$\boxed{\text{QM I-2}}$ **Formalismus kvantové teorie 1**

(diskrétní systémy)

• **Opakování pojmů z lineární algebry**

$\boxed{\text{LVP: lin. vekt. prostor}}$ ... Prostor stavů (ket space)

Def: LVP = $\{\text{vektorů } \phi\} \equiv V$ $+$; násob. $\mathbb{C}$ čísly

uzavř. vůči

$\forall\, \phi, \psi \in V : \; a\phi + b\psi \in V$ ... Princip superpozice

$a, b \in \mathbb{C}$ (kočka Schrödingerova)

další požad. (Axiomy): Asoc $+$, komut $+$, $\exists\, 0$, $\exists -\psi$

distrib. pro $\mathbb{C}\cdot$, asoc., $\exists\, 1\cdot\phi = \phi$

zopakujte: lineární závislost, dimenze V, báze

$\boxed{\text{Skalární součin}}$ (inner product) amplitudy pravd.

$\langle\psi|\phi\rangle \in \mathbb{C}$ ... $\forall\, \psi, \phi \in V$ ← $|\psi\rangle$ je stav

$\langle\psi|\phi\rangle$ ampl. pravd.

$|\langle\psi|\phi\rangle|^2$ pravd.

1) $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$

2) $\langle\phi|c_1\psi_1 + c_2\psi_2\rangle = c_1\langle\phi|\psi_1\rangle + c_2\langle\phi|\psi_2\rangle$

3) $\langle\phi|\phi\rangle \geq 0$; rovnost $\Leftrightarrow \phi = 0$

zopakujte: • ON báze ... $\{\phi_i\}_{i=1}^{d}$ $\quad \langle\phi_i|\phi_j\rangle = \delta_{ij}$

• rozklad do báze $\boxed{|\psi\rangle = \sum_{i=1}^{d} c_i|\phi_i\rangle}$ ... $c_i = \langle\phi_i|\psi\rangle$

• každý konečný LVP dimenze d je izomorfní s $\mathbb{C}^d$

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{pmatrix} = c_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{|\phi_1\rangle} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \dots$$

Reprezentace

$$|\psi\rangle \xleftrightarrow[\text{na bázi}]{\text{závisí}} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{pmatrix} \equiv \{c_i\}_{i=1}^{d}$$

Duální prostor V^* $\equiv$ prostor $\forall$ lin. funkcionálů nad V

bra-vektory

Tvrz. $V \equiv V^* \equiv \mathbb{C}^d$

DK: .. funkcionál $F(|\psi\rangle) = f \in \mathbb{C}$

$$F\left(\sum_i c_i |\phi_i\rangle\right) = \sum c_i\, F(|\phi_i\rangle) = \sum_i f_i^* c_i$$

def ... $f_i := F(|\phi_i\rangle)^*$

def $\underbrace{|\phi\rangle}_{\text{ket}} := \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}$ ... pak $F(|\psi\rangle) \equiv \langle\phi|\psi\rangle$

$\longleftrightarrow$ .. $F_\phi(\cdot) \equiv \langle\phi|\cdot\rangle$

Diracova notace ... $F \equiv \boxed{\langle\phi|}$ bra $\langle\phi| \equiv |\phi\rangle^+$

$= (f_1^*, f_2^* \dots f_d^*)$

pozn.: $\langle\phi|\psi\rangle$ ... chápeme jako matic. násob.

$$(f_1^*, f_2^* \dots f_d^*)\begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix} = \sum_i f_i^* c_i$$

Lineární operátory: $\hat{A} : V \longrightarrow V$

$\forall\, |\psi\rangle \in V \quad \exists\, |\phi\rangle = \hat{A}|\psi\rangle \in V$

$$\hat{A}(c_1|\phi_1\rangle + c_2|\phi_2\rangle) = c_1 \hat{A}|\phi_1\rangle + c_2 \hat{A}|\phi_2\rangle$$

ON báze + lin.: ... $a_{ij} \equiv \langle\phi_i|\hat{A}|\phi_j\rangle$ ... maticový element operátoru A

$\{\phi_i\}$

$|\psi\rangle \cdots \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}$ $\quad = \hat{A}|\psi\rangle \equiv \begin{pmatrix} p_1 \\ \vdots \\ p_d \end{pmatrix}$ $\quad p_i = \sum_j a_{ij} c_j$

$|\psi\rangle = \sum c_i |\phi_i\rangle \qquad p_j \equiv \langle\phi_j|\hat{A}|\psi\rangle$

$|\varphi\rangle = \hat{A}|\phi\rangle \quad \ldots \quad \begin{pmatrix} p_1 \\ \vdots \\ p_d \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & \\ \vdots & & & \\ a_{d1} & \cdots & & a_{dd} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}$

Dirac. notaci: $\langle\phi|\hat{A}|\psi\rangle \qquad \hat{A}|\psi\rangle \qquad \langle\psi|\hat{A}$

- rovnost oper. $\hat{A} = \hat{B} \iff \hat{A}|\phi\rangle = \hat{B}|\phi\rangle \quad \forall |\phi\rangle \in V$
- nulový $\hat{O}$ a $\hat{I}$ ... $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ... $\hat{O}|\phi\rangle = 0 \qquad \hat{I}|\phi\rangle = |\phi\rangle$
- sčítání $\hat{A} + \hat{B}$ ... $(\hat{A} + \hat{B})|\phi\rangle \qquad (A+B)+C = A+(B+C)$
- násobení operátorů: $\hat{A}\hat{B}$ ... $\hat{A}\hat{B}|\phi\rangle \equiv \hat{A}(\hat{B}|\phi\rangle)$
- $\hat{A}\hat{B} \neq \hat{B}\hat{A}$ ... def $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

  .. $\hat{A}\hat{B} = \hat{B}\hat{A} + [\hat{A}, \hat{B}]$
- mocniny: $\hat{A}^n = \underbrace{\hat{A} \cdot \hat{A} \cdots \hat{A}}_{n}$ ... $f(x) = \sum_n f_n x^n$

def.. $f(\hat{A}) \equiv \sum_n f_n \hat{A}^n$ ... př. $e^{\hat{A}} \equiv \sum_n \frac{1}{n!} \hat{A}^n$

pozn.: $|\psi\rangle \equiv A|\phi\rangle$! $(AB)^+ = B^+ A^+$

$\underbrace{A|\phi\rangle}_{|\psi\rangle}$ ket $\longleftrightarrow$ bra $\langle\phi|A^+ = \langle\psi|$

def $\hat{A}^+$

$\hat{A} \leftarrow \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ & & \\ a_{d1} & & a_{dd} \end{pmatrix} \qquad \hat{A}^+ \leftarrow \begin{pmatrix} a_{11}^* & a_{d1}^* \\ a_{1d}^* & a_{dd}^* \end{pmatrix}$

A^+ ... def -- $\langle\hat{A}^+\phi|\psi\rangle = \langle\phi|\hat{A}\psi\rangle$

(počítání s +)

① $(c\hat{A})^+ = c^* A^+$

② $(A+B)^+ = A^+ + B^+$

③ $(AB)^+ = B^+ A^+$

Vnější součin $|\psi\rangle\langle\phi|$ - Dirac $\begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}(f_1^* \dots f_d^*) = \begin{pmatrix} c_1 f_1^* & \dots \\ \vdots & \\ c_d f_1^* & \dots \end{pmatrix}$

$|u\rangle \in V$

def: $|\psi\rangle\langle\phi|\,|u\rangle = |\psi\rangle(\underbrace{\langle\phi|u\rangle}_{\mathbb{C}})$

platí: $(|\psi\rangle\langle\phi|)^+ = |\phi\rangle\langle\psi|$

Trik: Rozklad jedničky on báze $\{|\phi_i\rangle\}$ $\quad \hat{I} = \sum_i |\phi_i\rangle\langle\phi_i|$

$\to \hat{I}|\psi\rangle = |\psi\rangle = \sum_i |\phi_i\rangle\underbrace{\langle\phi_i|\psi\rangle}_{c_i} = \sum_i c_i|\phi_i\rangle \equiv |\psi\rangle$

$\hat{A} = \hat{I}\hat{A}\hat{I} = \sum_i |\phi_i\rangle\langle\phi_i|\hat{A}\sum_j |\phi_j\rangle\langle\phi_j| = \sum_{ij} |\phi_i\rangle\langle\phi_i|A|\phi_j\rangle\langle\phi_j|$

$$\boxed{\hat{A} = \sum_{ij} a_{ij}|\phi_i\rangle\langle\phi_j|}$$

OBR: geom. náhled projekce

(ortogonální) **Projekční operátor**:

Def: $\hat{P}^2 = \hat{P}$ (Idempotence) ... $|b\rangle = \hat{P}|a\rangle \quad \hat{P}|b\rangle = |b\rangle$

$\hat{P} = \hat{P}^+$ (ortogonalita) ... $|f\rangle = \hat{P}|\psi\rangle$

$|\tilde{f}\rangle = |\psi\rangle - |f\rangle = (\hat{I} - \hat{P})|\psi\rangle$

$\langle\tilde{f}|f\rangle = 0 = \langle\psi|(\hat{I} - \hat{P})^+\hat{P}|\psi\rangle = 0$

$\hat{P} - \hat{P}^2 = 0$

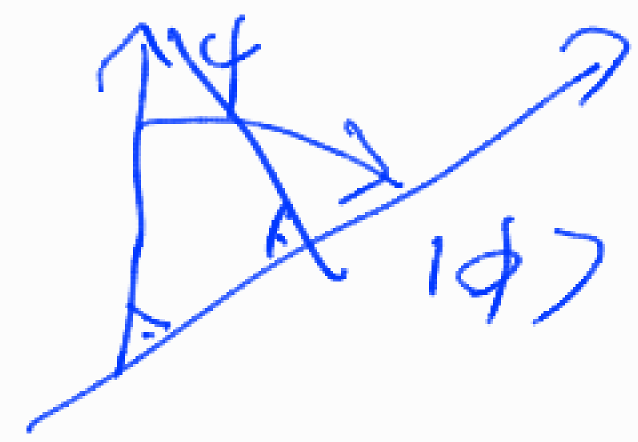
PŘ: $\hat{P} = |\phi\rangle\langle\phi|$ ... $\langle\phi|\phi\rangle = 1 \Leftarrow$

$= \|\phi\|^2$

$\|\phi\| = \sqrt{\langle\phi|\phi\rangle}$

je projektor

$$\hat{P}^2 = |\phi\rangle\underbrace{\langle\phi||\phi\rangle}_{1}\langle\phi| = |\phi\rangle\langle\phi| = \hat{P}$$



$$\hat{P}^+ = |\phi\rangle\langle\phi| = |\phi\rangle\langle\phi| = \hat{P}$$

PŘ: $$\hat{P} = \frac{1}{\langle\phi|\phi\rangle}|\phi\rangle\langle\phi| = \boxed{\frac{|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle}}$$