a_n = \frac{1}{2}((1+i)^n + (1-i)^n)\,.$$

Can you find a simple expression for a_n using only real numbers?

Types of Proofs

1. Prove that for all $n \in \mathbb{N}$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\,.$$

2. Prove that the sum of the arithmetic-geometric sequence for all $n \in \mathbb{N}$ is

$$s_n = \sum_{k=1}^{n} kq^k = q\frac{1-q^n}{(1-q)^2} - \frac{nq^{n+1}}{1-q}\,.$$

Hint: Subtract s_n and qs_n and use

$$\sum_{k=1}^{n} q^{k-1} = \frac{1-q^n}{1-q}\,.$$

3. If you know that

$$e^{x+y} = e^x \cdot e^y \qquad \text{for all } x, y \in \mathbb{R}$$

and $\ln x$ is the inverse function to e^x, prove the equality

$$\ln ab = \ln a + \ln b \qquad \text{for all } a, b \in (0, +\infty))\,.$$

4. Prove that $\sqrt{2}$ is an irrational number.