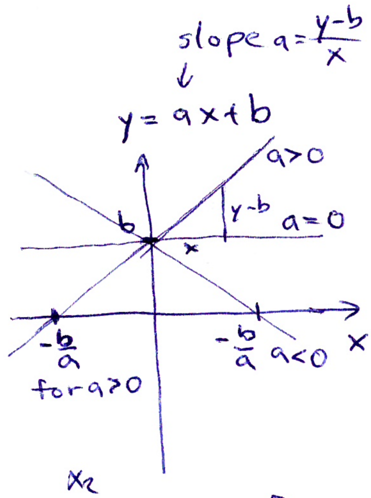
Equations and inequations - Formulas and Summary

1) linear equations

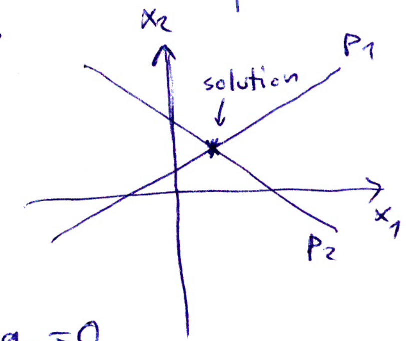


a) one unknown : $ax+b=0$ where $a,b \in \mathbb{R}$

solutions: $x = -\frac{b}{a}$ if $a \neq 0$

$x \in \mathbb{R}$ if $a=0$, $b=0$

no solution if $a=0$, $b \neq 0$



b) two unknowns: $a_{11}x_1 + a_{12}x_2 = b_1 \overset{\text{lines}}{\Rightarrow} P_1$

$a_{21}x_1 + a_{22}x_2 = b_2 \Rightarrow P_2$

solutions: if the equations define parallel lines, i.e.

$$\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} \Rightarrow a_{11}a_{22} - a_{12}a_{21} = 0$$

then there is no solution if $\frac{b_1}{b_2} \neq \frac{a_{11}}{a_{21}}$

or an infinite number of solutions (the same lines) given by $x_2 = \frac{b_1 - a_{11}x_1}{a_{12}}$ (for $a_{12} \neq 0$)

- if $D = a_{11}a_{22} - a_{12}a_{21} \neq 0$ then there is one solution

e.g. using Gauss elimination $x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{D}$, $x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{D}$

D is determinant of the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

c) more unknowns: a system of n linear equations for n unknows $x_1, \ldots x_n$ can be written in the matrix form $Ax = b$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

again, a unique solution exists if $\det A \neq 0$.

It can be solved using Gauss elimination (but it can be necesary to exchange rows if there is 0 on the diagonal in the process)

2) inequations: equations divide the plane, space etc.

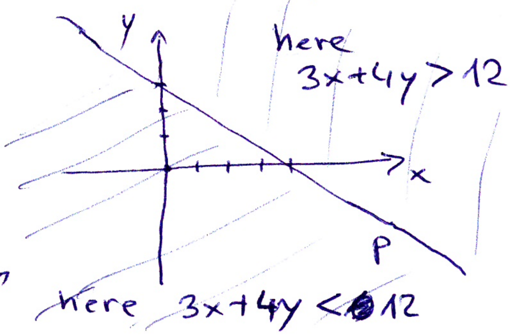
into two regions, e.g.

$$3x+4y=12$$

defines a line p

which divides the plane

into two half-planes



to find all solutions of e.g. $3x+4y\leq 12$

it is enough to draw a line and check one point

not lying on the line if it satisfies the inequation

to decide which half-plane is the solution

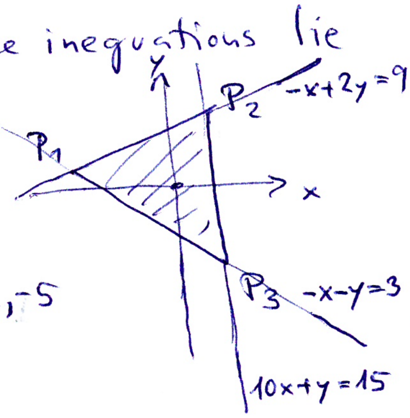
e.g. for the origin we get $0\leq 12$, thus the lower

half-plane is the solution

several inequations can define complex regions

where all points satisfying all the inequations lie

for example inequations



$-x+2y\leq 9$ give the triangle

$-x-y\leq 3$ given by points

and $10x+y\leq 15$ $P_1[-5,2]$, $P_2[1,5]$, $P_3[2,-5$

3) quadratic equation - one unknown

$$ax^2+bx+c=0 \quad \text{where } a\neq 0,\ b,\ c\in\mathbb{R}$$

it can be rewritten as

together = 0

$$a\Big[\underbrace{x^2+2\left(\tfrac{b}{2a}\right)x+\left(\tfrac{b}{2a}\right)^2}+\frac{c}{a}-\left(\tfrac{b}{2a}\right)^2\Big]=0$$

$$\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}=0$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}=\frac{\Delta}{4a^2}$$

discriminant must be ≥ 0 to have real solution

if $\Delta\geq 0$ we get

$$x_{1,2}=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{-b\pm\sqrt{b^2+4ac}}{2a}$$

and the original equation can be written as

$$a(x-x_1)(x-x_2)=0 \quad\Rightarrow\quad b=-a(x_1+x_2)$$

$$c=a\,x_1x_2$$

Equations and Inequations - Solved Examples

① Solve the system of linear equations

$$3x + 2y = p, \qquad (1)$$
$$px + 4y = 2p \qquad (2)$$

depending on a real parameter p.

Solution: By multiplying (1) by 2 and subtracting it from (2),

we get
$$(p-6)x = 0$$

Thus for $p \neq 6$ we get $x=0$ and $y = \frac{p}{2}$ (from (1)),

and for $p=6$ the equations are dependent (Eq. (2) = 2 × Eq. (1))

and there is an infinite number of solutions, for example

we can express $y = 3 - \frac{3}{2}x$ for any $x \in \mathbb{R}$.

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② Solve the quadratic inequation dependent on a real parameter p

$$px^2 - 2x + 2 > 0.$$

Solution: Using standard formulas for the discriminant and roots

of the quadratic equation $ax^2 + bx + c = 0$, we get

$$\Delta = b^2 - 4ac = 4 - 8p = 4(1-2p)$$

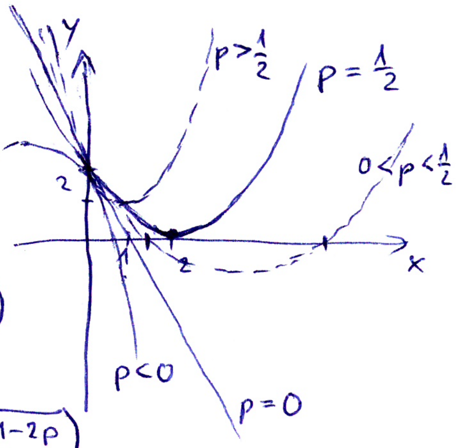
and for $\Delta \geq 0$, i.e. for $p \leq \frac{1}{2}$ and $p \neq 0$

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{2 \pm \sqrt{4(1-2p)}}{2p} = \frac{1 \pm \sqrt{1-2p}}{p}$$

Thus, we have the following cases:

p	roots	solution
$p > \frac{1}{2}$	no real roots	$x \in \mathbb{R}$
$0 < p \leq \frac{1}{2}$	$\frac{1 \pm \sqrt{1-2p}}{p}$	$x \in \left(-\infty, \frac{1-\sqrt{1-2p}}{p}\right) \cup \left(\frac{1+\sqrt{1-2p}}{p}, +\infty\right)$
$p = 0$	$x = 1$	$x \in (-\infty, 1)$
$p < 0$	$\frac{1 \pm \sqrt{1-2p}}{p}$	$x \in \left(\frac{1+\sqrt{1-2p}}{p}, \frac{1-\sqrt{1-2p}}{p}\right)$

root in $(0,1)$ root in $(-\infty, 0)$



③ Find $x \in \mathbb{R}$ satisfying the absolute value equation

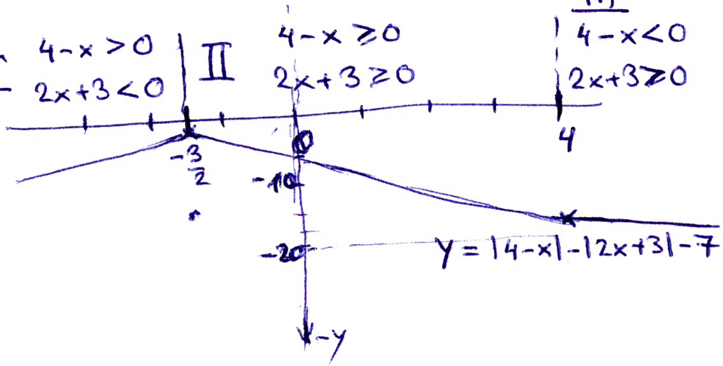
$$|4-x|-|2x+3|=7.$$

Solution: We devide $\mathbb{R}$ into three regions according to the sign of expressions in absolute values $4-x$ and $2x+3$ and replace

$$|4-x| = \begin{cases} 4-x & \text{for } x \le 4 \\ x-4 & \text{for } x>4 \end{cases}$$

and

$$|2x+3| = \begin{cases} 2x+3 & \text{for } x \ge -\frac{3}{2} \\ -2x-3 & \text{for } x<-\frac{3}{2} \end{cases}$$



In each region I, II, and III we have to solve a different equation:

I $(x<-\frac{3}{2})$: $4-x+2x+3=7 \Rightarrow x=0 > -\frac{3}{2} \Rightarrow$ no solution for $x<-\frac{3}{2}$

II $(-\frac{3}{2} \le x < 4)$: $4-x-2x-3=7 \Rightarrow x=-2 < -\frac{3}{2} \Rightarrow$ no solution in $\langle -\frac{3}{2}, 4)$

III $(x \ge 4)$: $x-4-2x-3=7 \Rightarrow x=-14 < 4 \Rightarrow$ no solution for $x \ge 4$

Thus the equation has no solution in real numbers.

Note that for the right-hand side smaller than 5.5, there would be two solutions. For example for $|4-x|-|2x+3|=5$ we would get solutions $x=-2$ (in the region I) and $x=-\frac{4}{3}$ (in the region II)

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④ Find all $x \in \mathbb{R}$ satisfying

$$x^2-3|x|+2>0.$$

Solution: Here $|x| = \begin{cases} x & \text{for } x \ge 0 \\ -x & \text{for } x<0 \end{cases}$

and thus for $x \ge 0$ we solve

$$x^2-3x+2>0$$

or

$$(x-1)(x-2)>0$$

which is valid for $x \in \langle 0,1) \cup (2,+\infty)$

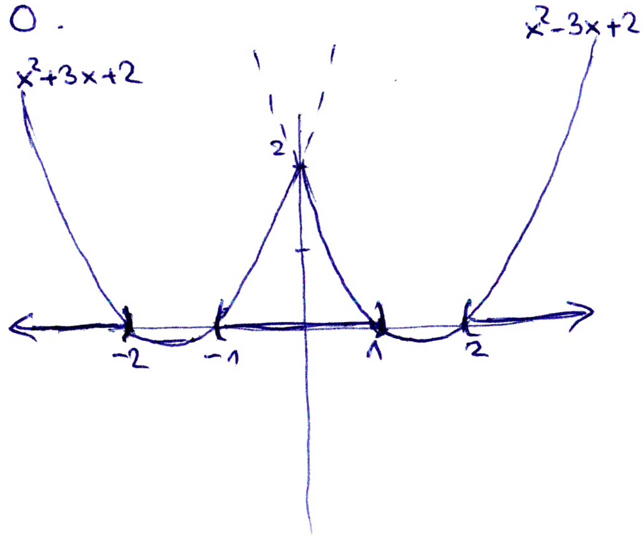
and for $x<0$ we solve

$$x^2+3x+2>0$$

or

$$(x+1)(x+2)>0$$

valid for $x \in (-\infty,-2) \cup (-1,0)$.



The solution is the combined intervals: $x \in (-\infty,-2) \cup (-1,1) \cup (2,+\infty)$.

(5) Find all $x \in \mathbb{R}$ satisfying

$$\sqrt{\frac{2x+1}{x-3}} + \sqrt{\frac{x-3}{2x+1}} = \frac{34}{15}.$$

Solution: Using the substitution $y = \sqrt{\frac{2x+1}{x-3}}$ we get

$$y + \frac{1}{y} = \frac{34}{15} \quad \text{or} \quad y^2 - \frac{34}{15}y + 1 = 0.$$

The discriminant $\Delta = \frac{34^2}{15^2} - 4 = \frac{16^2}{15^2}$

and we get two roots $y_{1,2} = \frac{34 \pm 16}{2 \cdot 15} = \left\langle \begin{array}{l} \frac{5}{3} \\ \frac{3}{5} \end{array}\right.$

From the first root $y_1 = \frac{5}{3}$ we have

$$\left(\frac{5}{3}\right)^2 = \frac{25}{9} = \frac{2x_1+1}{x_1-3} \Rightarrow x_1 = 12$$

and from $y_2 = \frac{3}{5}$ we have

$$\left(\frac{3}{5}\right)^2 = \frac{9}{25} = \frac{2x_2+1}{x_2-3} \Rightarrow x_2 = -\frac{52}{41}$$

The set of solutions is $\{12; -\frac{52}{41}\}$

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(6) Find all $x \in \mathbb{R}$ satisfying

$$\sqrt{2+x+2\sqrt{x+1}} + \sqrt{x+1} < 2.$$

Solution: Using the substitution $y = \sqrt{x+1}$ we have $y^2 = x+1$ for $x \geq -1$

and thus $\sqrt{y^2+1+2y} < 2-y$

and for $y<2$ we get $y^2+1+2y < (2-y)^2 = 4-4y+y^2$

or $6y < 3$

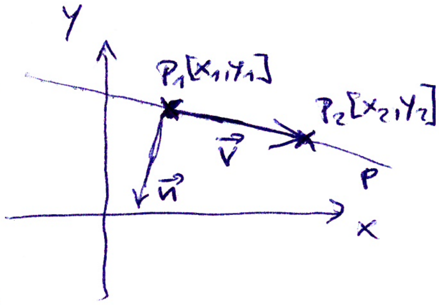
and thus $y = \sqrt{x+1} < \frac{1}{2}$

or for $x \geq -1$ $x+1 < \frac{1}{4} \Rightarrow x < -\frac{3}{4}$

The solution is $x \in \langle -1, -\frac{3}{4})$.

Analytic Geometry - Formulas

1) equations of a straight line in the plane



- a line is uniquely defined by two different points $P_1[x_1,y_1]$, $P_2[x_2,y_2]$

or by one point and one vector $\vec{v}=(v_x,v_y)$

(e.g. $\vec{v}$ can be $\vec{v}=\overrightarrow{P_1P_2}=(x_2-x_1,y_2-y_1)$)

this information gives the parametric form:

$$p: \left.\begin{aligned} x &= x_1 + t(x_2-x_1) \\ y &= y_1 + t(y_2-y_1) \end{aligned}\right\} \quad \vec{r} = P_1 + t\vec{v}$$

by eliminating t we get the standard form

$$ax + by = c \quad , a,b,c \in \mathbb{R}$$

or

$$\underbrace{(y_2-y_1)}_{a}x \underbrace{-(x_2-x_1)}_{b}y = \underbrace{x_1y_2-y_1x_2}_{c} \quad \text{if two points are given}$$

notice that $\vec{n}=(a,b)=(y_2-y_1,-(x_2-x_1))=(v_y,-v_x)$

is perpendicular to $\vec{v}$, thus being a normal vector of the line

finally expressing y as a function of x

we get the slope-intercept form

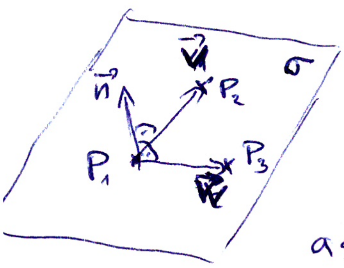
$$y = \underset{\text{slope}}{k}x + \underset{y\text{-intercept}}{y_0}$$

or from two points

$$y = \frac{y_2-y_1}{x_2-x_1}x + \frac{y_1x_2-x_1y_2}{x_2-x_1}$$

2) equations of a plane in the space

- a plane is given by three points (not on a line)



or by one point and two vectors

the parametric form

$$\sigma: \vec{r} = P_1 + t\vec{v} + s\vec{w} \quad \text{or} \quad \begin{aligned} x &= x_p + tv_x + sw_x \\ y &= y_p + tv_y + sw_y \\ z &= z_p + tv_z + sw_z \end{aligned}$$

again often $\vec{v}$ and $\vec{w}$ are given by points

$\vec{v}=P_2-P_1$, $\vec{w}=P_3-P_1$

- by eliminating t and s we get the standard form

$$ax + by + cz + d = 0$$

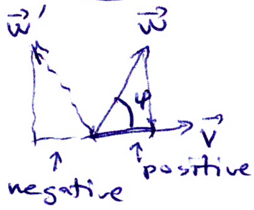
but more straightforward is to find a normal vector $\vec{n} \perp \vec{v}$ and $\vec{n} \perp \vec{w}$ by evaluating the cross (vector) product of $\vec{v}$ and $\vec{w}$ (works in 3D space)

$$\vec{n} = \vec{v} \times \vec{w} = (\underbrace{v_y w_z - v_z w_y}_{a}, \underbrace{v_z w_x - v_x w_z}_{b}, \underbrace{v_x w_y - v_y w_x}_{c})$$

giving

and d is determined by inserting one point of the plane into the equation of the plane.

③ scalar product of two vectors and its use



$$\left.\begin{array}{ll} \text{in 2D}: & \vec{v}\cdot\vec{w} = v_x w_x + v_y w_y \\ \text{in 3D}: & \vec{v}\cdot\vec{w} = \text{—} \;\text{''}\; \text{—} + v_z w_z \\ \text{in nD}: & \vec{v}\cdot\vec{w} = \sum_{i=1}^{n} v_i w_i \end{array}\right\} \text{algebraic definition}$$

or we can define it geometrically

$$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}| \cos\varphi \qquad (1)$$

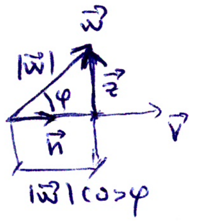
where $|\vec{v}| = \sqrt{v_1^2 + v_2^2 \cdots + v_n^2}$ is the norm of $\vec{v}$

and φ is the angle between $\vec{v}$ and $\vec{w}$

It can be shown that these two definitions are equivalent and thus the formula (1) can be used for determination of the angle φ if we calculate $\vec{v}\cdot\vec{w}$ algebraicly

i.e. $$\cos\varphi = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|} \Rightarrow \varphi \in \langle 0, \pi \rangle$$

If we use a normalized vector $\vec{n}_v = \frac{\vec{v}}{|\vec{v}|}$ then we get



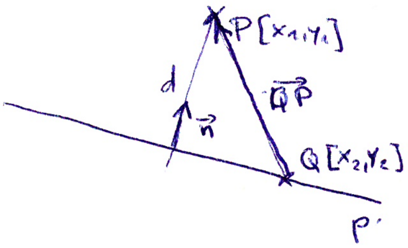
$$\vec{n}_v \cdot \vec{w} = |\vec{w}| \cos\varphi$$

i.e. a projection of $\vec{w}$ on the direction given by $\vec{v}$ or $\vec{n}_v$

This can be used to construct vectors perpendicular to $\vec{v}$ in the plane given by $\vec{v}$ and $\vec{w}$ in any dimension as $\vec{z} = \vec{w} - (\vec{w}\cdot\vec{n}_v)\vec{n}_v$

which can be useful in various applications.

④ distance of a point from a line in 2D and a plane in 3D



let p be a line given by $p: ax+by+c=0$

then the normal vector to this line

is $\vec{n} = \frac{(a,b)}{\sqrt{a^2+b^2}}$ ← normalization

let $Q[x_2,y_2]$ be any point on p, i.e.

$$ax_2+by_2+c=0$$

the distance d is the lenght of the projection of the vector $\vec{QP} = (x_1-x_2, y_1-y_2)$ on the normal vector $\vec{n}$

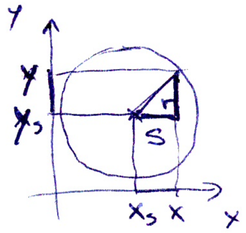
$$d = |\underbrace{\vec{n}\cdot\vec{QP}}_{\text{scalar product}}| = \frac{|a(x_1-x_2)+b(y_1-y_2)|}{\sqrt{a^2+b^2}} \underset{-ax_2-by_2=c}{=} \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

similarly, the distance of $P[x_1,y_1,z_1]$ from a plane

$\sigma: ax+by+cz+d=0$ is (derivation is exactly the same)

$$d = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

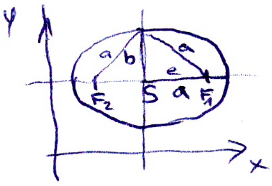
⑤ conic sections - quadratic curves in the plane



circle with the center $S[x_s,y_s]$

$$(x-x_s)^2+(y-y_s)^2=r^2$$

ellipse at $S[x_s,y_s]$ with axes in the direction of x-axis and y-axis

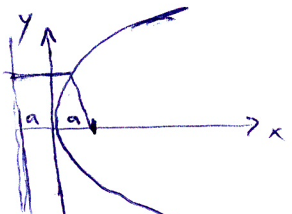


$$\frac{(x-x_s)^2}{a^2}+\frac{(y-y_s)^2}{b^2}=1$$

for $a>b$ the eccentricity (linear) $e=\sqrt{a^2-b^2}$

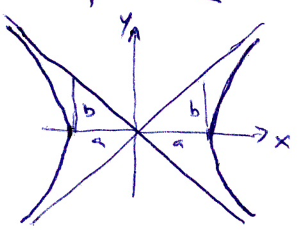
gives positions of foci $F_1[x_s+e, y_s]$

and $F_2[x_s-e, y_s]$



parabola $y^2=4ax$

hyperbola (centered at the origin)



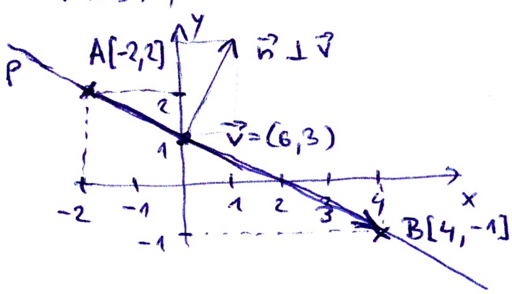
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Analytic Geometry - Solved Problems

(1) Write down different forms of the equation of a straight line passing through the points $A[-2;2]$ and $B[4;-1]$.

Solution:

First, we construct a vector from A to B: $\vec{v} = B - A = [4,-1] - [-2,2] =$
$\vec{v} = (6,-3)$



a) The parametric form:

$$\vec{r} = A + t\vec{v}$$

or

$$x = -2 + 6t$$
$$y = 2 - 3t$$

Note that there is an infinite number different parametrizations e.g. $\vec{r} = B - t'\vec{v}$ or $\vec{r} = A + t''\frac{\vec{v}}{3} = (-2+2t'', 2-t'')$ where $t'' = 3t$ giving the same line (we travel the line with different speed).

b) The standard form

$$ax + by = c, \quad a,b,c \in \mathbb{R}$$

or the general form

$$ax + by + c' = 0 \quad \text{where } c' = -c$$

We could express t from the parametric form and derive this form from it. But it is easier to take any vector $\vec{n}$ perpendicular to $\vec{v}$ and use its components (n_x, n_y) as (a,b). In general, we can use $\vec{n} = (-v_y, v_x)$ since then $\vec{n}.\vec{v} = -v_y v_x + v_x v_y = 0$. Note that this trick works only in the plane.

Thus in our case we get $\vec{n} = (3,6)$ and $3x + 6y = c$ where c can be determined by substituting any point on the line: $c = 3x_A + 6y_A = 3(-2) + 6.2 = 6$

The result is $3x + 6y = 6$.
or $x + 2y = 2$

[Again, there is an infinite number of equations differing by a multiplicative constant!]

c) The slope-intercept form

Just express y as a function of x, either from parametric or general (standart) form

$$y = kx + y_0 = -\frac{1}{2}x + 1$$

slope (k) y-intercept (y_0)

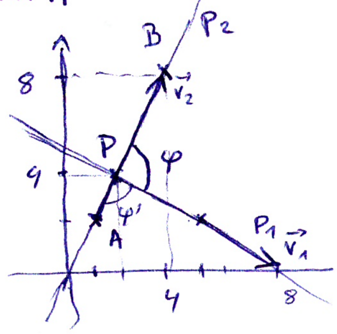
(2) Determine the intersection of and the angle between two straight lines $p_1: x = 5-3t$, $y = 2+2t$ and p_2 going through $A[1,2]$ and $B[4,8]$.

Solution: Probably the fastest way is to write p_2 in the standard form and substitute for x and y from p_1

The normal vector to p_2 from $\vec{v}_2 = B - A = (3,6)$ is $\vec{n}_2 = (-6,3)$

and thus
$$p_2: -6x + 3y = c = -6(1) + 3(2) = 0$$
point A

or
$$p_2: -2x + y = 0$$



By substituting p_1 into p_2 we get
$$-2(5-3t) + (2+2t) = 0$$
$$8t = 8$$
$$t = 1$$

which gives the parameter t of the intersection
$$P[5-3(1),\ 2+2\cdot 1] = P[2,4]$$

The angle φ between p_1 and p_2 can be obtained from the scalar product as
$$\cos\varphi = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|}$$
where $\vec{v}_1$ and $\vec{v}_2$ are directional vectors of p_1 and p_2.

We know $\vec{v}_2 = (3,6)$, $|\vec{v}_2| = \sqrt{3^2+6^2} = 3\sqrt{5}$

and $\vec{v}_1 = (-3,2)$, $|\vec{v}_1| = \sqrt{(-3)^2+2^2} = \sqrt{13}$

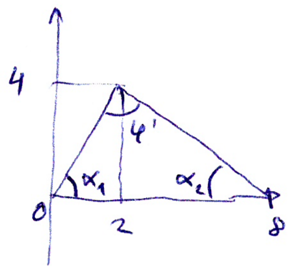
(coefficients next to t in the parametric form of p_1)

and thus
$$\cos\varphi = \frac{(-3)(3) + 2\cdot 6}{3\sqrt{5}\sqrt{13}} = -\frac{1}{\sqrt{65}}$$

$$\varphi \doteq 1{,}695 > \frac{\pi}{2}$$

A smaller angle is $\varphi' = \pi - \varphi \doteq 1{,}446 \doteq 83°$

Test: The same result should be obtained from
$$\varphi' = \pi - \alpha_1 - \alpha_2$$



where $\tan\alpha_1 = \frac{4}{2} = 2 \Rightarrow \alpha_1 \doteq 1{,}107$

$\tan\alpha_2 = \frac{4}{6} = \frac{2}{3} \Rightarrow \alpha_2 \doteq 0{,}588$

$\Rightarrow \varphi' = 1{,}447$ ✓

rounding error

(3) Write down the equation of a straight line passing through the point $A[2,3]$ and making the angle $45°$ with the line p given by the equation $p: 2x+5y-5=0$.

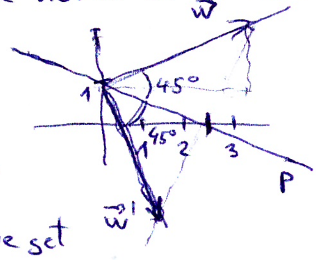
Solution: The direction vector of p is $\vec{v}=(5,-2)$ $\left(\begin{array}{l}\text{from } a=2,\ b=5 \\ \text{as } \vec{v}=(b,-a)\perp(a,b)\end{array}\right)$

The unknown direction of the searched line is $\vec{w}=(w_x,w_y)$ satisfying

$$\cos 45° = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}$$

We can set for example $w_x=1$ as the norm of $\vec{w}$ can be arbitrary

[Here we assume that $w_x \neq 0$ as one can guess from the sketch. We expect two solutions $\vec{w}$ and $\vec{w}'$.]



Thus $|\vec{w}|=\sqrt{1+w_y^2}$ and $|\vec{v}|=\sqrt{29}$ and we set

$$\cos 45° = \frac{\sqrt{2}}{2} = \frac{5\cdot 1-2w_y}{\sqrt{29}\sqrt{1+w_y^2}}$$

$$\sqrt{29}\sqrt{1+w_y^2} = \sqrt{2}(5-2w_y)$$

$$29+29w_y^2 = 2(25-20w_y+4w_y^2)$$

$$21w_y^2+40w_y-21=0 \quad \Rightarrow \quad \Delta = 1600+4\cdot 441 = 58^2$$

$$w_y = \frac{-40\pm\sqrt{58^2}}{2\cdot 21} = \begin{cases} \frac{3}{7} \\ -\frac{7}{3} \end{cases}$$

We got two solutions

line q_1 given by $A[2,3]$ and $\vec{w}=(1,\frac{3}{7})$: $x=2+t,\ y=3+\frac{3}{7}t$

and line q_2 given by $A[2,3]$ and $\vec{w}'=(1,-\frac{7}{3})$: $x=2+t,\ y=3-\frac{7}{3}t$

④ Determine the distance of the point $A[7,9,7]$ in space from the line given in the parametric form

$$p: x = 2+4t\ ,\ y = 1+3t\ ,\ z = 2t\ ,\ t \in \mathbb{R} \qquad (1)$$

Solution:

a) Trick: Construct a plane perpendicular to the given line containing the point A and then substitute (1) into it to get t corresponding to the intersection P of the plane with the line. Finally, calculate the distance $d = |AP|$.

The direction of the line is $\vec{v} = (4,3,2)$ (again, coefficients by t in (1)) which is also the normal vector for the plane $\perp p$

Thus, we can write for the plane $\varrho \perp p$

$$\varrho : 4x + 3y + 2z + d = 0$$

A must be in ϱ: $4 \cdot 7 + 3 \cdot 9 + 2 \cdot 7 + d = 0 \Rightarrow d = -69$

Now substitute (1) into ϱ:

$$4(2+4t) + 3(1+3t) + 2(2 \cdot t) - 69 = 0$$

$$29t = 58$$

$$t = 2$$

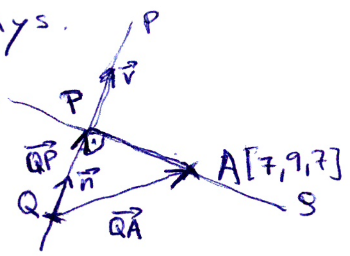
The intersection point is $P[10,7,4]$ (evaluate (1) using $t=2$)

and the distance is

$$d = |AP| = |\underbrace{(3,-2,-3)}_{\text{vector from A to P}}| = \sqrt{3^2 + (-2)^2 + (-3)^2} = \sqrt{22}$$

b) The same result can be obtained in other ways.

For example: If Q is any point on p, e.g. $[2,1,0]$, and $\vec{n} = \frac{\vec{v}}{|\vec{v}|}$ (normalised vector in the direction of p)



then $d = |\vec{PA}| = |\vec{QA} - \vec{QP}| = |\vec{QA} - \underbrace{(\vec{QA} \cdot \vec{n})\vec{n}}_{\text{projection of } \vec{QA} \text{ on the line } p}|$

and we have $\left.\begin{array}{l} \vec{QA} = (5,8,7) \\ \vec{n} = \frac{1}{\sqrt{29}}(4,3,2) \end{array}\right\} \Rightarrow (\vec{QA} \cdot \vec{n})\vec{n} = \frac{58}{\sqrt{29}} \frac{1}{\sqrt{29}}(4,3,2) = 2\vec{v}$

The result is

$$d = |(5,8,7) - (8,6,4)| = \sqrt{(-3)^2 + 2^2 + 3^2} = \sqrt{22}\ ✓$$

Or we can find the distance d by searching for a minimum of the function

$$d^2 = (\underbrace{x}_{\text{a point on the line}} - \overbrace{x_A}^{\text{the given point A}})^2 + (y - y_A)^2 + (z - z_A)^2 =$$

$$= (2+4t-7)^2 + (1+3t-9)^2 + (2t-7)^2$$

$$= (4t-5)^2 + (3t-8)^2 + (2t-7)^2 = f(t)$$

A minimum is where the derivative of d^2 with respect to t is zero, i.e.

$$\frac{df(t)}{dt} = \cancel{2}(4t-5)\cdot 4 + \cancel{2}(3t-8)\cdot 3 + \cancel{2}(2t-7)\cdot 2 = 0$$

$$\Rightarrow \quad 29t = 58$$

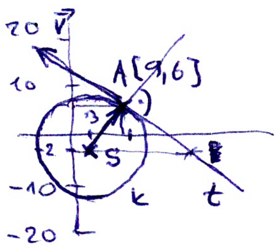
The same point P $\Leftarrow$ $t = 2$ as in the first method of solution

---

(5) Determine the equation of a tangent t to the circle k given by the equation $(x-3)^2 + (y+2)^2 = 100$

at the point $A[9,6]$.

Solution: As the point A is on the circle, vector $\vec{SA} = A - S = (6,8)$ must be the normal vector to t. Thus



$$t:\ 6x + 8y = c = 6\cdot 9 + 8\cdot 6 = 102$$

$\uparrow$ A is on the tangent

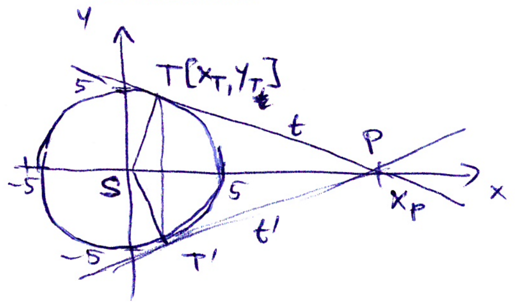
or $t:\ 3x + 4y = 51$

or we can write directly t in the parametric form since $\vec{v} \perp \vec{SA} = (-8,6)$ and thus

$$t:\ x = \underbrace{9}_{\text{point A}} - 8t,\quad y = 6 + 6t$$

⑥ Determine a tangent t to a circle $k: x^2+y^2=5^2$ passing through a point $P[x_P,0]$

Solution:



For $x_P>5$ or $x_P<-5$ we expect two solutions, for $x_P=5$ and $x_P=-5$ only one solution, otherwise none.

Vectors $\vec{ST}$ and $\vec{TP}$ must be perpendicular and T must be on k.

For $T[x_T, y_T]$ we get $\vec{ST}=(x_T, y_T)$

$\vec{TP}=(x_P-x_T, -y_T)$

and thus $\vec{ST}\cdot\vec{TP} = x_T(x_P-x_T)-y_T^2 = x_Tx_P-\overbrace{(x_T^2+y_T^2)}^{25}=0$

$$\Rightarrow \quad x_T=\frac{25}{x_P} \quad \text{and} \quad y_T=\pm\sqrt{25-\left(\frac{25}{x_P}\right)^2}=\pm 5\sqrt{1-\frac{25}{x_P^2}}$$

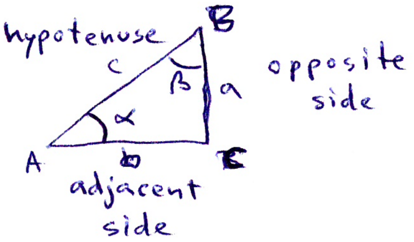
and the equation of t in the parametric form is

$$t: x=\frac{25}{x_P}-\left(x_P-\frac{25}{x_P}\right)t \quad , y=5\sqrt{1-\frac{25}{x_P^2}}\,(1-t)$$

$$\text{or } t': x=\frac{25}{x_P}-\left(x_P-\frac{25}{x_P}\right)t \quad , y=-5\sqrt{1-\frac{25}{x_P^2}}\,(1-t)$$

Trigonometry - Formulas

① basic definitions from the right(-angled) triangle



sine

$$\sin\alpha = \frac{a}{c}$$

cosine

$$\cos\alpha = \frac{b}{c}$$

tangent

$$\tan\alpha = \frac{a}{b} = \frac{\sin\alpha}{\cos\alpha}$$

cotangent

$$\cot\alpha = \frac{b}{a} = \frac{\cos\alpha}{\sin\alpha}$$

secant

$$\sec\alpha = \frac{c}{b} = \frac{1}{\cos\alpha}$$

cosecant

$$\csc\alpha = \frac{c}{a} = \frac{1}{\sin\alpha}$$

- as $\beta = \frac{\pi}{2} - \alpha$

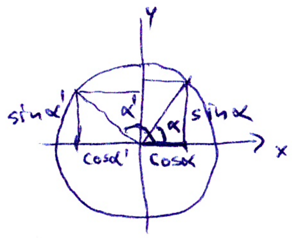
  we have e.g.

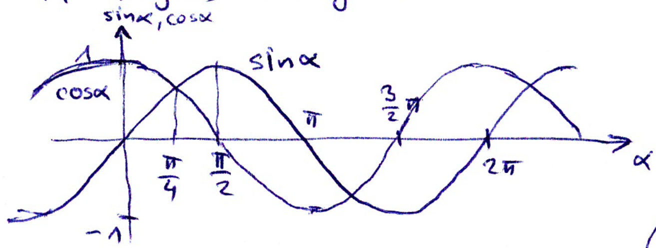
  $\sin\beta = \frac{b}{c} = \cos\alpha$

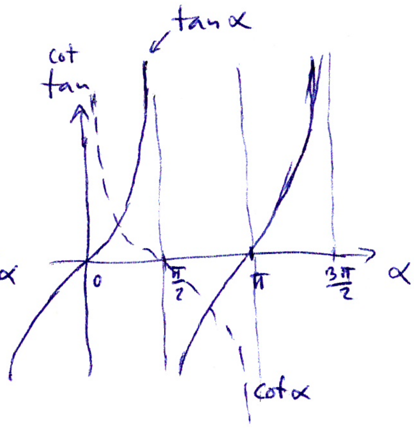
  or $\cos\alpha = \sin(\frac{\pi}{2} - \alpha)$ etc.

- from $a^2 + b^2 = c^2$ we get $\sin^2\alpha + \cos^2\alpha = 1$

② extension to all angles using a unit circle



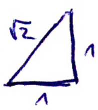




some values on $\langle 0, \frac{\pi}{2}\rangle$:

$\sin 0 = 0 = \cos\frac{\pi}{2}$ $(0°)$

$\sin\frac{\pi}{6} = \frac{1}{2} = \cos\frac{\pi}{3}$ $(30°)$

from  → $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$ $(45°)$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$ $(60°)$

$\sin\frac{\pi}{2} = 1 = \cos 0$ $(90°)$

(30°, 60°) from 

other values can be sometimes obtained from the half-angle formulas such as $\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$

or $\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$

③ periodicity and symmetry

from the unit circle we can see that for any $k \in \mathbb{Z}$

$\sin\alpha = \sin(\alpha + 2k\pi)$

$\cos\alpha = \cos(\alpha + 2k\pi)$

and

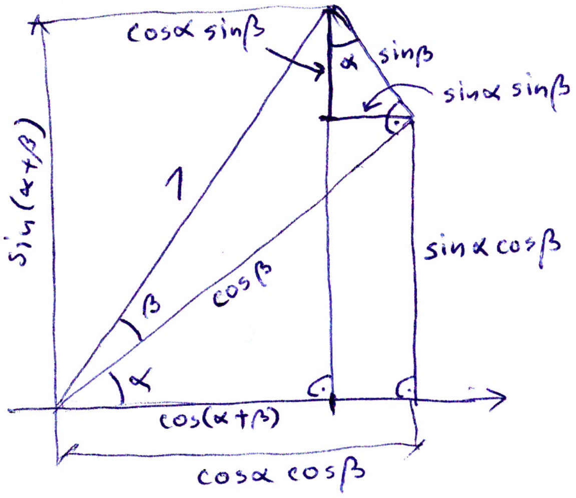
$\tan\alpha = \tan(\alpha + k\pi)$

$\cot\alpha = \cot(\alpha + k\pi)$

and $\sin\alpha = -\sin(-\alpha)$ $\tan(\alpha) = -\tan(-\alpha)$

$\cos\alpha = \cos(-\alpha)$ $\cot(\alpha) = -\cot(-\alpha)$

④ sum and difference formulas



$$\left.\begin{cases} \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{cases}\right\} \Leftarrow$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin(\alpha-\beta) = \cos\left(\left(\tfrac{\pi}{2}-\alpha\right)+\beta\right) =$$

$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

[or use symmetry properties for $\sin(\alpha+(-\beta))$]

similarly

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

especially

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$$

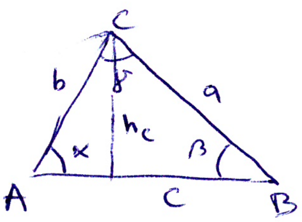
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = 2\cos^2\alpha - 1$$

half-angle formulas $\Leftarrow$

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$$

⑤ law of sines and law of cosines in an arbitrary triangles



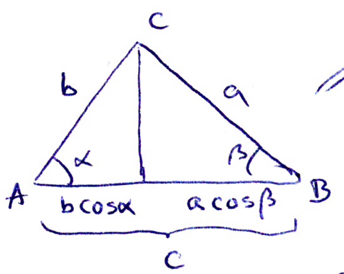
law of sines can be derived by comparing the area of $\triangle ABC$ expressed in various ways, e.g.

$$\frac{2S}{abc} = \frac{2\,c h_c}{abc\cdot 2} = \frac{2cb\sin\alpha}{2abc} = \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} = \frac{1}{2r}$$

in a similar way

radius of the circumcircle

law of cosines



$$c^2 = c(b\cos\alpha + a\cos\beta)$$

similarly for a and b

$$a^2 = a(c\cos\beta + b\cos\gamma)$$

$$b^2 = b(c\cos\alpha + a\cos\gamma)$$

and we get

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \Rightarrow \cos\gamma = \frac{a^2+b^2-c^2}{2ab}$$

or

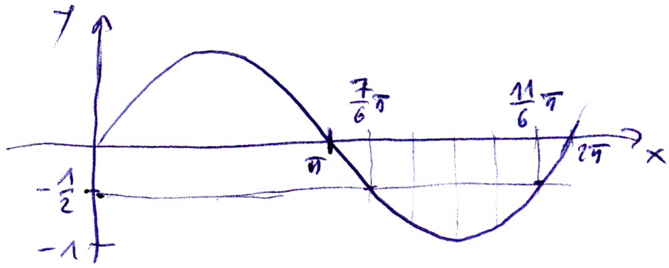
$$a^2 = b^2 + c^2 - 2bc\cos\alpha \Rightarrow \cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

or

$$b^2 = a^2 + c^2 - 2ac\cos\beta \Rightarrow \cos\beta = \frac{a^2+c^2-b^2}{2ac}$$

Trigonometry – Solved Examples

① Find all $x \in \mathbb{R}$ satisfying $2\sin(3x+\pi) = -1$.



As $\sin z = -\frac{1}{2}$ has two solutions

$$z_1 = \frac{7}{6}\pi + 2k\pi \qquad k \in \mathbb{Z}$$

and $$z_2 = \frac{11}{6}\pi + 2k\pi$$

we get solutions

$$z_1 = 3x_1 + \pi = \frac{7}{6}\pi + 2k\pi \Rightarrow x_1 = \frac{1}{18}\pi + \frac{2}{3}k\pi \quad \text{for all } k \in \mathbb{Z}$$

and $$z_2 = 3x_2 + \pi = \frac{11\pi}{6} + 2k\pi \Rightarrow x_2 = \frac{5}{18}\pi + \frac{2}{3}k\pi$$

---

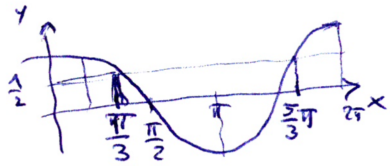
② Find all $x \in \mathbb{R}$ satisfying $2\cos^2 x - 7\cos x + 3 = 0$.

Using $y = \cos x$, we solve $2y^2 - 7y + 3 = 0$

to get $y_{1,2} = \frac{7 \pm \sqrt{49-24}}{2 \cdot 2} = \cancel{3}$, $\frac{1}{2}$

This root cannot be used as $|\cos x| \le 1$

Now, we solve $\cos x = \frac{1}{2}$



and the solutions are

$$x_1 = \frac{\pi}{3} + 2k\pi$$

and $$x_2 = \frac{5}{3}\pi + 2k\pi$$

---

③ Find all $x \in \mathbb{R}$ satisfying $\sin x + \sqrt{3}\cos x = \sqrt{2}$.

a) Trick: Use the angle sum identity $\sin(x+y) = \sin x \cos y + \cos x \sin y$ with suitable y. But we have to multiply the equation by some constant p first to be able to satisfy the identity

$$\sin^2 y + \cos^2 y = 1.$$

If we set $\cos y = p$ and $\sin y = p\sqrt{3}$ we get $3p^2 + p^2 = 1$ and thus $p = \pm\frac{1}{2}$ but we can use just one p to get all solutions, let us use $p = \frac{1}{2} = \cos y$ and $\frac{\sqrt{3}}{2} = \sin y$

These conditions are satisfied by $y = \frac{\pi}{3}$

and we get $\sin\left(x + \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2} = p\sqrt{2}$

with solutions $x_1 + \frac{\pi}{3} = \frac{\pi}{4} + 2k\pi \Rightarrow x_1 = -\frac{1}{12}\pi + 2k\pi$

$$x_2 + \frac{\pi}{3} = \frac{3}{4}\pi + 2k\pi \Rightarrow x_2 = \frac{5}{12}\pi + 2k\pi$$

b) Another possibility is to use $\cos x = \sqrt{1-\sin^2 x}$ and solve the quadratic equation

$$\sqrt{3}\sqrt{1-\sin^2 x} = \sqrt{2} - \sin x$$
$$3(1-\sin^2 x) = 2 - 2\sqrt{2}\sin x + \sin^2 x$$
$$4\sin^2 x - 2\sqrt{2}\sin x - 1 = 0$$

or $\quad 4z^2 - 2\sqrt{2}z - 1 = 0$

with roots $\quad z_{1,2} = \dfrac{\sqrt{2}\pm\sqrt{6}}{4} = \sin x$

which corresponds to angles $-\frac{\pi}{12}$ for $z_2 = \frac{\sqrt{2}-\sqrt{6}}{4}$

and $\frac{5}{12}\pi$ for $z_1 = \frac{\sqrt{2}+\sqrt{6}}{4}$

as previously.

---

(4) Find all $x \in \mathbb{R}$ satisfying $\quad \sin x + \cos x + \sin x \cos x = 1$

Solution: Using $y = \sin x + \cos x$

and $y^2 = \sin^2 x + 2\sin x\cos x + \cos^2 x = 1 + 2\sin x \cos x$

we get $\quad y^2 + 2y - 3 = 0 \quad \Rightarrow \quad y_{1,2} = \dfrac{-2 \pm \sqrt{4+4\cdot 3}}{2} = \begin{cases} 1 \\ \cancel{-3} \end{cases}$

$|\sin x + \cos x| \leq 2$!

Thus we have to solve

$$\sin x + \cos x = 1$$

Using the same trick as previously with $p = \frac{\sqrt{2}}{2}$

we get $\quad \frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x = \sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

and the solutions are $\quad x_1 = 2k\pi \quad$ and $\quad x_2 = \frac{\pi}{2} + 2k\pi$.

Complex Numbers - Formulas

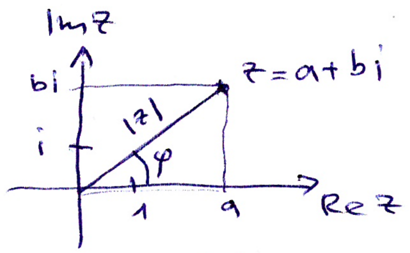
① basic definitions and operations

i ... imaginary unit satisfying the equation $i^2=-1$

thus solving $x^2=-1$ together with $(-i)^2=-1$

$z = a + bi$ ... general complex number

real part (for a), imaginary part (for bi)



$$z_1+z_2=(a_1+a_2)+(b_1+b_2)i$$

$$z_1\cdot z_2=(a_1+b_1 i)(a_2+b_2 i)=a_1a_2\overset{i^2}{-}b_1b_2+(a_1b_2+b_1a_2)i$$

we see that $z^2=z\cdot z$ is in general complex

$\bar{z}=\overline{a+bi}=a-bi$ ... complex conjugate of z

$z\cdot\bar{z}=(a+bi)(a-bi)=a^2+b^2=|z|^2$ ... complex absolute value squared

or $|z|=\sqrt{a^2+b^2}$ .... also called modulus or magnitude

$$\frac{z_1}{z_2}=\frac{z_1\cdot\bar{z}_2}{|z_2|^2}\quad,\quad\frac{1}{z}=\frac{\bar{z}}{|z|^2}=\frac{\bar{z}}{z\cdot\bar{z}}$$

we can see from the complex (Gauss) plane that

$$\left.\begin{aligned} a&=|z|\cos\varphi\\ b&=|z|\sin\varphi\end{aligned}\right\}\Rightarrow z=|z|(\cos\varphi+i\sin\varphi) = r(\cos\varphi+i\sin\varphi)$$

... polar form (in polar coordinates)

$\varphi=\arg z$ ... argument of z

② notice that

$$z_1\cdot z_2=|z_1||z_2|\left[\cos\varphi_1\cos\varphi_2-\sin\varphi_1\sin\varphi_2+i(\sin\varphi_1\cos\varphi_2+\cos\varphi_1\sin\varphi_2)\right]=$$

$$=|z_1||z_2|\left(\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)\right)$$

i.e. we multiply magnitudes

but add the arguments ⇒ this lead Euler

to represent complex numbers as $z=re^{i\varphi}$

because when multiplying exponentials we add up exponents $\quad e^x\cdot e^y=e^{x+y}$

and we get the famous Euler formula

$$e^{i\varphi}=\cos\varphi+i\sin\varphi$$

which can be proofed using Taylor series of e^x, $\cos x$ and $\sin x$

③ de Moivre's formula

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$$

which can be proved either by induction or successive use of the multiplication formula in the polar form or most effectively using Euler representation

$$(\cos\varphi + i\sin\varphi)^n = (e^{i\varphi})^n = e^{in\varphi} = \cos n\varphi + i\sin n\varphi$$

④ from $e^{i\varphi} = \cos\varphi + i\sin\varphi$

and $e^{-i\varphi} = \cos\varphi - i\sin\varphi$

we get $\cos\varphi = \dfrac{e^{i\varphi} + e^{-i\varphi}}{2}$ and $\sin\varphi = \dfrac{e^{i\varphi} - e^{-i\varphi}}{2i}$

Furthemore, we can express

$$\cos i\varphi = \frac{e^{\varphi} + e^{-\varphi}}{2} \equiv \cosh\varphi \quad \text{hyperbolic cosine}$$

$$-i\sin i\varphi = \frac{e^{\varphi} - e^{-\varphi}}{2} \equiv \sinh\varphi \quad \text{hyperbolic sine}$$

Complex Numbers – Solved Problems

① Evaluate the expressions:

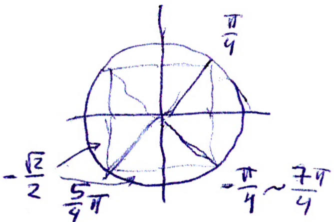
a) $(i-3)(\underset{(i^2)^3=(-1)^3=-1}{i^6}+5) = 4(i-3) = -12+4i$

b) $2\,\frac{2+i}{3-i} = 2\,\frac{(2+i)(3+i)}{\underbrace{(3-i)(3+i)}_{3^2-i^2=9+1}} = \frac{2}{10}(6+2i+3i-1) = \frac{1}{5}(5+5i) = 1+i$

c) $\left|\frac{2+3i}{3-3i}\right| = \frac{1}{3}\left|\frac{(2+3i)(1+i)}{\underbrace{(1-i)(1+i)}_{1^2-i^2=2}}\right| = \frac{1}{6}|-1+5i| = \frac{\sqrt{(-1)^2+5^2}}{6} = \frac{\sqrt{26}}{6}$

d) $(1+i)^{12} = [\underbrace{(1+i)(1+i)}_{1-1+2i}]^6 = (2i)^6 = \overset{i^6\cdot 2^6}{-64}$

② Write the following complex numers in the polar form
$$z = r(\cos\varphi + i\sin\varphi)$$



Solution:

a) $i+\sqrt3 = 2\left(\frac{\sqrt3}{2}+i\frac{1}{2}\right) = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$|i+\sqrt3| = \sqrt{1^2+3} = 2$

b) $\sqrt6 - i\sqrt6 = \sqrt{12}\left(\frac{\sqrt6}{\sqrt{12}} - i\frac{\sqrt6}{\sqrt{12}}\right) = \sqrt{12}\left(\frac{\sqrt2}{2} - i\frac{\sqrt2}{2}\right) = 2\sqrt3\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

$|\sqrt6 - i\sqrt6| = \sqrt{6+6} = \sqrt{12}$ or $= 2\sqrt3\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

c) $-1-i = \sqrt2\left(-\frac{\sqrt2}{2} - \frac{\sqrt2}{2}i\right) = \sqrt2\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

d) $31+27i = 13\sqrt{10}\left(\cos\left(\arctan\frac{27}{31}\right)+i\sin\left(\arctan\frac{27}{31}\right)\right)$:-)

③ Find both roots of the quadratic equation
$$x^2-2x+2=0$$

Solution: Using the standard formula we get
$$x_{1,2} = \frac{2\pm\sqrt{4-8}}{2} = 1\pm\sqrt{-1} = 1\pm i$$

④ Find all solutions of the equation

$$x^4 = 64$$

Solution: using de Moivre's formula

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$$

we can write

$$x^4 = r^4(\cos 4\varphi + i\sin 4\varphi) = 64(\cos 2\pi k + i\sin 2\pi k)$$

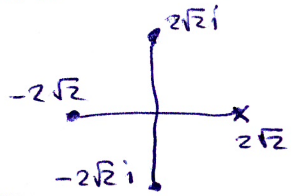
for any $k \in \mathbb{Z}$

and thus $r^4 = 64 \Rightarrow r = \sqrt{8} = 2\sqrt{2}$

and $4\varphi = 2\pi k \Rightarrow \varphi = \frac{\pi}{2}k \Rightarrow$ 4 different solutions

for $\varphi \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$

or $x \in \{2\sqrt{2}, 2\sqrt{2}i, -2\sqrt{2}, -2\sqrt{2}i\}$



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⑤ Find all solutions of the equation

$$x^3 = i$$

Solution: We could use again de Moivre's formula

but using exponential form $x = re^{i\varphi}$

we get immediately

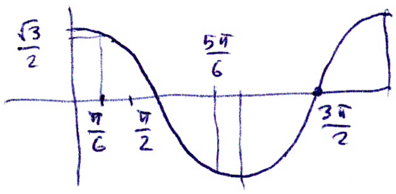
$$x^3 = r^3 e^{i3\varphi} = i = e^{i(\frac{\pi}{2} + 2k\pi)}$$

and thus $r^3 = 1 \Rightarrow r = 1$

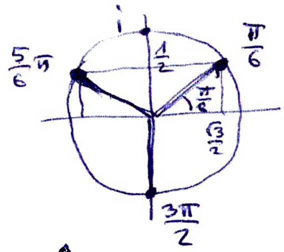
$$3\varphi = \frac{\pi}{2} + 2k\pi \Rightarrow \varphi = \frac{\pi}{6} + \frac{2}{3}k\pi$$

Now we have 3 different solutions for $k = 0, 1, 2$

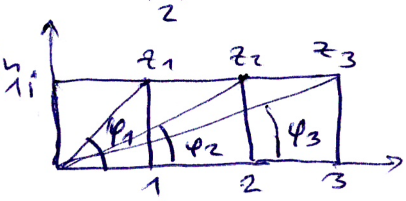
$$\varphi \in \{\frac{\pi}{6}, \frac{5}{6}\pi, \frac{9}{6}\pi = \frac{3}{2}\pi\}$$



$$x \in \{\frac{\sqrt{3}}{2} + \frac{i}{2}, -\frac{\sqrt{3}}{2} + \frac{i}{2}, -i\}$$



⑥ For fun :-) Find the value of $\varphi_1 + \varphi_2 + \varphi_3$ in



we know that arguments add up when multiplying complex numbers

Thus using $z_1 = 1+i$, $z_2 = 2+i$ and $z_3 = 3+i$

we get

$$\varphi_1 + \varphi_2 + \varphi_3 = \arg z_1 \cdot z_2 \cdot z_3 = \arg(1+3i)(3+i) = \arg 10i = \frac{\pi}{2}$$

Sequences of Real Numbers - Solved Examples

① Let $(a_k)_{k=1}^{n}$ be an arithmetic progression with the common difference d, i.e. $a_k = a_1 + (k-1)d$

(a) Show that the sum of all terms is

$$S_n = \sum_{k=1}^{n} a_k = \frac{n}{2}\left[2a_1 + (n-1)d\right] = n\frac{a_1+a_n}{2}$$

Solution:

$$S_n = \sum_{k=1}^{n}\left[a_1 + (k-1)d\right] = n a_1 + \underbrace{(1+2+\dots+n-1)}_{\frac{n(n-1)}{2}}d =$$

$$= n a_1 + \frac{n(n-1)}{2}d$$

$$= \frac{n}{2}\left[2a_1 + \underbrace{a_1 + (n+1)d}_{a_n}\right] = \frac{n}{2}\left[a_1 + a_n\right]$$

(b) Find d, a_1, a_8 and s_{11} if $a_4 = 6$ and $a_{11} = 34$.

Solution:

$$\left.\begin{aligned} a_4 &= a_1 + 3d = 6 \\ a_{11} &= a_1 + 10d = 34 \end{aligned}\right\} \Rightarrow 7d = 28$$

$$d = 4 \Rightarrow a_1 = -6,$$

now $a_8 = a_1 + 7d = -6 + 28 = 22$

$$s_{11} = n a_1 + \frac{n(n-1)}{2} d = 11(-6) + 55\cdot 4 = 154$$

$n = 11$

(c) Find d and a_1 if $s_5 = s_6 = 60$.

Solution:

$$\left.\begin{aligned} s_5 &= 5a_1 + 10d = 60 \\ s_6 &= 6a_1 + 15d = 60 \end{aligned}\right\} \Rightarrow a_1 + 5d = 0 \Rightarrow$$

$$\Rightarrow -25d + 10d = 60$$

$$a_1 = 20 \Leftarrow d = -4$$

Test: $s_5 = a_1 + \dots + a_5 = 20 + 16 + 12 + 8 + 4 = 60$ ✓

② Let $(a_k)_{k=1}^{n}$ be a geometric progression with the common ratio q, i.e. $a_k = a_1 q^{k-1}$.

a) Show that the sum of all terms is

$$s_n = \sum_{k=1}^{n} a_k = a_1\frac{1-q^n}{1-q}$$

Solution: $S_n = a_1 + a_1 q + \dots + a_1 q^{n-1} = a_1(1 + \cancel{q} + \cancel{q^2} + \dots + \cancel{q^{n-1}})$

Trick: subtract from it $q S_n = a_1(\cancel{q} + \cancel{q^2} + \dots + \cancel{q^{n-1}} + q^n)$

thus $$S_n - q S_n = a_1 - a_1 q^n$$

and $$S_n = a_1 \frac{1-q^n}{1-q} \quad \text{for } q \neq 1.$$

For $q=1$: $S_n = \sum_{k=1}^{n} a_1 = n a_1$.

Note that for $0<q<1$ the limit $n \to \infty$ gives $\sum_{k=1}^{\infty} q^{k-1} = \frac{1}{1-q}$

---

b) Find q, a_1 and S_6 if $a_2 = 48$ and $a_5 = 162$.

Solution: $\left.\begin{array}{l} a_2 = a_1 q = 48 \\ a_5 = a_1 q^4 = 162 \end{array}\right\}$ dividing the equations we get

$$q^3 = \frac{162}{48} = \frac{27 \cdot \cancel{6}}{8 \cdot \cancel{6}} = \left(\frac{3}{2}\right)^3$$

$$\boxed{q = \frac{3}{2}}$$

and thus $\boxed{a_1 = \frac{48}{q} = 32}$

Finally $$S_6 = a_1 \frac{q^6 - 1}{q-1} = \underset{2^5}{\cancel{32}} \frac{\frac{3^6 - 2^6}{\cancel{2^6}}}{\frac{1}{\cancel{2}}} = 729 - 64 = 665$$

Test: $\overset{a_1}{32} + 48 + 72 + 108 + 162 + \overset{a_6}{243} = 665$ ✓

---

c) Find a_6 and S_8 if $a_3 = 1$ and $q = \frac{1}{3}$.

Solution: $a_3 = a_1 q^2 = 1 \Rightarrow a_1 = \frac{1}{q^2} = 9$

and $a_6 = a_1 q^5 = 9 \cdot \left(\frac{1}{3}\right)^5 = \frac{1}{27}$

and finally $$S_8 = a_1 \frac{1-q^8}{1-q} = 9 \frac{\frac{3^8 - 1}{\cancel{3^8}\,3^5}}{\frac{2}{\cancel{3}}} = \frac{2}{3^5}(3^8 - 1)$$

---

(3) Let $(a_k)_{k=1}^{\infty}$ be the Fibonacci sequence given by

$$a_{n+2} = a_{n+1} + a_n \quad \text{for all } n \in \mathbb{N} \qquad (1)$$

with $a_1 = a_2 = 1$. Show that

$$a_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}, \quad \text{where } \varphi = \frac{1+\sqrt{5}}{2} \approx 1{,}618$$

(golden ratio)

Solution:

We can try a solution $a_n = \lambda^n$

from which we get $\lambda^{n+2} = \lambda^{n+1} + \lambda^n$

and $\lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$

(characteristic polynomial)

Because the recursive relation (1) is linear, we can combine independent solutions to get a general one

$$a_n = c_1 \lambda_1^n + c_2 \lambda_2^n$$

where c_1 and c_2 are constants given by initial values.

Thus we get

$$a_1 = 1 = c_1 \lambda_1 + c_2 \lambda_2 \quad | \cdot \lambda_2$$
$$a_2 = 1 = c_1 \lambda_1^2 + c_2 \lambda_2^2$$

$$\underset{\overset{\shortparallel}{-(\lambda_2 - 1)}}{\lambda_2 - 1} = c_1 \lambda_1 \underbrace{(\lambda_2 - \lambda_1)}_{+\sqrt{5}} \Rightarrow c_1 = \frac{1}{\sqrt{5}}$$

$$\lambda_2 - 1 = -\frac{1}{2} - \frac{\sqrt{5}}{2} \Rightarrow \lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$$

similarly we get for c_2

$$\underset{\overset{\shortparallel}{-(\lambda_1 - 1)}}{\lambda_1 - 1} = c_2 \lambda_2 \underbrace{(\lambda_1 - \lambda_2)}_{\sqrt{5}} \Rightarrow c_2 = \frac{-1}{\sqrt{5}}$$

Finaly, let us show that

$$\lambda_2 = -\frac{1}{\lambda_1} = -\frac{1}{\frac{1+\sqrt{5}}{2}} = \frac{-2(1-\sqrt{5})}{-4} = \frac{1-\sqrt{5}}{2} = \lambda_2 \checkmark$$

Thus leting $\varphi = \lambda_1$ we get

$$a_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \left(-\frac{1}{\varphi}\right)^n \right)$$

④ Let $(a_k)_{k=1}^{\infty}$ be a sequence given by the recursive relation $a_{n+2} = 2(a_{n+1} - a_n)$ for all $n \in \mathbb{N}$ with the initial values $a_1 = 1, a_2 = 0$

Show that
$$a_n = \frac{1}{2}\left[(1+i)^n + (1-i)^n\right] \qquad (1)$$

Solution: The characteristic polynomial is
$$\lambda^2 - 2\lambda + 2 = 0 \implies \lambda_{1,2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

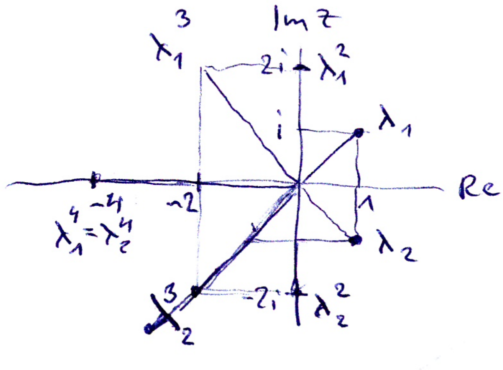
Thus the general solution is
$$a_n = c_1 \lambda_1^n + c_2 \lambda_2^n$$
where c_1 and c_2 are determined by $a_1 = 1, a_2 = 0$.

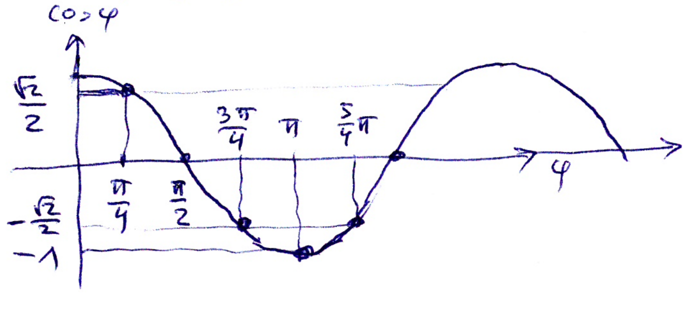
We get
$$a_1 = 1 = c_1(1+i) + c_2(1-i)$$
$$a_2 = 0 = c_1(1+i)^2 + c_2(1-i)^2 = c_1 2i + c_2(-2i)$$
$$\Downarrow$$
$$c_1 = c_2$$
and from the first $1 = 2c_1$
$$\implies c_1 = c_2 = \frac{1}{2} \quad \text{proving the formula (1).}$$

To rewrite a_n using only real numbers, we can use Euler representation of $1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$ and $1-i = \sqrt{2}\, e^{-i\frac{\pi}{4}}$

giving
$$a_n = (\sqrt{2})^n \underbrace{\frac{1}{2}\left(e^{i\frac{n\pi}{4}} + e^{-i\frac{n\pi}{4}}\right)}_{\cos\frac{n\pi}{4}} = (\sqrt{2})^n \cos\frac{n\pi}{4}.$$

You can check that you get
$$(a_k)_{k=1}^{\infty} = (1, 0, -2, -4, -4, 0, 8, 16, 16, 0, \dots)$$
1 2 3





(5) Find the formula for $\sum_{k=1}^{n} k^2$

Solution: A general technique to find sums of various powers is to write down a polynomial in n with the highest power given by the exponent in the sum plus 1, i.e. in our case $2+1=3$

The polynomial should be

$$a_0 + a_1 n + a_2 n^2 + a_3 n^3$$

and we can find the coefficients from a set of linear equations obtained for $n=1,2,3,4$:

$$n=1 : a_0 + a_1 + a_2 + a_3 = 1 \qquad (1)$$
$$n=2 : a_0 + 2a_1 + 4a_2 + 8a_3 = 5 \qquad (2)$$
$$n=3 : a_0 + 3a_1 + 9a_2 + 27a_3 = 14 \qquad (3)$$
$$n=4 : a_0 + 4a_1 + 16a_2 + 64a_3 = 30 \qquad (4)$$

and we subsequently subtract the equations to get rid of $a_0, a_1,$ and a_2

$$(2)-(1) = a_1 + 3a_2 + 7a_3 = 4 \qquad (\mathrm{I})$$
$$(3)-(2) = a_1 + 5a_2 + 19a_3 = 9 \qquad (\mathrm{II})$$
$$(4)-(3) = a_1 + 7a_2 + 37a_3 = 16 \qquad (\mathrm{III})$$

$$(\mathrm{II}) - (\mathrm{I}) = 2a_2 + 12a_3 = 5$$
$$(\mathrm{III}) - (\mathrm{II}) = 2a_2 + 18a_3 = 7$$

$$\Rightarrow \quad 6a_3 = 2 \Rightarrow a_3 = \frac{1}{3}$$
$$2a_2 + 12 \cdot \left(\frac{1}{3}\right) = 5 \Rightarrow a_2 = \frac{1}{2}$$
$$a_1 = 4 - \frac{3}{2} - \frac{7}{3} = \frac{1}{6}$$
$$a_0 = 1 - \frac{1}{6} - \frac{1}{2} - \frac{1}{3} = 0$$

Thus $$\sum_{k=1}^{n} k^2 = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} = \frac{n}{6}(n+1)(2n+1) \quad \checkmark$$

Types of Proofs - Solved Examples

① Prove that for all $n \in \mathbb{N}$ $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution: Proof by induction

a) For $n=1$ the formula is valid as

$$1 = \sum_{k=1}^{1} k^2 = \frac{1 \cdot 2 \cdot 3}{6} = 1$$

b) Let us assume that the formula is valid for n.

Then we can show that it is also valid for $n+1$

since

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 \overset{\text{by assumption}}{=} \frac{n(n+1)(2n+1)}{6} + (n+1)^2 =$$

$$= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} = \frac{(n+1)}{6}(2n^2+7n+6) =$$

$$= \frac{(n+1)}{6}(n+2)\underbrace{(2n+3)}_{(2(n+1)+1)}$$ as should be.

---

② Prove that the sum of the arithmetic-geometric sequence for all $n \in \mathbb{N}$ is $S_n = \sum_{k=1}^{n} k q^k = q\frac{1-q^n}{(1-q)^2} - \frac{nq^{n+1}}{1-q}$

Solution: We could prove it again by induction, but it is actually simpler to prove it directly using the trick in the hint:

$$S_n = q + 2q^2 + 3q^3 + \dots + nq^n$$

$$q\,S_n = \quad q^2 + 2q^3 + \dots + (n-1)q^n + nq^{n+1}$$

$$(1-q)S_n = q\,(\underbrace{1 + q + q^2 + \dots + q^{n-1}}_{\frac{1-q^n}{1-q}}) - nq^{n+1}$$

$$\Rightarrow \quad S_n = q\frac{1-q^n}{(1-q)^2} - \frac{nq^{n+1}}{1-q} \quad \checkmark$$

(3) If you know that $e^{x+y} = e^x e^y$ for all $x, y \in \mathbb{R}$

and $\ln x$ is the inverse function to e^x, prove the equality

$$\ln a \cdot b = \ln a + \ln b \quad \text{for all } a, b \in (0, +\infty)$$

Solution: Direct proof:

As $\ln x$ is the inverse to e^x, we can write

$$e^x = a \Rightarrow x = \ln a$$
$$e^y = b \Rightarrow y = \ln b$$

and we get $e^{\ln a + \ln b} = e^{\ln a} \cdot e^{\ln b} = a \cdot b$

and by taking logarithm of this equation

we get $\ln a + \ln b = \ln a \cdot b$ ✓

---

(4) Prove that $\sqrt{2}$ is an irrational number.

Solution: Proof by contradiction:

Let us assume that $\sqrt{2}$ is rational, i.e.

there exist p and q with no common factor

giving $\sqrt{2} = \frac{p}{q} \Rightarrow 2q^2 = p^2 \Rightarrow p = 2k \Rightarrow q^2 = 2k^2$

and as p and q have no common factor

we have contradiction because both should

be divisible by 2.

---

(5) Prove that if x^2 is even, then x is even.

Solution: Proof by contraposition (instead of $P \Rightarrow Q$

we proof $\neg Q \Rightarrow \neg P$)

not

Suppose x is odd (not even). Then x^2 is odd

as a product of two odd numbers $[(2k+1)^2 = 4(k^2+k)+1]$

Thus x^2 is not even. ✓

And thus if x^2 is even, x has to be even.