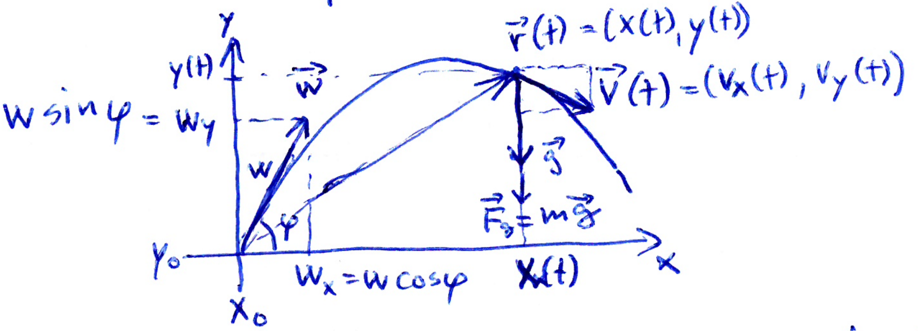
Projectile motion

(basic calculus, system of equations, planar geometry, quadratic equations, equations with trigonometric functions)

a) equations of motion for homogeneous gravitational field in the plane (x, y) (without air resistence or wind :-))



- force is given as a vector

$$\vec{F}_g = (0, -mg)$$

(standard acceleration of gravity $\approx 9{,}81 \frac{m}{s^2}$)

- initial conditions at time $t = 0$

position $\quad \vec{r}(0) = \vec{r}_0 = (x_0, y_0)$

velocity $\quad \vec{v}(0) = \vec{w} = (w_x, w_y) = (w\cos\varphi, w\sin\varphi)$

where $w = |\vec{w}|$ is the magnitude of velocity and φ is the angle under which the projectile is thrown/launched

- the motion is governed by Newton's second law

$$\vec{F}_g = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt}$$

where $\vec{a}(t) = (a_x(t), a_y(t)) =$ (in our case) $= (0, -g)$ is acceleration

this vector equation can be written in (x, y) coordinates as two ordinary differential equations first for velocities ($\vec{F}_g$ does not depend on $\vec{r}$)

$$F_x = 0 = m\frac{dv_x(t)}{dt} \Rightarrow v_x(t) \text{ must be constant}$$

$$\boxed{v_x(t) = w_x = w\cos\varphi}$$

$$F_y = -mg = m\frac{dv_y(t)}{dt}$$

or (by physicist's approach)

$$-g\,dt = dv_y$$

and by integration from 0 to time t

$$\int_{v_y(0)}^{v_y(t)} dv_y = -g\int_0^t dt \Rightarrow v_y(t) - \underbrace{v_y(0)}_{w_y = w\sin\varphi} = -g(t - 0)$$

or $\boxed{v_y(t) = w \cdot \sin\varphi - gt}$

- in a similar way we can get position $\vec{r}(t)$ as

$$v_x(t) = \frac{dx(t)}{dt} = w\cos\varphi$$

$$v_y(t) = \frac{dy(t)}{dt} = w\sin\varphi - gt$$

$\Rightarrow$ again by integration from 0 to time t we get

$$x(t) - x(0) = w\cos\varphi\,(t-0) = (w\cos\varphi)\,t$$

$$y(t) - y(0) = w\sin\varphi\,(t-0) - g\left(\frac{t^2}{2} - \frac{0^2}{2}\right) = (w\sin\varphi)\,t - \frac{1}{2}gt^2$$

integral of function t because $\frac{d}{dt}\left(\frac{t^2}{2}\right) = t$ and $\int_{t_0}^{t} t'\,dt' = \frac{t^2}{2} - \frac{t_0^2}{2}$

or

$$\boxed{\begin{aligned} x(t) &= x_0 + (w\cos\varphi)\,t \\ y(t) &= y_0 + (w\sin\varphi)\,t - \frac{1}{2}gt^2 \end{aligned}}$$

this is the parametric expression of the trajectory (time t plays a role of the parameter)

b) <u>the trajectory as a function $y = f(x)$</u> (without the parameter time t)

- we simply express time t from $x(t)$ and substitu it into $y(t)$, and we get

$$t = \frac{x - x_0}{w\cos\varphi}$$

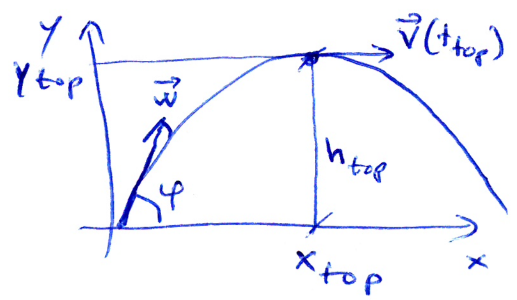
$$y = y_0 + \cancel{w}\sin\varphi\,\frac{x - x_0}{w\cos\varphi} - \frac{1}{2}g\,\frac{(x-x_0)^2}{w^2\cos^2\varphi}$$

or

$$\boxed{y = y_0 + \tan\varphi\,(x - x_0) - \frac{1}{2}g\,\frac{(x-x_0)^2}{w^2\cos^2\varphi}}$$

which is the equation of a parabola

c) <u>the maximal height the projectile can reach</u>



- we can determine this point in two ways:

1) velocity has zero v_y component
2) derivative of $y = f(x)$ is zero at this point

1) we first determine the time:

$$v_y(t_{top}) = w\sin\varphi - g\,t_{top} = 0 \Rightarrow t_{top} = \frac{w\sin\varphi}{g}$$

and thus

$$h_{top} = y(t_{top}) - y_0 = \frac{(w\sin\varphi)^2}{g} - \frac{1}{2}g\left(\frac{w\sin\varphi}{g}\right)^2 =$$

$$\boxed{h_{top} = \frac{1}{2}\frac{w^2\sin^2\varphi}{g}}$$

this height will be reached at

$$x_{top} = x_0 + w\cos\varphi\,\frac{w\sin\varphi}{g} = x_0 + \frac{w^2\sin\varphi\cos\varphi}{g}$$

2) we calculate the derivative of $y(x)$ and set it to zero

$$\frac{dy(x)}{dx} = \frac{d}{dx}\left[y_0 + \tan\varphi\,(x-x_0) - \frac{1}{2}\frac{g}{w^2\cos^2\varphi}(x-x_0)^2\right] =$$

constants

$$= \tan\varphi - \frac{1}{2}\frac{g}{w^2\cos^2\varphi}\,2\,(x-x_0) = 0$$

the solution is

$$\frac{g}{w^2\cos^2\varphi}(x-x_0) = \frac{\sin\varphi}{\cos\varphi}$$

$$x_{top} = x_0 + \frac{w^2\sin\varphi\cos\varphi}{g} \quad \text{(as above)}$$

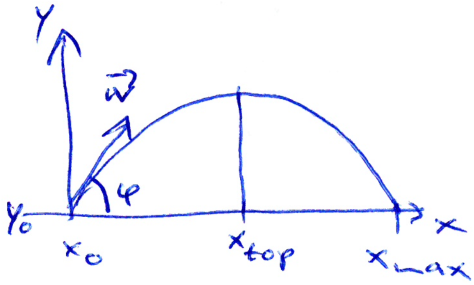
and thus

$$y_{top} - y_0 = h_{top} = \frac{\sin\varphi}{\cos\varphi}(x_{top}-x) - \frac{1}{2}\frac{g}{w^2\cos^2\varphi}(x_{top}-x)^2 =$$

$$= \frac{w^2\sin^2\varphi}{g} - \frac{1}{2}\frac{g}{\cancel{w^2\cos^2\varphi}}\left(\frac{w^2\sin\varphi\,\cancel{\cos\varphi}}{g}\right)^2 =$$

$$h_{top} = \frac{w^2\sin^2\varphi}{2g} \quad \text{(again as above)}$$

d) the horizontal distance $d = x_{max} - x_0$ where the projectile hits the ground (or the same value $y = y_0$ as at the beginnig)



- from $y = y_0$ we get

$$y_0 = y_0 + \tan\varphi \underbrace{(x_{max} - x_0)}_{d} - \frac{1}{2}\frac{g}{w^2\cos^2\varphi}\underbrace{(x_{max} - x_0)^2}_{d^2}$$

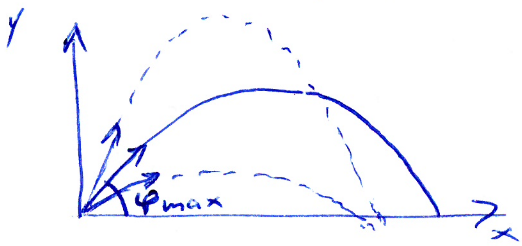
two solutions $d = 0$ (where also $y = y_0$)

and $\frac{1}{2}\frac{g}{w^2\cos^2\varphi} d = \tan\varphi = \frac{\sin\varphi}{\cos\varphi}$

thus $\boxed{d = \frac{w^2}{g} 2\sin\varphi\cos\varphi = \frac{w^2}{g}\sin 2\varphi}$

- or we could use the previous result for x_{top} and realize that $x_{max} - x_0 = d = 2(x_{top} - x_0)$ giving immediately $d = \frac{w^2}{g} 2\sin\varphi\cos\varphi$

For what angle we get the largest distance d_{max} if we throw the projectile at the same speed w to all directions?!



- clearly it is for the maximum of the function $\sin 2\varphi$ which is for $\varphi = 45° = \frac{\pi}{4}$

- the same result we would get by setting the derivative of $\sin\varphi\cos\varphi$ to zero

$$\frac{d}{d\varphi}(\sin\varphi\cos\varphi) = (\sin\varphi)'\cos\varphi + \sin\varphi(\cos\varphi)' =$$

Leibnitz rule

$$= \cos^2\varphi - \sin^2\varphi = 0$$

or $1 = \tan^2\varphi \Rightarrow$ again $\varphi = 45°$

e) For a given speed w, determine the angle φ to hit the point $(x_1, y_1) = ~~(d\cos\alpha, d\sin\alpha)~~$

$= (x_0 + d\cos\alpha, x_0 + d\sin\alpha)$