

REPRESENTATIONS OF LIE ALGEBRAS

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Def: Repre of of a CA G on lin. vect. space V
is homomorphism

$$\rho: G \rightarrow \text{End}(V) \quad (\text{ie, incl. } \text{Ker } V \neq \emptyset)$$

Def: d -dim matrix repre of G over field F
is association of every $X \in G$ with a $d \times d$ matrix
 $D(X)$ satisfying

$$\text{i}, D(\alpha X + \beta Y) = \alpha D(X) + \beta D(Y)$$

$$\text{ii}, D([X, Y]) = [D(X), D(Y)]$$

$$\forall X, Y \in G, \forall \alpha, \beta \in F$$

- it is sufficient to find matrices of the generators/basis of G
- null (trivial) repre: $D(X) = 0 \quad \forall X$
- reducibility & complete reducibility; equivalence of repres defined identically as for groups
- reducibility only needs to be checked for generators

Theorem: (Schur lemmas)

1, Let D & D' be two IRREPs of G of dims. d & d' .
& let A be $d \times d'$ matrix such that
 $D(X)A = AD(X) \quad \forall X \in G.$

Then either $A=0$ or $d=d'$ & $\det A \neq 0$.

2, Let D be a d -dim IRREP of G & B is a $d \times d$ matrix such that
 $D(X)B = BD(X) \quad \forall X \in G.$

Then $D = \lambda \mathbb{1}_d$

• IRREPs of Abelian G are 1-dim. (43)

Def: Analytical repres of a CG G is a repres such that the matrix elements of $D(g(x_1, \dots, x_n))$ are analytical functions of loc. coordinates on $U(e)$.

⇒ mat. elements can be expanded in Taylor series on $U(e)$ → left translat. → it holds on the whole group

Theorem: Let D_G is d-dim anal. repres of G with a CA G . Then

1, matrices D_g def. for every $x \in G$ as

$$D_g(x) = \frac{d}{dt} D_G(\exp tx) /_{t=0} \quad (*)$$

form d-dim repres of G & for $\forall x \in G \quad \forall t \in \mathbb{R}$,

$$\exp(t D_g(x)) = D_g(\exp(tx)) \quad (**)$$

(need not cover whole G but $\exists \forall t \in \mathbb{R}$)

2, let D_G & D'_G be two d-dim anal. repres of G & D_g, D'_g correps. repres of G def. through (*).

Then $D_G \sim D'_G \Rightarrow D_g \sim D'_g$. (\Leftrightarrow for connected G)

3, D_G reducible $\Rightarrow D_g$ reducible (\Leftrightarrow for $-n-$ G)

4, D_G fully red. $\Rightarrow D_g$ fully reducible ($-n-$)

5, let G be connected. Then D_G IR $\Leftrightarrow D_g$ IR.

6, D_G unitary $\Rightarrow D_g$ anti-hermitian (\Leftarrow for connected G)

Proof:

1, (***) is derived homomorphism $\beta_*: \mathcal{G} \rightarrow \text{End}(V)$
generated by $\beta: G \rightarrow \text{Aut}(V)$

(*) follows from (**) via derivation due to analyticity

$$2, \cdot \exp(S^{-1}gS) = S^{-1}\exp(g)S$$

$$\boxed{\Rightarrow} D_G \sim D'_G \Rightarrow D'_G(x) = \frac{d}{dt} (S^{-1}D_G(\exp(tx))S)_{t=0}$$

$$\stackrel{(**)}{=} \frac{d}{dt} S^{-1}\exp(tx)S = \frac{d}{dt} \exp(txS^{-1}D_G(x)S)$$

$$= S^{-1}D_G(x)S$$

$$\boxed{\Rightarrow} G \text{ connected} \Rightarrow g = \prod_i \exp(t_i x_i) \quad \forall g \in G$$

$$D'_G \sim D_G \stackrel{(**)}{\Rightarrow} D'_G(\exp(tx)) = \exp(txS^{-1}D_G(x)S) \quad \square$$

3&4, $\cdot \boxed{\Rightarrow}$ directly from (*)

$\cdot \boxed{\Leftarrow}$ block structure reproduced at all powers
of a matrix & (**) & $g = \prod_i \exp(t_i x_i)$

5, directly from 3

6, directly from (**)



Note: • not every D_g gives through \exp valid repn of G
• only if D_G is analytical, then it can be recovered
from the corresp. repn of g

\Rightarrow not all repn of G can be obtained by derivation
of some repn of G ; unless G is simply
connected (universal covering gr.)

Example 1, $SO(2)$, $SO(2)$

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$$\cdot SO(2) \text{ 1-dim, } e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \exp(te_1) = \begin{pmatrix} e^t & st \\ -st & e^{-t} \end{pmatrix} \quad (7)$$

• str. constants: $[e_1, e_1] = 0$

$\Rightarrow D_G(e_1) = p$ is 1-dim repn of G for arb. $p \in \mathbb{C}$

$$\Rightarrow \exp(tD_G(e_1)) = \exp(tp) \quad (7+)$$

$$\cdot (7+) \Rightarrow \exp((t + z\bar{u})e_1) = \exp(te_1)$$

$$(7+) \Rightarrow \exp((t + z\bar{u})p) = \exp(z\bar{u}p) \exp(tp)$$

$\Rightarrow \exp(tp)$ is repn of $SO(2)$ only for $p = ik, k \in \mathbb{Z}$

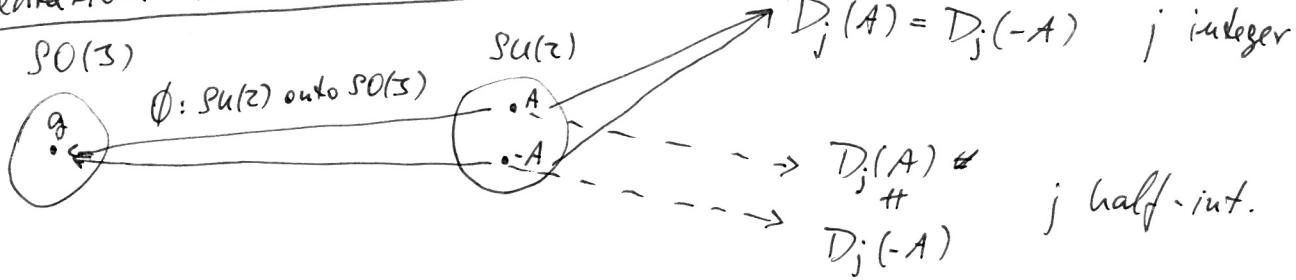
2, $SO(3) \otimes SO(3) \cong SU(2)$

• 3-dim repn $SO(3)$ given by Pauli matrices

$$D_G(e_1) = \frac{1}{2}\sigma_1 ; D_G(e_2) = \frac{1}{2}\sigma_2 ; D_G(e_3) = \frac{1}{2}\sigma_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\Rightarrow \exp(tD_G(e_3)) = \begin{pmatrix} \exp(\frac{1}{2}ti) & 0 \\ 0 & \exp(\frac{1}{2}ti) \end{pmatrix} \stackrel{t=t+z\bar{u}}{\Rightarrow} \begin{pmatrix} \cdot & 0 \\ 0 & \cdot \end{pmatrix} \quad y$$

\Rightarrow representation is multi-valued



• let \tilde{G} be univ. cover. group of $G \Rightarrow D_{\tilde{G}}$ gives repn of $G \cong \tilde{G}/K$ through the homomorphism $\phi: \tilde{G} \rightarrow G$ with discrete kernel K ,

$$D_G(\phi(g)) = D_{\tilde{G}}(g), \text{ if for all } k \in K \quad D_{\tilde{G}}(kg) = D_{\tilde{G}}(g)$$

$$kg \in \overline{G}$$

ADJOINT REPRESENTATION

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NB: for a (A, g) , $[X, e_j] = \sum_{k=1}^n (\text{ad}(X))_j^k e_k$ defines
 adjoint repree $X \mapsto \text{ad}(X)$ $\text{ad}: g \times g \rightarrow g$
 $\text{ad}(X)Y = [X, Y]$

Lemma: Let G be linear LG & g its LA . Then

$$1, g X g^{-1} \in G \quad \forall g \in G \text{ & } \forall X \in g$$

$$2, \text{ for } g = \exp(tY), Y \in g$$

$$F(t) = \exp(tY) X [\exp(tY)]^{-1} = X + t [YX] + \frac{t^2}{2!} [Y, [Y, X]] + \dots$$

Proof: 1. lin. G has faithful mat. repree $\Rightarrow g X g^{-1}$
 is well defined (product of matrices)

• $\mathcal{J}(s) = g \exp(sX) g^{-1} \subset G$ is a one-param. subgr.

$$(g \in G, X \in g, s \in \mathbb{R})$$

$\Rightarrow \mathcal{J}(s) = \exp(s g X g^{-1}) \rightarrow g X g^{-1} \in g$ is generator of \mathcal{J}

2, $F(t) = \exp(tY) X e^{-tY}$ is analytical function

$$F'(t)|_{t=0} = YX - XY = [Y, X]$$

$$F''(t)|_{t=0} = Y(YX - XY) - (YX - XY)Y = [Y, [Y, X]]$$

$\Rightarrow 2,$ is Taylor exp. of $F(t)$

Theorem: Let G be lin. $\mathcal{L}(G)$ of dim. n and e_1, \dots, e_n basis of its $\mathcal{L}(G)$. Define

$$\text{Ad}(g) : g e_i g^{-1} = \sum_{k=1}^n (\text{Ad}(g))_i^k e_k \quad i=1, \dots, n$$

Then

1, matrices $\text{Ad}(g)$ form n -dim analytical repres of G , called adjoint repres

2, corresponding repres of $\mathcal{L}(G)$, defined through

$$\text{ad}(X) = \frac{d}{dt} \text{Ad}(\exp tX) \Big|_{t=0} \quad \forall X \in \mathcal{L}(G)$$

is adjoint repres of $\mathcal{L}(G)$, satisfying

$$\exp(t \text{ad}(X)) = \text{Ad}(\exp(tX))$$

Proof: • $g e_i g^{-1} \in \mathcal{L}(G) \Rightarrow$ /lemma/ $\Rightarrow g e_i g^{-1} = c_i^k e_k$
 $\Rightarrow \text{Ad}(X)$ is well defined

$$\begin{aligned} 1, (gg') e_i (gg')^{-1} &= g (g' e_i g'^{-1}) g^{-1} = \sum_l \text{Ad}(g')_i^l g e_l g^{-1} \\ &= \sum_{kl} \text{Ad}(g')_i^l \text{Ad}(g)_l^k e_k = \sum_k [\text{Ad}(g) \text{Ad}(g')]_i^k e_k \quad \square \end{aligned}$$

• analyticity follows from the lemma

($g = \exp(tY)$ exists for $\forall g \in \mathcal{U}(e)$)

$$\begin{aligned} 2, \text{Ad}(\exp(tY))X &= \exp(tY) X \exp(-tY) = / \text{lemma} / = \\ &= X + t[Y, X] + o(t) \\ &= [I + t \text{ad}(Y)]X + o(t) \end{aligned}$$

$$\Rightarrow \frac{d}{dt} (\text{Ad}(\exp(tY))) \Big|_{t=0} = \text{ad}(Y) \quad \square$$

• $\text{Ad}(g)$ for general G :

- def. automorphism

$$I_g : G \rightarrow G \quad h \mapsto g^{-1}hg \quad (\text{conjugation})$$

- def. $\text{Ad}(g) \equiv (I_g)_* : g \exp(X) g^{-1} \equiv \exp(\text{Ad}(g)X)$

$$\Downarrow$$

$$I_g \circ \exp = \exp \circ (I_g)_*$$

$$\begin{array}{ccc} G & \xrightarrow{I_g} & G \\ \exp \uparrow & & \uparrow \exp \\ G & \xrightarrow[\equiv \text{Ad}(g)]{(I_g)_*} & G \end{array}$$

$\Rightarrow \text{Ad}(g) : G \rightarrow \text{Aut}(G)$ is repres acting on G

\Rightarrow matrix group:

$$g \exp(X) g^{-1} = \exp(g X g^{-1}) = \exp(\text{Ad}(g)X)$$

corresp. to the above definition

• significance of $\text{Ad}(g), \text{ad}(x)$: semi-simple CA/G

- we have already seen KC form/metric

Theorem: Adjoint space of a semisimple CA G is faithful.

Proof: need to show that $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ is injective:

- $\exists X \neq Y \in \mathfrak{g} : \text{ad}(X) = \text{ad}(Y)$

$$\Rightarrow 0 = \text{ad}(X) - \text{ad}(Y) = \text{ad}(X-Y), X-Y \neq 0$$

- $\text{ad}(X-Y)\text{ad}(z) = 0 \quad \forall z \in \mathfrak{g} \Rightarrow B(X-Y, z) = 0$

$\Rightarrow B$ degenerate $\Rightarrow \mathfrak{g}$ not semisimple \downarrow (Cartan Z.)

Theorem: Adjoint repres of a simple Lie \mathfrak{g} is irreducible. (49)

Proof: let ad be reducible $\Rightarrow \exists \mathcal{G}' \subsetneq \mathcal{G} :$

$$\text{ad}(x)y' \in \mathcal{G}' \quad \forall x \in \mathcal{G} \quad \forall y' \in \mathcal{G}'$$

$$\Rightarrow [x, y'] \in \mathcal{G}' \Rightarrow \mathcal{G}' \text{ is invariant subalgebra}$$

$$\Rightarrow \mathcal{G} \text{ not simple } \quad \square$$

NB: simple is also semi-simple

• is the non-inv. subspace $\mathcal{G}' \subsetneq \mathcal{G}$ subalgebra?

$\Rightarrow \text{ad}(x)$ is very powerfull tool, used in proofs of theorems about simple/semi-simple Lie

\Rightarrow furthermore, it can be shown that every semi-simple Lie is direct sum of simple Lie

$$1, V = V_1 + V_2 \quad \forall V \in \mathcal{G}, V_1 \in \mathcal{G}_1, V_2 \in \mathcal{G}_2 \text{ unique}$$

$$2, [V_1, V_2] = 0 \quad \forall V_1 \in \mathcal{G}_1 \text{ & } \forall V_2 \in \mathcal{G}_2$$

$$\Rightarrow \mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$$

COMPLEXIFICATION OF A CT

- how to make $\mathbb{R}\text{CT}$ out of $\mathbb{R}\text{CT}$?
- motivation: construction of IRREPS
cf. ladder operators $L_{\pm} = L_1 \pm iL_2$

CASE 1: generators of $\mathbb{R}\text{CT}$ are CN also over \mathbb{C} :

$$\sum_i d_i e_i = 0 \Rightarrow d_i = 0 \quad \forall i, d_i \in \mathbb{C}$$

\Rightarrow straight forward

Example: $\cdot \underline{\text{su}(2)}$ as $\mathbb{R}\text{CT}$ of anti-herm. matrices

has basis $e_i = \pm \frac{i}{2} \sigma_i$

$$e_1 = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad e_2 = -\frac{i}{2} \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad e_3 = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow ([\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k) \Rightarrow [e_i, e_j] = \epsilon_{ijk} e_k$$

$\Rightarrow e_i$ CN also over \mathbb{C} , but $\text{su}(2)\mathbb{C}$ is no longer CT of anti-herm. matrices (i x anti-herm \neq anti-herm.)

CASE 2: generators of $\mathbb{R}\text{CT}$ lin. dependent over \mathbb{C} :

Example: $\cdot \underline{\text{sl}(2, \mathbb{C})}$ as $\mathbb{R}\text{CT}$ is 6-dim (traceless mat.):

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = ie_1 \quad e_5 = ie_2 \quad e_6 = ie_3$$

$\Rightarrow \text{sl}(2, \mathbb{C})^{\mathbb{C}} \sim \text{su}(2)\mathbb{C}$ is only 3-dim

• however, for $\mathbb{R}\text{CT}$ $\text{sl}(2, \mathbb{C}) \sim O(1, 3)$ & direct complexification $O(1, 3)\mathbb{C}$ is 6-dim

\Rightarrow there must be 6-dim complex CT, which is complexification of $\text{sl}(2, \mathbb{C})$

CONSTRUCTION:

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- let e_1, \dots, e_n be basis of a \mathbb{R} CAG, which can be lin. dep. over \mathbb{C}

- construct $2n$ -dim \mathbb{R} vec. space with elements

$$(x, y) \quad x, y \in G \quad \text{with operations}$$

$$1, \alpha(x, y) = (\alpha x, \alpha y) \quad \alpha \in \mathbb{R}$$

$$2, (x, y) + (x', y') = (x+x', y+y')$$

- transform to \mathbb{C} vect. space G_c with scalar multiplication

$$(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x) \quad \alpha, \beta \in \mathbb{R}$$

$$\begin{aligned} \text{Ex: } & 1, z_1(z_2(x, y)) = (z_1 z_2)(x, y) \\ & 2, (z_1 + z_2)(x, y) = z_1(x, y) + z_2(x, y) \\ & 3, z[(x, y) + (x', y')] = z(x, y) + z(x', y') \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow G_c \text{ is lin. vect. space}$$

- $\dim G_c = n$: basis is $(e_1, 0), \dots, (e_n, 0)$

$$(0, e_j) = i(e_j, 0)$$

- commutator on G_c :

$$[(x, y), (x', y')] = ([x, x'] - [y, y'], [x, y'] - [y, x'])$$

$\Rightarrow G_c$ forms \mathbb{C} CA

Exercise: verify anti-sym. & Jacobi identity

- structure constants on G_c :

$$[(e_i, 0), (e_j, 0)] = [(e_i, e_j), 0] = \sum_k c_{ij}^k (e_k, 0)$$

$\Rightarrow G_c$ has the same c_{ij}^k as G

- if e_i are CN over \mathbb{C} , $G_\mathbb{C}$ is isomorphic to straightform complexification through the map

$$\phi((X, Y)) = X + iY$$

- c_{ij}^k the same $\Rightarrow \text{ad}_g(e_i) = \text{ad}_{G_\mathbb{C}}((e_i, 0))$
 $\Rightarrow B_g(e_i, e_j) = B_{G_\mathbb{C}}((e_i, 0), (e_j, 0))$

Cartan II.
 $\Rightarrow \underline{G \text{ semi-simple} \Leftrightarrow G_\mathbb{C} \text{ eucl-simple}}$

- simple CT - only one implication
 $G_\mathbb{C}$ simple $\Rightarrow G$ simple

Ex: $\cdot SO(1, 3)$ simple, $SO(1, 3)_\mathbb{C}$ not simple

- however, if G simple & $G_\mathbb{C}$ not simple, then
 $G_\mathbb{C}$ is direct sum of two isomorphic CTs:
 $SO(1, 3)_\mathbb{C} \sim SU(2)_\mathbb{C} \oplus SU(2)_\mathbb{C}$ (see (53))

Representations of complexified CT

- if $D_g(e_i)$ is rep of G , then $D_{G_\mathbb{C}}((e_i, 0)) = D_g(e_i)$
is rep of $G_\mathbb{C}$:

$$\rightarrow [D_{G_\mathbb{C}}((e_i, 0)), D_{G_\mathbb{C}}((e_j, 0))] = [D_g(e_i), D_g(e_j)] = D([e_i, e_j]) //$$

$$\rightarrow D_{G_\mathbb{C}}([(e_i, 0), (e_j, 0)]) = D_{G_\mathbb{C}}([e_i, e_j], 0) = D_g([e_i, e_j]) \quad \checkmark$$

- through same reasoning, rep of $G_\mathbb{C}$ generates rep of G

$$\bullet \text{ obviously } \boxed{D_g \text{ IRREP} \Leftrightarrow D_{G_\mathbb{C}} \text{ IRREP}}$$

\Rightarrow IRREPs of G can be constructed via complexification

Example : $\mathfrak{so}(3) \sim \mathfrak{su}(2)$

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• ladder operators $J_{\pm} = J_x \pm iJ_y$ \exists only in $\mathfrak{su}(2)_C$

$\mathbb{R}/\mathfrak{so}(1,3) \sim \mathfrak{so}(1,3) \sim \mathfrak{sl}(2,\mathbb{C})$... see Z-cover tutorial

J_i ... generators of rotations

K_i ... generators of boosts

\Rightarrow 6-dim \mathbb{R} LA

• complexification (straightforw.)

$$N_i^{\pm} = \frac{1}{2} (J_i \pm iK_i)$$

$$\left. \begin{array}{l} [J_i, J_j] = -\epsilon_{ijk} J_k \\ [J_i, K_j] = -\epsilon_{ijk} K_k \\ [K_i, K_j] = \epsilon_{ijk} J_k \end{array} \right\} \Rightarrow \begin{array}{l} [N_i^+, N_j^+] = -\epsilon_{ijk} N_k^+ \\ [N_i^-, N_j^-] = -\epsilon_{ijk} N_k^- \\ [N_i^+, N_j^-] = 0 \end{array}$$

$\Rightarrow \mathfrak{so}(1,3)_C \sim \mathfrak{su}(2)_C \oplus \mathfrak{su}(2)_C \sim \mathfrak{sl}(2,\mathbb{C})^2 \oplus \mathfrak{sl}(2,\mathbb{C})^2$

$\Rightarrow \mathfrak{so}(1,3)_C$ not simple while $\mathfrak{so}(1,3)$ is

(cf. Example V of Chapter 2, Section 7). For such a transformation

$$c^2 t'^2 - x'^2 - y'^2 - z'^2 = c^2 t^2 - x^2 - y^2 - z^2, \quad (17.4)$$

and, conversely, any linear transformation of the form of Equation (17.1) for which Equation (17.4) holds must satisfy Equation (17.2). L is a non-compact linear Lie group (see Example IV of Chapter 3, Section 3).

The set of transformations T for which $\det \Lambda(T) = 1$ form an invariant subgroup of L called the "proper homogeneous Lorentz group", which will be denoted by L_+ . The subgroup of L_+ consisting of transformations T such that $\Lambda(T)_{44} > 0$ forms the component of L (and of L_+) that is connected to the identity. This subgroup is known as the "proper orthochronous Lorentz group", and is denoted by L_+^\dagger (cf. Chapter 3, Section 6). The groups L , L_+ and L_+^\dagger are isomorphic to $O(3, 1)$, $SO(3, 1)$ and $SO_0(3, 1)$ respectively (see Chapter 3, Section 6). The real Lie algebra of $SO_0(3, 1)$ is $so(3, 1)$, which consists of the set of 4×4 real matrices a such that

$$\bar{a}g + ga = 0. \quad (17.5)$$

As noted in Example IV of Chapter 13, Section 3, a convenient basis of $so(3, 1)$ is provided by

$$\begin{aligned} J_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ J_3 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad J_4 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \\ L_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \end{aligned} \quad (17.6)$$

(cf. Example IV of Chapter 13, Section 3). This implies that the complexification $\tilde{\mathcal{L}}$ of $\mathcal{L} = so(3, 1)$ is semi-simple but is *not* simple, and that $\tilde{\mathcal{L}}$ is actually isomorphic to $A_1 \oplus A_1$ (cf. Chapter 14, Section 5).

It is sometimes convenient to use a different notation for the basis elements of $so(3, 1)$ and define $L_{\lambda\mu}(\lambda, \mu = 1, 2, 3, 4)$ by

$$\begin{aligned} L_{12} &= a_3, & L_{23} &= a_1, & L_{31} &= a_2, \\ L_{14} &= a_4, & L_{24} &= a_5, & L_{34} &= a_6, \end{aligned} \quad (17.10)$$

and

$$L_{\lambda\mu} = -L_{\mu\lambda} \quad (17.11)$$

The one-parameter subgroups generated by a_1 , a_2 and a_3 correspond to rotations in \mathbb{R}^3 about the axes Ox , Oy and Oz respectively, while the one-parameter subgroups generated by a_4 , a_5 and a_6 correspond to standard Lorentz transformations along Ox , Oy and Oz respectively.

For $p, q = 1, 2, 3$,

$$\left. \begin{aligned} J_i \cdot J_j &= \epsilon_{ijk} J_k \\ [a_p, a_q] &= -\sum_{r=1}^3 \epsilon_{par} a_r, \end{aligned} \right\} \quad (17.7)$$

$$\left. \begin{aligned} [J_i, J_j] &= -\epsilon_{ijk} J_k \\ [J_i, J_j] &= \epsilon_{ijk} J_k [a_p, a_{q+3}] = \sum_{r=1}^3 \epsilon_{pqr} a_r, \end{aligned} \right\}$$

where ϵ_{par} is defined in Equations (10.20). The real Lie algebra $so(3, 1)$ is simple, but not compact. However, with the 4×4 matrices a'_1, a'_2, \dots, a'_6 defined by

$$\left. \begin{aligned} N^+ & a'_p = \frac{1}{2}(a_p + ia_{p+3}), \\ N^- & a'_{p+3} = \frac{1}{2}(a_p - ia_{p+3}), \end{aligned} \right\} \quad (17.8)$$

for $p = 1, 2, 3$, it follows that

$$\left. \begin{aligned} N_i^+ N_j^+ [a'_p, a'_q] &= -\sum_{r=1}^3 \epsilon_{pqr} a'_r, \\ N_i^+ N^- [a'_p, a'_{q+3}] &= 0, \\ N_i^- N^- [a'_{p+3}, a'_{q+3}] &= -\sum_{r=1}^3 \epsilon_{pqr} a'_{r+3}, \end{aligned} \right\} \quad (17.9)$$

(which implies that $L_{11} = L_{22} = L_{33} = L_{44} = 0$). Comparison of Equations (17.6) and (17.10) shows that

$$(L_{\lambda\mu})_{\text{box}} = -\delta_{\beta\lambda} g_{\alpha\mu} + \delta_{\beta\mu} g_{\alpha\lambda} \quad (17.12)$$

REDUCIBLNÍ REPREZENTACE LIEOVÝCH ALGEBR

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Def: Casimirův operátor (element) je operační, tedy je funkcií generacií (tj. a který když komutuje s všemi pevnými algebry:

$$C = C(X_i) \quad \text{a} \quad [C, X_i] = 0 \quad \forall X_i = S_g(e_i)$$

obecná def.
cf. univ.
enveloping
algebra

\Rightarrow jedná se o operační na reprezentaci prostoru A , které jsou funkcemi (obvykle polynomy) operační X_i reprezentujících bázi/generátory A

- pro R , $C = \lambda I$ (Schur)
- $IR(A)$ lze indexovat hodnotami Casimirových operačních elaní A v daném R

Př: kvadratický Casimirov operační poloprostý A

- poloprostý A má nezágenerovanou $C-L$ metriku $g_{ij} = B(e_i, e_j) \Rightarrow$ inverzní matice $g^{ij} = (g^{-1})_{ij}$
- nechť S je reprezentační generátor g a $X_i = S(e_i)$ jsou operační reprezentující generátory g

$$\Rightarrow C = g^{ij} X_i X_j \text{ je Casimirov operační}$$

$$D\kappa: [C, X_a] = g^{ij} [X_i, X_j, X_a]$$

$$\begin{aligned}
 &= g^{ij} (X_i X_j X_a - X_a X_i X_j - X_i X_a X_j + X_i X_j X_a) \\
 &= g^{ij} X_i [X_j, X_a] + g^{ij} [X_i, X_a] X_j \\
 &= g^{ij} C_{jk}^e X_i X_e + g^{ij} C_{ik}^e X_e X_j = / \text{II: } i \leftrightarrow j, g_{ij} \text{ sym} \\
 &= g^{ij} C_{jk}^e (X_i X_e + X_e X_i) = 0 = c_{ijk} g^{ij} g^{el} p(X_i X_e + X_e X_i)
 \end{aligned}$$

NB: • bez ohledu na reprezentační Casimirov. op. (která je definována pomocí dualního báze ohledem na g)

$C = X^i X_i$

neboli $C_{jk}^e = g^{el} p c_{ijkl} /$

$g^{ij} g^{el} (X_i X_e + X_e X_i)$ je symetrické a $c_{ijkl} = -c_{jilk}$ (Dle pending)

$g_{ij} = c_{i\alpha}^\alpha c_{j\beta}^\beta$

$$\cdot C_{ijk} = -C_{jik} :$$

$$\begin{aligned}
 C_{ijk} &= g_{il} C_{jk}^l = \underset{\checkmark}{C_{id}} \underset{\checkmark}{C_{pl}} \underset{\checkmark}{C_{jk}^l} = -\underset{\checkmark}{C_{id}} \underset{\checkmark}{C_{pl}} \underset{\checkmark}{C_{jk}^l} = -\underset{\checkmark}{C_{id}} \underset{\checkmark}{C_{jk}} \underset{\checkmark}{C_{pl}} \\
 &= \text{Jacobi pro } C^\ell C^\alpha / = \underset{\checkmark}{C_{id}} \underset{\checkmark}{C_{pl}} \underset{\checkmark}{C_{jk}^l} + \underset{\checkmark}{C_{id}} \underset{\checkmark}{C_{pj}} \underset{\checkmark}{C_{kl}^\alpha} \\
 \Rightarrow C_{ijk} &= \underset{\checkmark}{C_{id}} \underset{\checkmark}{C_{pl}} \underset{\checkmark}{C_{jk}^l} - \underset{\checkmark}{C_{id}} \underset{\checkmark}{C_{pj}} \underset{\checkmark}{C_{kl}^\alpha} \quad (*)
 \end{aligned}$$

$$\begin{aligned}
 \stackrel{i \neq j}{\Rightarrow} C_{jik} &= \underset{\checkmark}{C_{jx}} \underset{\checkmark}{C_{pl}} \underset{\checkmark}{C_{ki}^l} - \underset{\checkmark}{C_{jx}} \underset{\checkmark}{C_{pi}} \underset{\checkmark}{C_{ki}^l} \\
 &= -\underset{\checkmark}{C_{il}} \underset{\checkmark}{C_{pk}} \underset{\checkmark}{C_{jk}^x} + \underset{\checkmark}{C_{ip}} \underset{\checkmark}{C_{jk}} \underset{\checkmark}{C_{ki}^l} \\
 &= \underset{\checkmark}{C_{ie}} \underset{\checkmark}{C_{aj}} \underset{\checkmark}{C_{pk}^l} - \underset{\checkmark}{C_{ip}} \underset{\checkmark}{C_{ki}^l} \underset{\checkmark}{C_{aj}^x} = -(*) \quad \square
 \end{aligned}$$

Def: Rank Lieovy grupy je největší počet nezájemně komutujících operačních políslusiné C^A = dimenze maximální abelovské podalgebry

- Př:
- Translace ve \mathbb{R}^3 - rank 3 ($p_i = -i \frac{\partial}{\partial x_i}$, komutují t_i)
 - $SO(3)$ - rank 1 ($[J_3, J_3] = 0$, ale $[J_3, J_i] \neq 0 \quad i=1,2$)
 - $SU(N)$ - rank $N-1$ (algebra bezesporužných antikomut. matic
 \Rightarrow největší abelovskou podalgebru tvoří matice

$\begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}$ kde i je na lib. pozici ne diag. a $i \neq n$
 poslední pozici

- $SO(3) \otimes SO(3)$ - rank 2 ($J_3^{(1)}, J_3^{(2)}$... odpovídají nezávislostem cískám)

Veta: (Pacahův leorem)

Pro každou poloprostou (A) ranku l existuje l Casimirových operačních, které komutují mezi sebou i proti G , ale i mezi sebou:

$$[C_k, X] = 0, [C_k, C_j] = 0 \quad \forall X \in G, t_{jik} = 1, \dots, \text{rank } G$$

Vlastní hodnoty těchto operačních jednoznačně charakterizují irreducibilní reprezentace příslušné Lieovy grupy.

- pro R , $C_k = \lambda_k \mathbb{1} \Rightarrow$ volání $R(S, V)$, $C_k \Psi = \lambda_k \Psi \quad \forall \Psi \in V$ (64)
 $\Rightarrow \lambda_k$ lze považit jako «pojmenování» reprezentace
- bázi příslušného invariantského podprostoru lze volit
jako vektory Ψ kommutujících operátorů C_k ($k=1, \dots, \ell$) a X_i ($i=1, \dots, \ell$)
(předložením toho X_i , které dohromady jednoznačně
vzdí $\dim S_R$ různých vektorech)

Př: $su(2)$ - rank 1 $\Rightarrow C = J_1^2 + J_2^2 + J_3^2 = J^2$ je jeho j

Casimirova operační

$\Rightarrow J_3$ má max. abstraktnou subalgebrou

- pomocí posuvovacích operátorů zjistíme, že dimenze R
mohou být $d_{jk} = 2j+1$, kde j je lib. polocelé číslo:
(viz kurz QM; jedná se o komplektifikaci $\mathfrak{su}(2)$!)
- $J_{\pm} = J_1 \pm i J_2 \Rightarrow [J_3, J_{\pm}] = \pm J_{\pm}$
 $[J_+, J_-] = 2J_3$

- vektory J_1^2, J_3 značíme $|b, m\rangle$

$$J_3 (|J_{\pm}, b, m\rangle) = J_{\pm} J_3 |b, m\rangle \pm J_{\pm} |bm\rangle = (m \pm 1) J_{\pm} |bm\rangle$$

$\Rightarrow J_{\pm} |b, m\rangle$ je vektor J_3 s vektorem $(m \pm 1)$

- jedná se o konečně-dimensionální reprezentaci $\mathfrak{su}(2)$
- \Rightarrow musí J_{\pm} mít: $J_{\pm} |b, m\rangle = 0$ a $m^-: J_- |b, m\rangle = 0$

$$J_- J_+ = J^2 - J_3^2 - J_3$$

$$\Rightarrow J_- J_+ |b, m^+\rangle = 0 = b - (m^+)^2 - m^+ \Rightarrow b = (m^+ + 1) m^+$$

$$\Rightarrow \text{značíme } m^+ = j \Rightarrow J^2 = j(j+1) \mathbb{1}_{\text{dim } \mathcal{S}}$$

$$\Rightarrow |b = j(j+1), m\rangle \equiv |j, m\rangle$$

$$\Rightarrow \text{opakování aplikuj: } J_- \text{ na } |j, j\rangle \Rightarrow m^- = -j \Rightarrow 2j \in \mathbb{N}$$

$$\Rightarrow \dim \mathcal{S} = 2j + 1$$

Příklad: $j=0 \Rightarrow \dim \mathfrak{J}_0 = 1$, $J_i = 0 \quad \forall i$, $J^2 = 0$

$j=\frac{1}{2} \Rightarrow \dim \mathfrak{J}_{\pm\frac{1}{2}} = 2$, $J_i = \frac{i}{2} \sigma_i$, $J^2 = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$, $J_3 = i \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

$j=1 \Rightarrow \dim \mathfrak{J}_1 = 3$, $J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$, $J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$, $J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\rightarrow J^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ (diag)}$$

Příklad: $so(1,3)$ - reprezentace $so(1,3)$ - algebra \mathcal{L}_+^\uparrow a představu univ. pokryvající grupy $SL(2, \mathbb{C})$ - dle které je jako reprezentace zkomplexifikované $so(1,3)_{\mathbb{C}} \sim su(2)_{\mathbb{C}} \oplus su(2)_{\mathbb{C}}$

• $(0,0)$ - triviální reprezentace $N_i^+ = N_i^- = 0 \Rightarrow g = e^0 = 1$ ($\dim = 1$, skalární reprezentace)

• $(\frac{1}{2}, 0)$ - 2-dim. reprezentace, $N_i^+ = \frac{1}{2} \sigma_i$, $N_i^- = 0$

$$N_i^- = 0 \Rightarrow \sum (J_i - iK_i) = 0 \Rightarrow \boxed{J_i = iK_i}$$

$$N_i^+ = \frac{1}{2} \sigma_i = \frac{1}{2} (J_i + iK_i) = iK_i = J_i$$

$$\Rightarrow J_i = \frac{1}{2} \sigma_i, \quad K_i = -\frac{i}{2} \sigma_i$$

- reprezentacemi prostor - (levé) chirální spinory (Weyl/ony)

• $(0, \frac{1}{2})$ - 2-dim. repre., $\boxed{J_i = -iK_i} = \frac{1}{2} \sigma_i$

- (pravo.) chirální Weyl/ony spinory

• $(\frac{1}{2}, \frac{1}{2})$ - 4-dim. reprezentace

- reprezentacemi prostor mohou být buď hermitovské matice 2×2 nebo 4-vektory - viz homomorfismus $SL(2, \mathbb{C}) \xrightarrow{\text{na}} \mathcal{L}_+^\uparrow$
 \Leftrightarrow je kó-repre na Mink. časoprostoru

podrobně viz
Garnett, vol. 2

NB: • (j, j') repre pro polohy sítí $j+j'$ jsou dojednací

repre \mathcal{L}_+^\uparrow (sítí se podílí na podgrupu rotací)

• proto také $(\frac{1}{2}, \frac{1}{2})$ musí být 4-vektor - je kó-sítina jednoznačná 4-dim. reprezentace