

DODATEK: ZÁKLADNÍ MATEMATICKÉ "TRIKY"

1, Implicitní funkce

$f(x, y, z) = 0$... implicitní funkcí mezi proměnnými x, y, z :

$$x = x(y, z) \Rightarrow dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz \quad (1)$$

$$z = z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy \quad (2)$$

• dosadíme (2) \rightarrow (1)

$$\left[\left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial x}\right)_y - 1\right] dx + \left[\left(\frac{\partial x}{\partial y}\right)_z + \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_x\right] dy = 0$$

• dx, dy nezávislé \Rightarrow

$$a) \left(\frac{\partial x}{\partial z}\right)_y = \left[\left(\frac{\partial z}{\partial x}\right)_y\right]^{-1}$$

$$b) \left(\frac{\partial x}{\partial y}\right)_z = - \frac{\left(\frac{\partial z}{\partial y}\right)_x}{\left(\frac{\partial z}{\partial x}\right)_y}$$

2, Úplně diferenciální funkce více proměnných

• $f = f(x_1, \dots, x_n)$ & $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$

$$\Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \Leftrightarrow \oint df = 0$$

• integrální faktor:

$d\omega = \sum_i A_i(x_1, \dots, x_n) dx_i$ nemá úplný diferenciál

$$\Rightarrow \frac{\partial A_i}{\partial x_j} \neq \frac{\partial A_j}{\partial x_i}$$

• může $\exists \mu = \mu(x_1, \dots, x_n) : \frac{\partial(\mu A_i)}{\partial x_j} = \frac{\partial(\mu A_j)}{\partial x_i}$

$$\Rightarrow d\sigma = \mu d\omega \text{ je úplný dif. a } \exists \sigma = \sigma(x_1, \dots, x_n)$$

$$! d\sigma = 0 \Leftrightarrow d\omega = 0 !$$

Príklad: $(x^2+y)dx - xdy = 0$

$$\frac{\partial}{\partial y}(x^2+y) = 1 \neq \frac{\partial}{\partial x}(-x) = -1$$

$$\frac{\partial}{\partial y}[\mu(x^2+y)] = \frac{\partial \mu}{\partial y}(x^2+y) + \mu = \frac{\partial}{\partial x}(-\mu x) = -x \frac{\partial \mu}{\partial x} - \mu$$

$$\Rightarrow \text{môže voliť } \mu = \mu(x) \Rightarrow -x \frac{d\mu}{dx} = 2\mu$$

$$\Rightarrow \frac{d\mu}{\mu} = -2 \frac{dx}{x} \Rightarrow \log \mu = -2 \log x \Rightarrow \mu = \frac{1}{x^2}$$

$$\Rightarrow \boxed{d\sigma = \left(1 + \frac{y}{x^2}\right)dx - \frac{1}{x}dy}$$

integrácia

$$1) \int dy \Rightarrow \sigma = -\frac{y}{x} + \tilde{C}(x) \Rightarrow \left/ \frac{\partial}{\partial x} \right/ \Rightarrow \frac{\partial \sigma}{\partial x} = \frac{y}{x^2} + \frac{d\tilde{C}}{dx}$$

$$\Rightarrow \frac{y}{x^2} + \frac{d\tilde{C}}{dx} = 1 + \frac{y}{x^2} \Rightarrow \frac{d\tilde{C}}{dx} = 1 \Rightarrow \tilde{C} = x + C \quad \leftarrow \begin{array}{l} \text{opravdu} \\ \text{konstanta} \end{array}$$

$$\Rightarrow \boxed{\sigma = x - \frac{y}{x} + C}$$

3) Jacobiho determinanta

$$\frac{\partial(u, v, \dots, w)}{\partial(x, y, \dots, z)} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \dots & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \dots & \frac{\partial v}{\partial z} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial w}{\partial x} & \dots & \dots & \frac{\partial w}{\partial z} \end{pmatrix} \quad \begin{array}{l} \bullet u, v, \dots, w \text{ jsou} \\ \text{funkce } x, y, \dots, z \end{array}$$

parciální derivace: $\left(\frac{\partial u}{\partial x}\right)_{y, \dots, z} = \frac{\partial(u, y, \dots, z)}{\partial(x, y, \dots, z)} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \dots & \frac{\partial u}{\partial z} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{pmatrix}$

řetězové pravidlo: $\frac{\partial(u, v, \dots, w)}{\partial(x, y, \dots, z)} = \frac{\partial(u, v, \dots, w)}{\partial(r, s, \dots, t)} \frac{\partial(r, s, \dots, t)}{\partial(x, y, \dots, z)}$

- řetězové pravidlo $\frac{\partial f_i}{\partial y_k} \frac{\partial y_k}{\partial x_j} = \frac{\partial f_i}{\partial x_j}$ je "soudim matice"

a platí $A = B \cdot C \Rightarrow |A| = |B| |C|$

• první důsledek: $\frac{\partial(x, y, \dots, z)}{\partial(a, v, \dots, w)} = \left[\frac{\partial(a, v, \dots, w)}{\partial(x, y, \dots, z)} \right]^{-1}$

příklad:

$$\begin{aligned} \left(\frac{\partial S}{\partial T} \right)_{V, N} &= \frac{\partial(S, V, N)}{\partial(T, V, N)} = \frac{\partial(S, V, N)}{\partial(T, P, N)} \frac{\partial(T, P, N)}{\partial(T, V, N)} = \\ &= \begin{pmatrix} \left(\frac{\partial S}{\partial T} \right)_{P, N} & \left(\frac{\partial S}{\partial P} \right)_{T, N} \\ \left(\frac{\partial V}{\partial T} \right)_{P, N} & \left(\frac{\partial V}{\partial P} \right)_{T, N} \end{pmatrix} \begin{pmatrix} \left(\frac{\partial P}{\partial V} \right)_{T, N} \end{pmatrix} \stackrel{\text{N u z}}{\text{apř í s u}} = \begin{pmatrix} \left(\frac{\partial S}{\partial T} \right)_P \left(\frac{\partial V}{\partial P} \right)_T - \left(\frac{\partial S}{\partial P} \right)_T \left(\frac{\partial V}{\partial T} \right)_P \end{pmatrix} \begin{pmatrix} \left(\frac{\partial P}{\partial V} \right)_T \end{pmatrix} \\ &= \left(\frac{\partial S}{\partial T} \right)_P - \left(\frac{\partial S}{\partial P} \right)_T \frac{\left(\frac{\partial V}{\partial T} \right)_P}{\left(\frac{\partial V}{\partial P} \right)_T} = \left(\frac{\partial S}{\partial T} \right)_P + \left(\frac{\partial S}{\partial P} \right)_T \left(\frac{\partial P}{\partial T} \right)_V \end{aligned}$$

4) Homogenní funkce p-tého řádu

$$F(\lambda x_1, \dots, \lambda x_k) = \lambda^p F(x_1, \dots, x_k)$$

$$\Rightarrow p \lambda^{p-1} F(x_1, \dots, x_k) = \frac{dF}{d\lambda} = \sum_j \frac{\partial F}{\partial(\lambda x_j)} \frac{\partial(\lambda x_j)}{\partial \lambda} = \sum_j \frac{\partial F}{\partial(\lambda x_j)} x_j$$

$$\Rightarrow \lambda = 1 / \Rightarrow \boxed{p F(x_1, \dots, x_k) = \sum_j \frac{\partial F}{\partial x_j} x_j} \quad \text{Eulerovo} \\ \text{věťe}$$

5) Legendre's transform