QMI-2 Formalismus QM – 1. diskrétní systémy

OPAKOVÁNÍ: kvantová teorie .. principy (postuláty)

1. Stavy .. LVP .. $\mathcal{H}$ ... stav $\equiv \lambda|\psi\rangle$; $|\psi\rangle \in \mathcal{H}$; $\lambda \in \mathbb{C}$ libovolné

   normování $\|\psi\|^2 = \langle\psi|\psi\rangle = 1$

2. Pozorovatelné (měřitelné veličiny) .. lineární Hermiteovské operátory $\hat{A}$

   přípustné hodnoty ... spektrum: $a \in \sigma_A$ ... $\hat{A}|a,k\rangle = a|a,k\rangle$

   spektr. rozklad $\hat{A} = \sum_{a,k} a|a,k\rangle\langle a,k| = \sum_a a\hat{P}_a$

3. skalární součin .. $\langle a|\psi\rangle$ .. amplituda pravd. přechodu $|\psi\rangle \to |a\rangle$ z příslušného měření

   $|\langle a|\psi\rangle|^2$

podrobněji:

systém ve stavu $|\psi\rangle \xrightarrow[\hat{A}]{\text{měření}}$ hodnota a $p(a,\psi) = \langle\psi|\hat{P}_a|\psi\rangle = \sum_k |\langle a,k|\psi\rangle|^2$

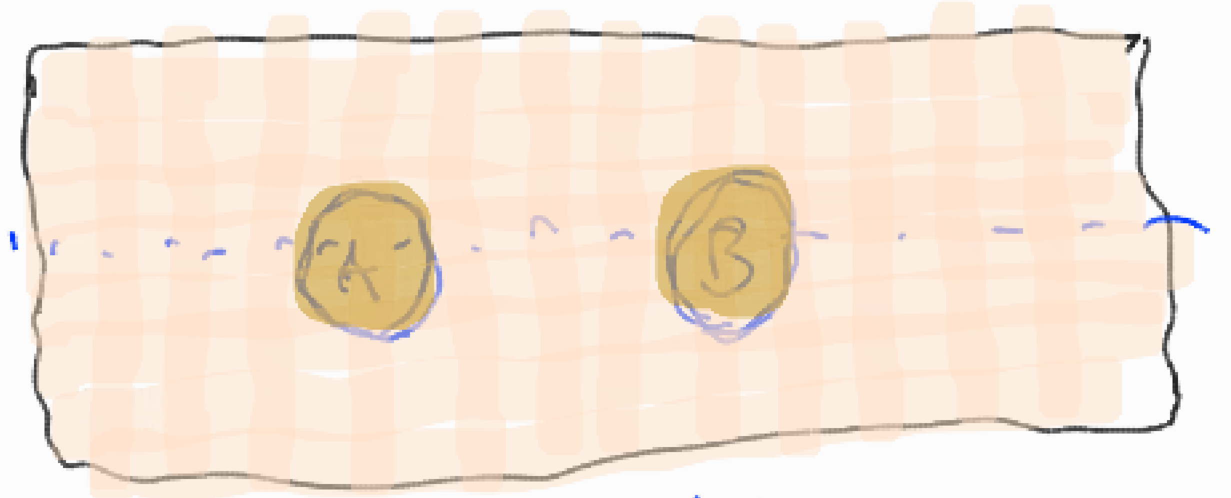
systém přejde do stavu $|\psi'\rangle = \frac{1}{\sqrt{p}}\hat{P}_a|\psi\rangle$

4. $i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle$

PŘÍKLAD 1: Kvantové tečky

$\mathcal{H} = \mathcal{L}\{|A\rangle, |B\rangle\}$ $\langle i|j\rangle = \delta_{ij}$, $i,j \in \{A,B\}$

$|\psi\rangle = \alpha|A\rangle + \beta|B\rangle$ ... $\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$



$\hat{X} = 1 \cdot |A\rangle\langle A| + 2|B\rangle\langle B|$

$\hat{X}|A\rangle = 1|A\rangle$

$\hat{X}|B\rangle = 2|B\rangle$

$$\hat{X} \leftrightarrow \begin{array}{cc} A & B \end{array} \quad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{array}{c} A \\ B \end{array}$$

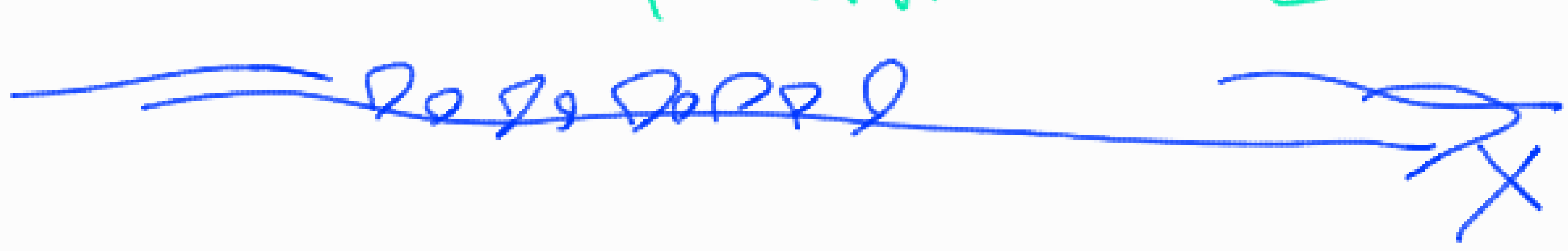
různé způsoby zadání operátoru

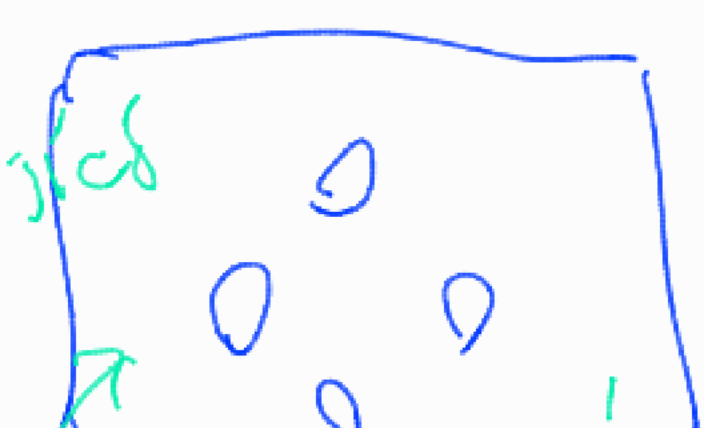
$\langle A|\hat{H}|A\rangle = \langle B|\hat{H}|B\rangle = \alpha$

$\langle A|H|B\rangle = \beta$

$$\hat{H} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha \end{pmatrix}$$

OBECNÝ operátor respektující symetrii $A \leftrightarrow B$

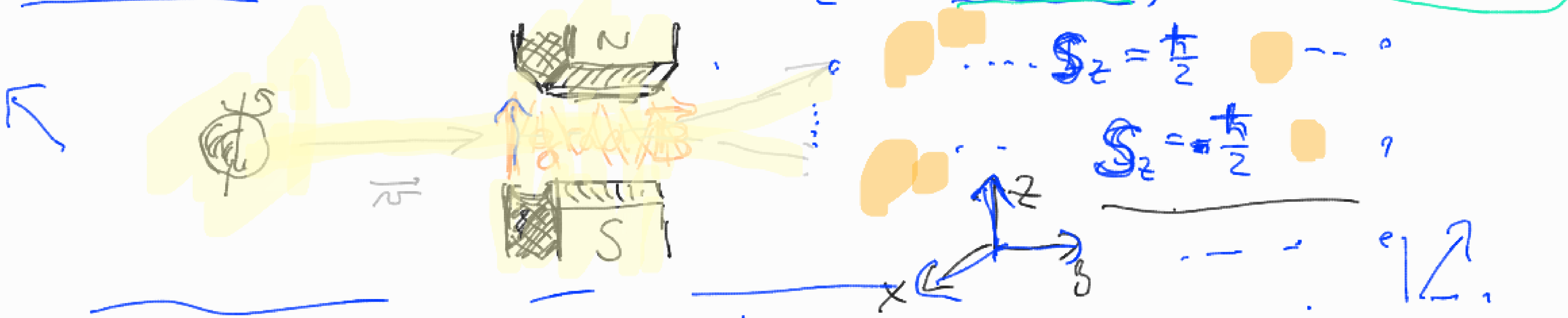


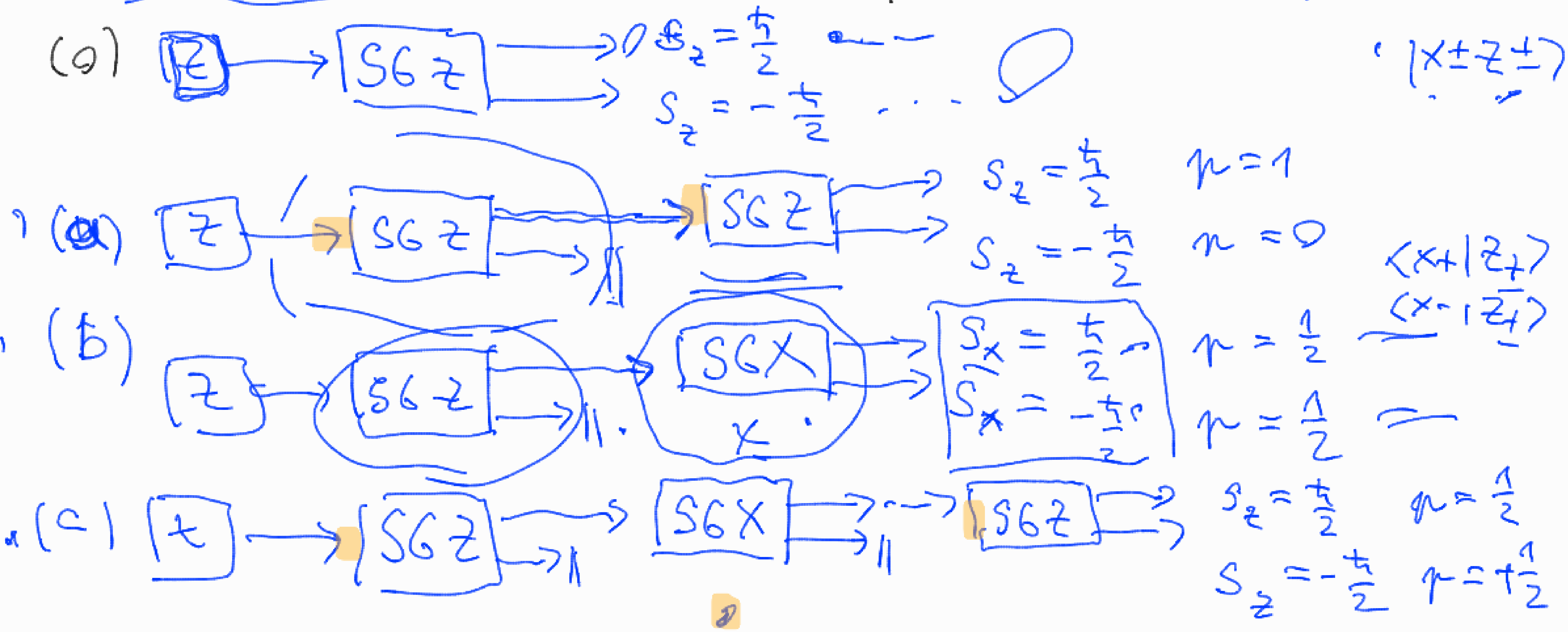


TOTO BUDE DÚ
sledujte web

PŘÍKLAD 2: částice se spinem 1/2

odvození popisu – zde náznak, (Sakurai) ← podrobnosti





Nejjednodušší možný model: (Pauli) S_z $|\uparrow\rangle$ $|\downarrow\rangle$ S_x S_y

$\mathcal{H} = \mathcal{L}\{|z+\rangle, |z-\rangle\}$ ... $\hat{S}_z = \frac{\hbar}{2}\left(|z+\rangle\langle z+| - |z-\rangle\langle z-|\right)$

$\langle z_i|z_j\rangle = \delta_{ij}$ $S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\hbar}{2}\sigma_z$

(a) před ... $|\psi\rangle = |z+\rangle$

měření $\hat{S}_z = \frac{\hbar}{2}\underbrace{|z+\rangle\langle z+|}_{P_{z+}} - \frac{\hbar}{2}\underbrace{|z-\rangle\langle z-|}_{P_{z-}} = \sum_a a\hat{P}_a$

$p_{z+} = |\langle z+|\psi\rangle|^2 = |\langle z+|z+\rangle|^2 = 1$

$p_{z-} = |\langle z-|\psi\rangle|^2 = |\langle z-|z+\rangle|^2 = 0$

po měření $|\tilde{\psi}\rangle = \hat{P}_+|\psi\rangle$

$|\tilde{\psi}\rangle = |z+\rangle\underbrace{\langle z+|z+\rangle}_{1} = |z+\rangle$

(b) před .. $|\psi\rangle = |z+\rangle$ .. $\begin{pmatrix}1\\0\end{pmatrix}$ .. S

měření .. $\hat{S}_x = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$

$\hat{S}_x |x+\rangle = \frac{\hbar}{2}|x+\rangle$

$\hat{S}_x |x-\rangle = -\frac{\hbar}{2}|x-\rangle$

$\hat{S}_x = \frac{\hbar}{2}\left(|x+\rangle\langle x+| - |x-\rangle\langle x-|\right)$

$\frac{1}{2} = p_{x+} = \langle\psi|\hat{P}_{x+}|\psi\rangle = |\langle x+|\psi\rangle|^2 = |\langle x+|z+\rangle|^2 = |\langle z+|x+\rangle|^2$

$|x_+\rangle = \begin{pmatrix}\cdot\\\cdot\end{pmatrix} \leftarrow \langle z+|x+\rangle = \frac{1}{\sqrt{2}} e^{i\alpha}$

(c) $\longrightarrow$ $\langle z-|x+\rangle = \frac{1}{\sqrt{2}} e^{i\beta}$ $\quad p_{x-} = \frac{1}{2}$

$|x_+\rangle = \frac{1}{\sqrt{2}}|z+\rangle + \frac{1}{\sqrt{2}} e^{i\delta_1}|z-\rangle$ $\quad | e^{-i\alpha + i\beta} = e^{i\delta_1}$

$|x_-\rangle = \frac{1}{\sqrt{2}}|z+\rangle - \frac{1}{\sqrt{2}} e^{i\delta_1}|z-\rangle$ $\quad \hat{S}_x$ $\quad \langle x+|x-\rangle = 0$

$$\boxed{\hat{S}_x = \frac{\hbar}{2}\left[e^{-i\delta_1}|z+\rangle\langle z-| + e^{i\delta_1}|z-\rangle\langle z+|\right]}$$

$\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\e^{i\delta_1}\end{pmatrix}}_{x+}$ $\quad \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-e^{i\delta_1}\end{pmatrix}}_{x-}$

$\frac{1}{\sqrt{2}}(1 \;\; e^{-i\delta}) \quad \frac{1}{2}\begin{pmatrix}1\\-e^{i\delta}\end{pmatrix} = 0$

$|x+\rangle\langle x+| = \frac{1}{2}\begin{pmatrix}1\\e^{i\delta}\end{pmatrix}(1 \;\; e^{-i\delta}) = \frac{1}{2}\begin{pmatrix}1 & e^{-i\delta}\\ e^{i\delta} & 1\end{pmatrix}$

$|y-\rangle\langle y-|$

podobný exp. $\boxed{z} \xrightarrow{\uparrow} \boxed{y} \rightarrow \boxed{z}$ $\quad$ 1/2, 1/2

S_y

$$\boxed{\hat{S}_y = \frac{\hbar}{2}\left[e^{-i\delta_2}|z+\rangle\langle z-| + e^{i\delta_2}|z-\rangle\langle z+|\right]}$$

$\boxed{x} \xrightarrow{\uparrow} \boxed{y} \rightarrow \boxed{x}$ $\quad$ 1/2, 1/2

$|\langle y\pm|x+\rangle|^2 = 1/2$

$|y+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\e^{i\delta_2}\end{pmatrix}$

$|y-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-e^{i\delta_2}\end{pmatrix}$

$i\,|y\pm\rangle$

$$\frac{1}{2}\left|1 \pm e^{i(\delta_1-\delta_2)}\right| = \frac{1}{\sqrt{2}}$$

$$\boxed{\left|1 \pm e^{i(\delta_1-\delta_2)}\right| = \sqrt{2}}$$

$$\delta_1-\delta_2 = \pm\frac{\pi}{2}$$

ZAPAMATUJTE:

$\boxed{\text{Záver}}$: častice so spinom 1/2

1) $\mathcal{H} = \mathbb{C}^2 \ \dots \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \ \dots \ \mathcal{L}(|z+\rangle, |z-\rangle)$, $|z+\rangle \equiv |\uparrow\rangle$, $|z-\rangle \equiv |\downarrow\rangle$

$|z\pm\rangle$, $|\pm\rangle$, $|\uparrow\rangle |\downarrow\rangle$

2) $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $\quad \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$

$$\hat{\vec{S}} \equiv (\hat{S}_x, \hat{S}_y, \hat{S}_z) = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z) = \frac{\hbar}{2}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$$

$\boxed{\hat{S}_+ = \hbar |+\rangle\langle -|}$ $\quad \left(|z+\rangle\langle z-| \right)$ $\quad \boxed{\hat{S}_+ = \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}}$

$\hat{S}_+|-\rangle = |+\rangle$ $\quad \hat{S}_+|+\rangle = 0$

$\hat{S}_- = \hat{S}_+^\dagger = \begin{pmatrix}0&0\\1&0\end{pmatrix} = \hbar |-\rangle\langle +|$

pomocné operátory $\hat{S}_+$ a $\hat{S}_-$ ... niekedy užitočné, ale nepozorovateľné!

POZN: častice s iným spinom

Spin: 0, 1/2, 1, 3/2, 2, ...

$\mathcal{H} = \mathbb{C}^1$ $\quad \boxed{\mathcal{H} = \mathbb{C}^2}$ $\quad \mathcal{H} = \mathbb{C}^3$ $\quad \boxed{\mathcal{H} = \mathbb{C}^4}$

$\to \boxed{SG} \to$

viz VYŠE

$S_z = \hbar$, $S_z = 0$, $S_z = -\hbar$

OBECNEJŠÍ S-G experiment

? $\quad S_x \quad S_y \quad \boxed{S_z = \hbar\begin{pmatrix}1&&\\&0&\\&&-1\end{pmatrix}}$

Přechod k jiné bázi:

$\hat{A}$ .... báze $\{|a_i\rangle\}$ $i = 1, \dots, d$ .. $|a_i\rangle$

$\hat{B}$ .... $\{|b_i\rangle\}$ $i = 1, \dots, d$ .... $|b_i\rangle$

$|\psi\rangle \underset{A}{\dots} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix}$

A-reprezentace

$|\psi\rangle = \hat{I}|\psi\rangle = \sum_i |a_i\rangle \underbrace{\langle a_i|\psi\rangle}_{\alpha_i}$

$\hat{I} \equiv \sum_i |a_i\rangle\langle a_i|$

$|\psi\rangle = \sum_i \alpha_i |a_i\rangle$

B-reprezentace

$|\psi\rangle = \sum_i \alpha_i |a_i\rangle = \sum_j \underbrace{\left[\sum_i \langle b_j|a_i\rangle \alpha_i\right]}_{\beta_j} |b_j\rangle$

$\hat{I} = \sum_j |b_j\rangle\langle b_j|$

$\beta_j = \langle b_j|\psi\rangle$

$\hat{I} = \sum_i |a_i\rangle\langle a_i|$

$\beta_i = \underbrace{\sum_i \langle b_j|a_i\rangle}_{U} \underbrace{\alpha_i}_{\begin{pmatrix} \alpha \\ \vdots \end{pmatrix}}$

unitární matice přechodu $A \to B$

$$\boxed{\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_d \end{pmatrix} = U \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix}}$$

$U_{ji} = \langle b_j|a_i\rangle$

$U^+ U = \sum_j \underbrace{(U^+)_{ij}}_{U^*_{ji}} U_{ji'}$

$U^+ U = \sum_j U^*_{ji} U_{ji'} = \sum_j \underbrace{\langle b_j|a_i\rangle^*}_{\langle a_i|b_j\rangle} \langle b_j|a_{i'}\rangle = \langle a_i| \Big(\underbrace{\sum_j |b_j\rangle\langle b_j|}_{\hat{I}} \Big) |a_{i'}\rangle$

$= \langle a_i|a_{i'}\rangle = \delta_{ii'}$

$\hat{C}$ ... A-repr.

B-repr.

$C_{ij} \equiv \langle a_i|\hat{C}|a_j\rangle$

$\hat{C}_B \dots C^{(B)}_{jj'} \equiv \langle b_j|\hat{C}|b_{j'}\rangle = \sum_{ii'} \underbrace{\langle b_j|a_i\rangle}_{U} \underbrace{C_{ii'}}_{\langle a_i|C|a_{i'}\rangle} \underbrace{\langle a_{i'}|b_{j'}\rangle}$

$\hat{I}$ $\hat{I} = \sum_i |a_i\rangle\langle a_i|$

$C^{(B)} = U \cdot C^{(A)} U^+$