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Conformal Einstein equation and symplectic flux with a positive cosmological constant

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Abstract

We analyze the conformal Einstein equation with a positive cosmological constant to extract fall-off conditions of the gravitational fields. The fall-off conditions are consistent with a finite, non-trivial presymplectic current on the future boundary of de Sitter. Hence our result allows a non-zero gravitational flux across the boundary of the de Sitter. We present an explicit gauge-free computation to show that the Gibbons–Hawking boundary term, counterterm in the action, and fall-off condition of gravitational field in conformal Einstein equation are crucial to reproduce the finite symplectic flux.

Keywords: de Sitter, symplectic flux, conformal Einstein equation

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1. Introduction

Ever since LIGO's first detection of gravitational waves [1], a breakthrough that was itself the result of years of growing activity in the field, the field has continued to grow into one of the most dynamic areas of research. At the theoretical level, the formalism describing gravitational radiation has a long tradition in the zero cosmological constant setting. The first complete formalism describing gravitational radiation in the full non-linear theory is the well-known Bondi-Sachs formalism [2–5]. This is based on the existence of a particular coordinate gauge which allows one to define the energy flux of gravitational radiation in terms of an object known as *Bondi News*.

Another approach due to Penrose and Newman [6, 7] is based on a frame formalism to give a description of radiation at null future (conformal) infinity $\mathcal{I}^+$. A remarkable result by Geroch [8] is the definition of a tensor quantity at $\mathcal{I}^+$, which is additionally invariant under conformal scalings, and that matches the Bondi News. This therefore receives the name of *News tensor*. The News tensor depends entirely on the intrinsic quantities defined on $\mathcal{I}^+$. It is worth to remark that the coordinate and conformal invariance of the News makes it an optimal candidate for describing a physical quantity such as the gravitational energy flux.

The results by Wald and Zoupas in [9] also provide further mathematical structure to understand the News tensor and gravitational radiation. By giving a general definition of conserved quantities (for arbitrary field theories) in terms of Hamiltonian or equivalently a symplectic form defined at the boundary manifold, they find out that in the case of general relativity with zero cosmological constant with conformal infinity $\mathcal{I}^+$ as a boundary, such conserved quantity is actually determined by the News tensor. The non-vanishing nature of the News tensor or equivalently the symplectic structure on the boundary is attributed to the gravitational flux. The symplectic form depends on the linear fields δg over a certain gravitational field g . Thus, in order to evaluate it at the conformal boundary, one must perform a conformal rescaling of both the background $\tilde{g} = \Omega^2 g$ and the linear field $\delta \tilde{g} = \Omega^2 \delta g$, then analyze the fall-off behavior of the unphysical linear field $\delta \tilde{g}$. An essential result in order to relate the symplectic structure and the News tensor is the fact that $\delta \tilde{g}$ vanishes to first order at $\mathcal{I}^+$. This fall-off behavior was previously proven by Geroch and Xanthopoulos in [10] and, as mentioned by Wald and Zoupas, it can be seen as a consequence of the rigidity⁵ of the geometry at $\mathcal{I}^+$. The asymptotic behavior of the linearized fields given by Geroch and Xanthopoulos have also been explored by Ashtekar *et al* [12, 13] to understand the symplectic geometry of radiative modes of general relativity.

Our aim in this paper is to address the above problem in the context of the positive cosmological constant Λ . Our approach will be based on Wald-Zoupas formalism. Namely, by calculating a presymplectic potential at $\mathcal{I}^+$, which in turn determines the symplectic form, we will define a conserved quantity that is a good candidate for gravitational energy flux [9]. The Wald-Zoupas formalism has been used in recent literature [14–20] to understand the gravitational radiation in presence of positive cosmological constant. Several [14, 17–19, 21] carry out the analysis in a particular conformal and coordinate gauge, named after Fefferman and Graham's work [22, 23]. This is the conformal gauge in which $\nabla \Omega$ is a geodesic vector field and Gaussian coordinates $\{\Omega, x^i\}$ adapted to $\Omega = \text{const}$ foliation. In a nutshell, there is an asymptotic formal series expansion in the Fefferman–Graham gauge of the unphysical metric

⁵ Note that this happens only in four spacetime dimensions. For a higher dimensional approach see [11].

$\tilde{g}_{ab} = \Omega^2 g_{ab}$, where g is the physical metric (i.e. solving the Einstein equation). In this gauge, $\tilde{g}$ is written as [24]

$$\tilde{g} = -\ell^2 d\Omega^2 + \tilde{\gamma}_{ab} d\tilde{x}^a d\tilde{x}^b = -\ell^2 d\Omega^2 + \left(\tilde{\gamma}_{ab}^{(0)} + \Omega \tilde{\gamma}_{ab}^{(1)} + \Omega^2 \tilde{\gamma}_{ab}^{(2)} + \Omega^3 \tilde{\gamma}_{ab}^{(3)} + \mathcal{O}(\Omega^4) \right) d\tilde{x}^a d\tilde{x}^b, \quad (1.1)$$

where $\ell^{-2} = \frac{\Lambda}{3}$ and $\tilde{\gamma}_{ab}^{(j)}$ are coefficients⁶ in the Ω -expansion of the spatial metric $\tilde{\gamma}$ ‘near’ $\Omega = 0$. The Λ -vacuum Einstein equation determines a recursive relation for these coefficients and in four dimensions⁷ one obtains [25],

$$\tilde{\gamma}_{ab}^{(1)} = 0, \quad (1.2)$$

$$\tilde{\gamma}_{ab}^{(2)} = \ell^2 \left(\tilde{r}_{ab}^{(0)} - \frac{1}{4} \tilde{r}^{(0)} \tilde{\gamma}_{ab}^{(0)} \right) =: \ell^2 \tilde{s}_{ab}^{(0)}, \quad (1.3)$$

$$\tilde{\gamma}_{ab}^{(3)} = -\frac{2\ell^2}{3} \tilde{e}_{ab}^{(0)}. \quad (1.4)$$

Here $\tilde{\gamma}^{(0)}$ is the induced metric at $\mathcal{I}^+$, $\tilde{r}_{ab}^{(0)}$ and $\tilde{r}^{(0)}$ its Ricci tensor and scalar, $\tilde{s}^{(0)}$ its three-dimensional Schouten tensor, and $\tilde{e}^{(0)}$ is the rescaled electric Weyl tensor of $\tilde{g}$ at the boundary. *All terms in the expansion to infinite order can be written solely in terms of $\tilde{\gamma}^{(0)}$ and $\tilde{e}^{(0)}$.* The pair $(\tilde{\gamma}^{(0)}, \tilde{e}^{(0)})$ can be thus understood as Cauchy data for the spacetime metric specified in the conformal setting at the infinity $\mathcal{I}^+$.

We also note that in Fefferman–Graham gauge, $\tilde{\gamma}_{ab}^{(3)}$ can also be identified with a holographic stress-energy tensor [14, 16, 26, 27]

$$T_{ab} := \frac{2}{|\gamma|^{1/2}} \frac{\delta S}{\delta \gamma_{ab}} = \frac{3}{16\pi G\ell} \tilde{\gamma}_{ab}^{(3)}, \quad (1.5)$$

which is trace-free and divergence-free with respect to the boundary metric $\tilde{\gamma}_{ab}^{(0)}$.

In the Fefferman and Graham gauge, the fall-off behavior of the linear fields is obtained by perturbing the seed data $(\tilde{\gamma}^{(0)}, \tilde{e}^{(0)})$ of the expansion (1.1)

$$\delta \tilde{g}_{ab} \equiv \delta \tilde{\gamma}_{ab} = \delta \tilde{\gamma}_{ab}^{(0)} + \Omega^2 \ell^2 \delta \tilde{s}_{ab}^{(0)} - \frac{2\ell^2}{3} \Omega^3 \delta \tilde{e}_{ab}^{(0)} + \mathcal{O}(\Omega^4). \quad (1.6)$$

The above behavior of the linear fields is general for all conformally extendable metrics. This is a consequence of the strong non-linear stability result that stems from Friedrich’s conformal field equations with positive Λ [28] (see also [29]). By strong non-linear stability we mean that metrics initially close remain close in the long term (wrt suitable Sobolev norms, cf appendix A), and also that the conformal extendibility property is preserved. This implies that the asymptotic data (i.e. initial data at $\mathcal{I}^+$) are also close. Interestingly, the asymptotic data coincide with the degrees of freedom of the Fefferman–Graham expansion, namely $(\tilde{\gamma}^{(0)}, \tilde{e}^{(0)})$. Thus, given a background metric $\tilde{g}$ with data $(\tilde{\gamma}^{(0)}, \tilde{e}^{(0)})$, a perturbed metric $\tilde{g} + \delta \tilde{g}$ has data $(\tilde{\gamma}^{(0)} + \delta \tilde{\gamma}^{(0)}, \tilde{e}^{(0)} + \delta \tilde{e}^{(0)})$, so that (1.6) follows.

The result obtained in four dimensions is that the presymplectic structure is determined by the holographic stress-energy tensor at the boundary [14, 16]. From (1.4), and (1.5), it follows that the holographic stress-energy tensor is related to rescaled electric Weyl tensor

⁶ These coefficients can be understood as tensors on the surface $\Omega = 0$, i.e. at the infinity $\mathcal{I}^+$, and they are exported into spacetime by the time flow, or, equivalently, using the Gaussian coordinates.

⁷ This can be similarly performed in arbitrary higher dimensions [22, 23].

in Fefferman–Graham gauge, $T_{ab} = -\frac{\ell}{8\pi G}\tilde{e}_{ab}$. Therefore, one expects to rewrite the presymplectic structure in terms of the rescaled electric Weyl tensor. Although this implies a remarkable coordinate independent identification of the symplectic form, there is no guarantee that it is a conformal gauge invariant. This is because kinematical quantities (depending on the lapse and shift of the foliation determined by Ω) may appear, which in the Fefferman and Graham gauge vanish. One of the contributions in this paper is to carry out a coordinate and conformal gauge independent definition of the presymplectic potential at $\mathcal{I}^+$. Our result matches previous calculations of the presymplectic potential in the Fefferman and Graham gauge, and additionally shows its conformal gauge invariant nature. Without imposing any gauge, we have shown that the natural extension of the holographic stress-energy tensor via conformal Einstein equation can be obtained, and one can write the presymplectic structure in terms of electric part of the rescaled Weyl tensor. Our approach is novel, and to the best of our knowledge we are not aware of explicit derivation of symplectic structure of linearized field in de Sitter in terms of rescaled electric Weyl tensor without imposing any gauge.

The fall-off behavior in (1.6) is general in the sense that for every conformally extendible physical metric g there exists a Fefferman–Graham gauge such that the linear perturbations of the unphysical metric $\tilde{g} = \Omega^2 g$ satisfy (1.6). Although in a strongly gauge dependent way, this solves the problem of analyzing the fall-off behavior of the linear fields with considerable generality. There exists, however, the above mentioned interesting approach in the $\Lambda = 0$ case by Geroch and Xanthopoulos [10], which would be interesting to extend to the $\Lambda > 0$ case. The idea in [10] (see section 3 for a review) is to regularize the linearized Einstein equation (with $\Lambda = 0$) for $\delta\tilde{g}$ by first guessing the fall-off behavior of its components and rescaling them accordingly. Then, by introducing auxiliary fields and defining equations for them, one obtains a regular hyperbolic partial differential equations (PDE) system in a suitable linear gauge, extending up to $\mathcal{I}^+$. The fall-off of the linear fields follows then by the well-posedness of the hyperbolic system.

What is remarkable from [10] is the relative simplicity of the whole procedure with respect to other approaches such as Friedrich’s conformal field equations [28, 30], (see [31, 32] for exhaustive review.) In essence, Friedrich framework also consists in supplementing the Einstein equation with additional fields and equations, but the amount of auxiliary equations and fields is bigger than in [10]. Note that, as a counterpart, Friedrich equations work in the full non-linear regime. Linearizing Friedrich’s equations is also a possible approach for the problem of determining the decay of the linear fields which, however, will not be considered here. For a linearized treatment of these equations in the $\Lambda = 0$ case and their application to gravitational radiation theory, we refer the reader to [33].

We wish to investigate the possibility of extending Geroch and Xanthopoulos results [10] to $\Lambda > 0$ setting. We find that a major difference between $\Lambda = 0$ and $\Lambda > 0$ is that in the first case, linear fields can be generally chosen (in a suitable gauge) to always vanish to the first order at $\mathcal{I}^+$, namely $\delta\tilde{g} = O(\Omega)$, while this is not the case for $\Lambda > 0$. This appears to be an obstacle to the regularization of the field equation, which seems to prevent the construction of hyperbolic PDE system in the spirit of [10].

Over the last few years, the study of gravitational waves in presence of positive cosmological constant has gained lot of attentions. In the context of gravitational wave generation from spatially compact sources, quadrupolar truncated linearized solution of gravitational fields around de Sitter background have been obtained in a generalized harmonic gauge [34–39]. For the linearized quadrupolar solution, quadratic flux-balance laws associated with de Sitter isometries have also been obtained [40]. By now, Bondi-Sachs formalism for gravitational radiation in de Sitter is also well understood [17, 21, 41–44]. Bondi-Sachs formalism also plays a crucial role in understanding asymptotic symmetries and memory effect in de Sitter

[34, 45–49]. Recently, there has also been work [50–53] related to the computation of symplectic potential in partial Bondi gauge (the leading order fall-off for gravitational fields are more generic than Bondi gauge) with non-vanishing cosmological constant. There are also attempts [54–58] to understand gravitational radiation with a positive cosmological constant in Penrose’s conformal completion technique. Our present work contributes in the direction of understanding gravitational radiation with a positive cosmological constant in conformal completion formalism. In particular, we extract the fall-off behavior of gravitational fields from conformal Einstein equation and show that these fall-off behavior induces non-zero gravitational flux in the future infinity of de Sitter.

The paper is organized as follows. In section 2 we analyze, in an arbitrary conformal gauge, the fall-off behavior of the linear fields with positive Λ directly from the Einstein equation. In section 3, we study the differences with respect to the $\Lambda = 0$ case. Section 4 revisits the infinitesimal diffeomorphisms group from a conformal perspective. Then, in section 5 we study where the gauge degrees of freedom are localized within the linear fields, which aims to help understanding why the gauge (1.6) is actually general. We analyze the asymptotic behavior for background gravitational fields in section 6. We conclude the paper with section 7, where we apply our previous analysis to compute the presymplectic potential in a gauge-independent manner. Relevant mathematical tools are given in the appendices.

Notation and conventions

Throughout this paper, we shall consider 4-dimensional manifold (M, g) . The spacetime metric, as well as their related objects, will be referred to as ‘physical’ and always denoted without a tilde. The formalism employed to study asymptotic properties will be that of Penrose’s conformal completions. Namely, we consider a smooth positive function Ω nowhere vanishing on M , in such a way that the boundary $\mathcal{S} := \partial M = \{\Omega = 0\}$ must satisfy $d\Omega|_{\mathcal{S}} \neq 0$. We define the conformally rescaled metric

$$\tilde{g}_{ab} = \Omega^2 g_{ab}, \quad (1.7)$$

which we require to be smooth on $\tilde{M} := M \cup \partial M$. The manifold $(\tilde{M}, \tilde{g})$ is called a conformal extension of (M, g) or simply the ‘unphysical manifold’. All unphysical objects are denoted with a tilde, and their indices are raised or lowered by $\tilde{g}$.

We will often make use of the foliation defined on $\tilde{M}$ with $\{\Omega = \text{const}\}$ leaves. Therefore, we define the following objects:

$$\tilde{\nu}_a := \tilde{\nabla}_a \Omega, \quad -\tilde{N}^{-2} := \tilde{g}^{cd} \tilde{\nu}_c \tilde{\nu}_d, \quad \tilde{n}_a := \tilde{N} \tilde{\nu}_a, \quad (1.8)$$

where $\tilde{N}$ is the lapse function, which we define by the positive root of $\tilde{N}^2$.

We now have the following decomposition of the metric $\tilde{g}$

$$\tilde{g}_{ab} = -\tilde{n}_a \tilde{n}_b + \tilde{\gamma}_{ab}, \quad (1.9)$$

where $\tilde{\gamma}_{ab}$ is the induced metric at the leaves $\{\Omega = \text{const}\}$, with associated Levi-Civita covariant derivative $\tilde{D}$. We introduce also

$$\tilde{K}_{ab} := \tilde{\nabla}_a \tilde{n}_b, \quad \tilde{k}_{ab} := \tilde{\gamma}^c{}_a \tilde{\gamma}^d{}_b \tilde{K}_{cd}, \quad (1.10)$$

$\tilde{k}_{ab}$ being the extrinsic curvature of the leaves $\{\Omega = \text{const}\}$. Note that typically we assume $\Omega > 0$ in the physical spacetime M . It implies that $\tilde{\nu}_a$ and $\tilde{n}_a$ are inside oriented normal one-forms, i.e. the past-oriented at the future infinity and the future-oriented at the past infinity, while $\tilde{\nu}^a$ and $\tilde{n}^a$ are outward oriented normal vectors.

We focus on the positive cosmological constant $\Lambda > 0$ and we introduce the corresponding cosmological length ℓ ,

$$\ell^{-2} = \frac{\Lambda}{3}. \quad (1.11)$$

2. Fall-off analysis of linearized field in presence of cosmological constant

In this section, we wish to extract a fall-off condition for linearized gravitational fields in presence of positive cosmological constant. For this purpose, we will follow the method of Geroch and Xanthopoulos [10], originally developed for $\Lambda = 0$. Their approach (see section 3 for a review) regularizes the linearized conformal Einstein equations by first guessing the fall-off behavior of the field components *a priori* and rescaling them accordingly. Then, by introducing auxiliary fields and imposing suitable gauge conditions, they obtain a well-posed hyperbolic PDE system that extends smoothly to $\mathcal{I}$. Given the success of this method for $\Lambda = 0$, it is worth exploring its applicability to $\Lambda > 0$.

We will only focus on extracting the fall-off condition of the linearized fields from the conformal linearized Einstein field equation, and also explore whether the field variables produce a finite symplectic structure at the boundary, $\Omega = 0$. The well-posedness of the system of equations for a positive cosmological constant is beyond the scope of our paper.

Assume now that the physical metric g_{ab} satisfies the Einstein equation for some stress-energy tensor T_{ab} ,

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi GT_{ab}. \quad (2.1)$$

It is convenient and customary to write down Einstein equation in terms of unphysical Schouten tensor,

$$\tilde{S}_{ab} := \tilde{R}_{ab} - \frac{1}{6}\tilde{R}\tilde{g}_{ab}. \quad (2.2)$$

Using the conformal transformation between physical and unphysical Ricci tensor we obtain,

$$\tilde{S}_{ab} = \left(R_{ab} - \frac{1}{6}Rg_{ab} \right) - 2\Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}_b\Omega + \Omega^{-2}\tilde{g}_{ab}\tilde{g}^{cd}\tilde{\nabla}_c\Omega\tilde{\nabla}_d\Omega. \quad (2.3)$$

Substituting for the curvature terms from the Einstein equation, we get

$$\Omega\tilde{S}_{ab} + 2\tilde{\nabla}_a\tilde{\nabla}_b\Omega - \Omega^{-1}\left(\tilde{g}^{cd}\tilde{\nabla}_c\Omega\tilde{\nabla}_d\Omega + \ell^{-2}\right)\tilde{g}_{ab} = 8\pi G\Omega^{-1}\left(\tilde{T}_{ab} - \frac{1}{3}\tilde{T}\tilde{g}_{ab}\right), \quad (2.4)$$

where we define $\tilde{T}_{ab} := \Omega^2 T_{ab}$.

Since $\tilde{g}_{ab}$ and Ω are smooth on $\tilde{M}$, the first two terms in the left-hand side of (2.4) are smooth on $\mathcal{I}$. If the stress-energy tensor on the right-hand side vanishes asymptotically to order one (i.e. $\Omega^{-1}\tilde{T}_{ab} \equiv \Omega T_{ab}$ has a smooth limit on $\mathcal{I}$), the remaining term on the left-hand side of (2.4) must be also smooth on $\mathcal{I}$. We denote it as $\tilde{f}$,

$$\tilde{f} = \Omega^{-1}\left(\tilde{g}^{ab}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega + \ell^{-2}\right) = \Omega^{-1}\left(\tilde{g}^{ab}\tilde{\nu}_a\tilde{\nu}_b + \ell^{-2}\right). \quad (2.5)$$

Hence, the requirement of regularity for the most divergent term in (2.4) implies the asymptotic behavior of the lapse function (cf equation (1.8)),

$$\tilde{N}^{-2} = \ell^{-2} - \Omega\tilde{f}. \quad (2.6)$$

Although the analysis could be continued with a non-vanishing stress-energy tensor satisfying suitable fall-off conditions, we will consider only a vacuum case and assume $T_{ab} = 0$ throughout the paper. Then (2.4) becomes,

$$\Omega \tilde{\nabla}_{ab} + 2 \tilde{\nabla}_a \tilde{\nabla}_b \Omega - \tilde{f} \tilde{g}_{ab} = 0. \quad (2.7)$$

For convenience, we also write down $\Lambda > 0$ -vacuum Einstein equation in terms of unphysical Ricci tensor,

$$\tilde{R}_{ab} = -\frac{2}{\Omega} \tilde{\nabla}_a \tilde{\nabla}_b \Omega - \frac{\tilde{g}^{cd} \tilde{\nabla}_c \tilde{\nabla}_d \Omega}{\Omega} \tilde{g}_{ab} + \frac{3}{\Omega^2} \left(\tilde{g}^{cd} \tilde{\nabla}_c \Omega \tilde{\nabla}_d \Omega + \ell^{-2} \right) \tilde{g}_{ab}. \quad (2.8)$$

We shall refer to (2.8) as conformal Einstein field equations.

Throughout our paper, we consider that $\tilde{f}$ has, in general, a smooth non-vanishing contribution on the boundary, i.e. $\tilde{f} \sim \mathcal{O}(1)$. However, we note that the conformal Einstein equation (2.7) is invariant under a rescaling freedom, $\Omega \rightarrow \omega \Omega$ and $\tilde{g} \rightarrow \omega^2 \tilde{g}$. This rescaling of the conformal factor is a gauge freedom. One can use this gauge freedom to set $\tilde{\nabla}_a \tilde{\nu}_b = 0$ on $\mathcal{I}$ [8, 55, 59]. This implies $\tilde{f} \sim \mathcal{O}(\Omega)$, and, as a consequence, $\tilde{N}^{-2} = \ell^{-2} + \mathcal{O}(\Omega^2)$.

To introduce the perturbation, we consider a one-parameter family of physical metrics, $g_{ab}(\epsilon)$ which is differentiable with respect to ϵ at $\epsilon = 0$. The first-order perturbation of the physical metric is given by $\delta g_{ab} := \frac{dg_{ab}(\epsilon)}{d\epsilon}|_{\epsilon=0}$. We assume that the conformal rescaling is independent of the perturbation procedure, i.e. the foliation $\Omega = \text{const}$ does not change, and $\delta \Omega = 0$, $\delta \nu_a = 0$. The linearized Einstein equation for δg_{ab} , expressed in terms of corresponding unphysical perturbation $\delta \tilde{g}_{ab} = \Omega^2 \delta g_{ab}$, is given by

$$\begin{aligned} \square \delta \tilde{g}_{ab} = & 2 \tilde{\nabla}_{(a} \tilde{\nabla}^m \delta \tilde{g}_{b)m} - \tilde{\nabla}_a \tilde{\nabla}_b \delta \tilde{g}_m^m + 2 \tilde{R}^m{}_a{}^n{}_b \delta \tilde{g}_{mn} + 2 \tilde{R}_{(a}{}^m \delta \tilde{g}_{b)m} + \tilde{R}^{mn} \delta \tilde{g}_{mn} \tilde{g}_{ab} \\ & - \frac{\tilde{R}}{3} \left(\delta \tilde{g}_{ab} + \frac{1}{2} \delta \tilde{g}_m^m \tilde{g}_{ab} \right) - 2 \Omega^{-1} \tilde{\nu}^m \left(2 \tilde{\nabla}_{(a} \delta \tilde{g}_{b)m} - \tilde{\nabla}_m \delta \tilde{g}_{ab} \right) \\ & - \Omega^{-1} \left(2 \tilde{\nu}^m \tilde{\nabla}^n \delta \tilde{g}_{mn} - \tilde{\nu}^m \tilde{\nabla}_m \delta \tilde{g}_n^n \right) \tilde{g}_{ab} \\ & - 2 \Omega^{-2} (\ell^{-2} + \tilde{\nu}_c \tilde{\nu}^c) \left(\delta \tilde{g}_{ab} + \frac{1}{2} \delta \tilde{g}_m^m \tilde{g}_{ab} \right) + 6 \Omega^{-2} \tilde{g}_{ab} \delta \tilde{g}_{cd} \tilde{\nu}^c \tilde{\nu}^d, \end{aligned} \quad (2.9)$$

where $\delta \tilde{g}_m^m := \delta \tilde{g}_{ma} \tilde{g}^{ma}$ ⁸.

We will assume a generic fall-off for gravitational perturbation as,

$$\delta \tilde{g}_{ab} = \Omega^\alpha \tilde{\tau}_{ab}, \quad \tilde{\tau}_{ab} \tilde{\nu}^b = \Omega^\beta \tilde{\tau}_a, \quad \tilde{\tau}_a \tilde{\nu}^a = \Omega^\gamma \tilde{\tau}, \quad (2.10)$$

with $\alpha \geq 0, \beta \geq 0, \gamma \geq 0$, and shall denote the trace of the $\tilde{\tau}_{ab}$ field as,

$$\tilde{\phi} := \tilde{g}^{ab} \tilde{\tau}_{ab}. \quad (2.11)$$

The fall-off condition introduced in (2.10), can also be written as

$$\delta \tilde{g}_{ab} = \Omega^\alpha \tilde{\tau}_{ab}, \quad \delta \tilde{g}_{ab} \tilde{\nu}^b = \Omega^{\alpha+\beta} \tilde{\tau}_a, \quad \delta \tilde{g}_{ab} \tilde{\nu}^a \tilde{\nu}^b = \Omega^{\alpha+\beta+\gamma} \tilde{\tau}. \quad (2.12)$$

This choice of ansatz is inspired by the analysis of Geroch and Xanthopoulos [10]. For clarity, we present the decomposition of the perturbations of the metric (1.9),

$$\delta \tilde{g} = \begin{bmatrix} \tilde{N}^{-4} \Omega^{\alpha+\beta+\gamma} \tilde{\tau} & -\tilde{N}^{-2} \Omega^{\alpha+\beta} \tilde{\tau}_i \\ -\tilde{N}^{-2} \Omega^{\alpha+\beta} \tilde{\tau}_i & \Omega^\alpha \tilde{\tau}_{ij} \end{bmatrix}. \quad (2.13)$$

⁸ Beware that we also use the standard convention $\delta \tilde{g}^{ab} \equiv \delta g^{-1ab} = -g^{ac} g^{bd} \delta g_{cd}$ which is however in a conflict with the standard convention for raising indices.

We assume that $\tilde{\tau}_{ab}, \tilde{\tau}_a, \tilde{\tau}$ are regular⁹ of the order $\mathcal{O}(1)$ on the conformal boundary $\mathcal{I}$. Our goal is to find a suitable choice of α, β, γ so that the linearized conformal Einstein equation with a positive cosmological constant becomes regular. In terms of these variables, the linearized Einstein equation becomes,

$$\begin{aligned} \tilde{\square}\tilde{\tau}_{ab} = & 2\tilde{\nabla}_{(a}\tilde{\nabla}^m\tilde{\tau}_{b)m} - 2\tilde{R}_{ambn}\tilde{\tau}^{mn} - \frac{\alpha\tilde{\phi}}{2}\tilde{R}_{ab} - \frac{\alpha\tilde{\phi}}{12}\tilde{R}\tilde{g}_{ab} - \tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} + \frac{\alpha}{6}\tilde{R}\tilde{\tau}_{ab} \\ & + \Omega^{-2}\left(\ell^{-2}\alpha(\alpha-3)\tilde{\tau}_{ab} - \alpha(\alpha-1)\tilde{\phi}\tilde{\nu}_a\tilde{\nu}_b - \ell^{-2}\alpha\tilde{\phi}\tilde{g}_{ab}\right) \\ & + \Omega^{-1}\left(2\alpha\tilde{\nu}_{(a}\tilde{\nabla}^m\tilde{\tau}_{b)m} - 2\alpha\tilde{\nu}_{(a}\tilde{\nabla}_{b)}\tilde{\phi} - \alpha(\alpha-1)\tilde{f}\tilde{\tau}_{ab} + 2(1-\alpha)\tilde{\nu}^m\tilde{\nabla}_m\tilde{\tau}_{ab} + \tilde{g}_{ab}\tilde{\nu}^m\tilde{\nabla}_m\tilde{\phi}\right) \\ & + 2\Omega^{\beta-2}\left[(\alpha+\beta)(\alpha-2) - \alpha\right]\tilde{\nu}_{(a}\tilde{\tau}_{b)} + 2\Omega^{\beta-1}\left[(\alpha-2)\tilde{\nabla}_{(a}\tilde{\tau}_{b)} - \tilde{g}_{ab}\tilde{\nabla}^m\tilde{\tau}_m\right] \\ & + 2\Omega^{\beta+\gamma-2}\left[3 - (\alpha+\beta)\right]\tilde{g}_{ab}\tilde{\tau}. \end{aligned} \quad (2.14)$$

We now attempt to regularize equation (2.14). By ‘regularize’ we mean a choice of parameters α, β, γ such that factors multiplying the negative powers of Ω in (2.14) cancel out or can be gauged away, so that the resulting equation is regular at $\{\Omega = 0\}$. Note that one can always divide by the most negative power of Ω and get a regular equation at $\{\Omega = 0\}$, but then the principal symbol of the equation vanishes at that hypersurface and therefore fails to be hyperbolic. For $\Lambda = 0$ such choice was shown by Geroch and Xanthopoulos [10] to be $\alpha = 1 = \beta, \gamma = 0$ (we review this in section 3). As we shall next see, for non-zero Λ this ansatz does not make the leading order divergent term (of order Ω^{-2}) vanish. Moreover, this term contains $\tilde{\tau}_{ab}$ and therefore there is no chance that this can be gauged away. This is a remarkable difference entirely due to the presence of a cosmological constant.

Let us start by examining the leading order divergent term at $\Omega = 0$ for non-zero Λ . From equation (2.14) this is

$$\begin{aligned} \Omega^{-2}\left(\ell^{-2}\alpha(\alpha-3)\tilde{\tau}_{ab} - \alpha(\alpha-1)\tilde{\phi}\tilde{\nu}_a\tilde{\nu}_b - \ell^{-2}\alpha\tilde{\phi}\tilde{g}_{ab}\right) \\ + 2\Omega^{\beta-2}\left((\alpha+\beta)(\alpha-2) - \alpha\right)\tilde{\nu}_{(a}\tilde{\tau}_{b)} + 2\Omega^{\beta+\gamma-2}\left(3 - (\alpha+\beta)\right)\tilde{g}_{ab}\tilde{\tau}, \end{aligned} \quad (2.15)$$

where $\Omega^{\beta-2}$ and $\Omega^{\beta+\gamma-2}$ terms contribute only if $\beta=0$ and $\beta=\gamma=0$, respectively. We need (2.15) to vanish or at least, be such that it can be gauged away. Thus, it cannot contain the complete field $\tilde{\tau}_{ab}$, but there may only appear certain components, such as a trace or a divergence of it. Also note that the term $\tilde{\tau}_{ab}$ cannot be removed using the terms of $\Omega^{\beta-2}$ and $\Omega^{\beta+\gamma-2}$ either. Thus, in the case of non-zero Λ one is forced to choose $\alpha=0$ ¹⁰. This is a central difference between the zero and non-zero Λ cases.

Therefore, the highest order divergent terms Ω^{-2} vanishes for $\alpha=0$. A possible choice for the regularization of the Ω^{-1} term in (2.14) is $\alpha=0, \beta=0, \gamma=1$. The regularity condition then reads

$$2\tilde{\nu}^m\tilde{\nabla}_m\tilde{\tau}_{ab} + \tilde{g}_{ab}\tilde{\nu}^m\tilde{\nabla}_m\tilde{\phi} - 4\tilde{\nabla}_{(a}\tilde{\tau}_{b)} - 2\tilde{g}_{ab}\tilde{\nabla}^m\tilde{\tau}_m + 6\tilde{g}_{ab}\tilde{\tau} = \mathcal{O}(\Omega). \quad (2.16)$$

⁹ From the perspective of PDE theory, the regularity condition (interpreted as smoothness of the fields) can, in principle, be relaxed to require only an optimal degree of differentiability. However, we will not explore this issue in detail and will simply assume that all fields involved are sufficiently differentiable.

¹⁰ The case $\alpha=3$ faces problems at Ω^{-1} order and leads to vanishing symplectic structure. Therefore we do not consider this option here.

Taking the trace of this equation, we obtain

$$\tilde{\nu}^m \tilde{\nabla}_m \tilde{\phi} - 2 \tilde{\nabla}^m \tilde{\tau}_m + 4 \tilde{\tau} = \mathcal{O}(\Omega). \quad (2.17)$$

Subtracting (2.17) from (2.16) multiplied by $\tilde{g}_{ab}$, gives

$$-2 \tilde{\nabla}_{(a} \tilde{\tau}_{b)} + \tilde{\nu}^m \tilde{\nabla}_m \tilde{\tau}_{ab} + \tilde{g}_{ab} \tilde{\tau} = \mathcal{O}(\Omega). \quad (2.18)$$

Note that this condition is fully equivalent to (2.16). Similar to the discussion of the Ω^{-2} order above, this term has no chance to be canceled out by a choice of gauge, because that would imply a too strong condition on $\tilde{\nu}^m \tilde{\nabla}_m \tilde{\tau}_{ab}$. Thus we must require that $\tilde{\tau}_a$ satisfies the constraint

$$-2 \tilde{\nabla}_{(a} \tilde{\tau}_{b)} + \tilde{\nu}^m \tilde{\nabla}_m \tilde{\tau}_{ab} + \tilde{g}_{ab} \tilde{\tau} = \Omega \tilde{A}_{ab} \quad (2.19)$$

with some regular field $\tilde{A}_{ab}$. Equation (2.14) then reads

$$\tilde{\square} \tilde{\tau}_{ab} = 2 \tilde{\nabla}_{(a} \tilde{\nabla}^m \tilde{\tau}_{b)m} - 2 \tilde{R}_{ambn} \tilde{\tau}^{mn} - \tilde{\nabla}_a \tilde{\nabla}_b \tilde{\phi} + \tilde{A}_{ab}. \quad (2.20)$$

We now have a smooth equation at $\Omega = 0$. However, the system is incomplete as there is no dynamical equation for $\tilde{A}_{ab}$. In this situation, one needs to check if the core equation (2.20) propagates the field $\tilde{A}_{ab}$ too. There is no well-defined prescription on how to do this, although this usually involves taking derivatives of (2.20) and using again definition (2.19) and (2.20) to try to obtain a new propagation equation for $\tilde{A}_{ab}$. Unfortunately, this seems to introduce new divergent terms that must be eliminated by defining additional auxiliary fields, leading to an infinite process of solving the equations order-by-order in Ω .

In summary, we have found that the choice of parameters $\alpha = \beta = 0$ and $\gamma = 1$ leads to a constraint of the form (2.18). Such a constraint does not appear to be removable by any other choice of parameters. Since the choice $\alpha = \beta = 0$ and $\gamma = 1$ is minimal in the sense that it includes all other cases as subcases, we shall refer to this condition as the minimal fall-off behavior of the linearized fields in de Sitter spacetimes.

If we wish to extend this fall-off behavior up to the conformal boundary $\mathcal{I}$, additional results are required to guarantee the extendibility of linear fields to $\mathcal{I}$. As the well-posedness of the full system of equations lies beyond the scope of this paper, we instead rely on the nonlinear stability results derived from Friedrich's conformal field equations with $\Lambda > 0$ [28]. This implies linear stability of the fields and thus extendibility of the linear fields up to $\mathcal{I}$ (see section A.2 of appendix A). With this result at hand, the condition $\alpha = \beta = 0$, $\gamma = 1$ does indeed yield the general fall-off behavior of linear fields with $\Lambda > 0$. In the remainder of this paper, we will assume the minimal fall-off behavior and examine its implications.

We also emphasize that the regularization condition in (2.18) can also be understood as the regularization of $\tilde{f} = \Omega^{-1}(\tilde{g}^{cd} \tilde{\nu}_c \tilde{\nu}_d + \ell^{-2})$ under the first order perturbation. Indeed, thanks to $\delta\Omega = 0 = \delta\tilde{\nu}_a$ we obtain

$$-\delta \tilde{g}_{cd} \tilde{\nu}^c \tilde{\nu}^d = \Omega \delta \tilde{f}. \quad (2.21)$$

Hence $\delta \tilde{g}_{ab} \tilde{\nu}^a \tilde{\nu}^b = \mathcal{O}(\Omega)$, which corresponds to $\alpha + \beta + \gamma = 1$. To match with the notation in (2.10), we obtain $\delta \tilde{f} = -\tilde{\tau}$. From (2.7), we also have

$$\tilde{\nabla}_a \tilde{\nu}_b|_{\mathcal{I}} = \frac{\tilde{f}}{2} \tilde{g}_{ab}. \quad (2.22)$$

Under the linearized perturbation, this equation transforms as

$$\begin{aligned} -\frac{\tilde{\nu}^c}{2} \left(\tilde{\nabla}_a \delta \tilde{g}_{bc} + \tilde{\nabla}_b \delta \tilde{g}_{ac} - \tilde{\nabla}_c \delta \tilde{g}_{ab} \right) &= \frac{\delta \tilde{f}}{2} \tilde{g}_{ab} + \frac{\tilde{f}}{2} \delta \tilde{g}_{ab} + \mathcal{O}(\Omega), \\ \implies \tilde{\nabla}_a (\tilde{\nu}^c \delta \tilde{g}_{bc}) + \tilde{\nabla}_b (\tilde{\nu}^c \delta \tilde{g}_{ac}) - \tilde{\nu}^c \tilde{\nabla}_c \delta \tilde{g}_{ab} &= -\delta \tilde{f} \tilde{g}_{ab} + \mathcal{O}(\Omega). \end{aligned} \quad (2.23)$$

With the ‘minimal’ fall-off condition, $\alpha = 0 = \beta, \gamma = 1$, (2.23) reduces to (2.18). We emphasize that this constraint equation is trivially satisfied for Geroch and Xanthopoulos’ anstaz $\alpha = 1 = \beta, \gamma = 0$, in the context of asymptotically flat space-times.

3. Regularized conformal Einstein equation for vanishing cosmological constant

We next review the regularization of (2.14) carried out by Geroch and Xanthopoulos for the $\Lambda = 0$ case in [10]. *A priori*, $\Lambda = 0$ case of the equation (2.14) is not clear. For this case it is convenient to write down (2.14) in a different form,

$$\begin{aligned} \tilde{\square} \tilde{\tau}_{ab} &= 2\tilde{\nabla}_{(a} \tilde{\nabla}^m \tilde{\tau}_{b)m} - 2\tilde{R}_{ambn} \tilde{\tau}^{mn} + \tilde{R}^{mn} \tilde{\tau}_{mn} \tilde{g}_{ab} + \frac{\alpha \tilde{\phi}}{2} \tilde{R}_{ab} - \frac{\tilde{R}}{6} \tilde{\phi} \tilde{g}_{ab} \left(1 + \frac{\alpha}{2} \right) - \tilde{\nabla}_a \tilde{\nabla}_b \tilde{\phi} + \frac{\alpha}{6} \tilde{R} \tilde{\tau}_{ab} \\ &+ \Omega^{-2} \alpha (1 - \alpha) \tilde{\phi} \tilde{\nu}_a \tilde{\nu}_b + \Omega^{-1} \left(2\alpha \tilde{\nu}_{(a} \tilde{\nabla}^m \tilde{\tau}_{b)m} - 2\alpha \tilde{\nu}_{(a} \tilde{\nabla}_{b)} \tilde{\phi} - \alpha (\alpha - 1) \tilde{f} \tilde{\tau}_{ab} \right. \\ &+ 2(1 - \alpha) \tilde{\nu}^m \tilde{\nabla}_m \tilde{\tau}_{ab} + \tilde{g}_{ab} \tilde{\nu}^m \tilde{\nabla}_m \tilde{\phi} - 2\tilde{g}_{ab} \tilde{\nu}^m \tilde{\nabla}^n \tilde{\tau}_{mn} - \tilde{\phi} \tilde{f} \tilde{g}_{ab} \left(1 - \frac{\alpha}{2} \right) \Big) \\ &+ 2\Omega^{\beta-2} [(\alpha + \beta)(\alpha - 2) - \alpha] \tilde{\nu}_{(a} \tilde{\tau}_{b)} + 2\Omega^{\beta-1} [(\alpha - 2) \tilde{\nabla}_{(a} \tilde{\tau}_{b)}] + 2\Omega^{\beta+\gamma-2} (3 - \alpha) \tilde{g}_{ab} \tilde{\tau}. \end{aligned} \quad (3.1)$$

In deriving equation (3.1), we have used the identity

$$\Omega^{-1} \tilde{\nabla}_a \tilde{\nu}_b = -\frac{\tilde{R}_{ab}}{2} + \frac{1}{12} \tilde{R} \tilde{g}_{ab} + \Omega^{-1} \frac{\tilde{f}}{2} \tilde{g}_{ab}, \quad (3.2)$$

which follows by writing the trace components of (2.8) in terms of R . The suitable choice of parameters for $\Lambda = 0$ made in [10] is $\alpha = 1 = \beta$, and $\gamma = 0$. Then (3.1) becomes

$$\begin{aligned} \tilde{\square} \tilde{\tau}_{ab} &= 2\tilde{\nabla}_{(a} \tilde{\nabla}_{b)} \tilde{\tau} - 2\tilde{R}_{ambn} \tilde{\tau}^{mn} + \tilde{R}^{mn} \tilde{\tau}_{mn} \tilde{g}_{ab} + \frac{\tilde{R}}{6} \tilde{\tau}_{ab} + \tilde{\nabla}_a \tilde{\nabla}_b \tilde{\phi} - \frac{\tilde{R}}{4} \tilde{\phi} \tilde{g}_{ab} + \frac{\tilde{\phi}}{2} \tilde{R}_{ab} + 4\tilde{\nabla}_{(a} \tilde{\tau}_{b)} \\ &+ 2\Omega^{-1} \left(\tilde{\nu}_{(a} \tilde{\tau}_{b)} - \tilde{g}_{ab} \tilde{\nu}^m \tilde{\tau}_m - \tilde{g}_{ab} \left(\frac{1}{2} \tilde{\nu}^c \tilde{\nabla}_c \tilde{\phi} + \tilde{\nu}^c \tilde{\tau}_c + \frac{\tilde{f}}{4} \tilde{\phi} \right) \right), \end{aligned} \quad (3.3)$$

where we have used the definition

$$\tilde{y}_a := \tilde{\nabla}^m \tilde{\tau}_{ma} - \tilde{\nabla}_a \tilde{\phi} - 3\tilde{\tau}_a. \quad (3.4)$$

Introducing the auxiliary field variable

$$\tilde{\sigma} := \Omega^{-1} \left(\tilde{n}^a \tilde{\tau}_a + \frac{1}{2} \tilde{n}^a \tilde{\nabla}_a \tilde{\phi} + \frac{1}{4} \tilde{f} \tilde{\phi} \right), \quad (3.5)$$

and using the identity between Riemann tensor and Weyl tensor, we rewrite equation (3.3) as

$$\begin{aligned}\tilde{\square}\tilde{\tau}_{ab} = & 2\tilde{\nabla}_{(a}\tilde{y}_{b)} + \tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} + 4\tilde{\nabla}_{(a}\tilde{\tau}_{b)} - 2\tilde{C}_{ambn}\tilde{\tau}^{mn} - \frac{1}{6}\tilde{R}\tilde{\tau}_{ab} + \frac{1}{12}\tilde{R}\tilde{\phi}\tilde{g}_{ab} \\ & - \frac{1}{2}\tilde{R}_{ab}\tilde{\phi} + 2\tilde{\tau}_{(a}\tilde{R}_{b)m} - 2\tilde{\sigma}\tilde{g}_{ab} + 2\Omega^{-1}(\tilde{\nu}_{(a}\tilde{y}_{b)} - \tilde{g}_{ab}\tilde{\nu}^m\tilde{y}_m)\end{aligned}\quad (3.6)$$

For $\Lambda = 0$ case, we can choose the following gauge conditions (see [10])

$$\tilde{y}_a := \tilde{\nabla}^b\tilde{\tau}_{ab} - \tilde{\nabla}_a\tilde{\phi} - 3\tilde{\tau}_a = 0, \quad (3.7)$$

$$\left(\tilde{n}^a\tilde{\nabla}_a + \frac{1}{6}\Omega\tilde{R} + \frac{3}{2}\tilde{f}\right)\tilde{\nabla}^2\tilde{\phi} = \frac{1}{12}\tilde{R}\tilde{f}\tilde{\phi} - \frac{1}{2}\tilde{\phi}\tilde{\nabla}^2\tilde{f} - \frac{1}{3}\tilde{R}\tilde{n}^a\tilde{\tau}_a + 4\Omega^{-1}\tilde{C}_{ambn}\tilde{\tau}^{mn}\tilde{n}^a\tilde{n}^b. \quad (3.8)$$

Therefore, from (3.7), equation (3.6) reduces to

$$\tilde{\square}\tilde{\tau}_{ab} = \tilde{\nabla}_a\tilde{\nabla}_b\tilde{\phi} + 4\tilde{\nabla}_{(a}\tilde{\tau}_{b)} - 2\tilde{C}_{ambn}\tilde{\tau}^{mn} - \frac{1}{6}\tilde{R}\tilde{\tau}_{ab} + \frac{1}{12}\tilde{R}\tilde{\phi}\tilde{g}_{ab} - \frac{1}{2}\tilde{R}_{ab}\tilde{\phi} + 2\tilde{\tau}_{(a}\tilde{R}_{b)m} - 2\tilde{\sigma}\tilde{g}_{ab}. \quad (3.9)$$

We have now removed all divergent terms, but we still need to introduce some dynamical equation for the other components of the fields σ and $\tilde{\tau}_a$. Contracting (3.9) with $\tilde{\nu}^b$ leads (after doing some algebra) to

$$\begin{aligned}\tilde{\square}\tilde{\tau}_a = & 2\tilde{\nabla}_a\tilde{\sigma} + \frac{1}{2}\tilde{R}_{am}\tilde{\nabla}^m\tilde{\phi} + \frac{1}{12}\tilde{R}\tilde{\nabla}_a\tilde{\phi} - \tilde{R}^{mn}\tilde{\nabla}_m\tilde{\tau}_{an} - \frac{1}{3}\tilde{\tau}_{ab}\tilde{\nabla}^b\tilde{R} \\ & + 2\tilde{\tau}^{mn}\tilde{\nabla}_{[m}\tilde{R}_{a]n} + 2\tilde{\tau}^m\tilde{R}_{ma} + \frac{1}{2}\tilde{R}\tilde{\tau}_a + \frac{1}{2}\tilde{\phi}\tilde{\nabla}_a\tilde{R}.\end{aligned}\quad (3.10)$$

Similarly, a second contraction with $\tilde{\nu}^a$, together with gauge condition (3.8) gives

$$\begin{aligned}\tilde{\square}\tilde{\sigma} = & -\frac{1}{2}\tilde{R}^{mn}\tilde{\nabla}_m\tilde{\nabla}_n\tilde{\phi} - 2\tilde{R}^{mn}\tilde{\nabla}_m\tilde{\tau}_n - \frac{1}{12}(\tilde{\nabla}_m\tilde{R})(\tilde{\nabla}^m\tilde{\phi}) + \tilde{R}\tilde{\sigma} \\ & + \frac{1}{72}\tilde{R}^2\tilde{\phi} - \frac{1}{2}\tilde{\tau}_{ab}\tilde{R}^{ab} - \frac{1}{3}\tilde{\tau}^m\tilde{\nabla}_m\tilde{R}.\end{aligned}\quad (3.11)$$

The system of equations (3.9)–(3.11), supplemented with the gauge conditions (3.7), (3.8) with some additional variables satisfying wave equations and transport equations, turns out to give a hyperbolic PDE system [10]. A higher (even) dimensional generalization of this formalism with different choice of gauge and field variables is also available in [11].

4. Gauge transformation of perturbation

Linearized perturbations are defined up to gauge transformations corresponding to infinitesimal diffeomorphisms generated by a vector field ξ^a

$$\delta g_{ab} \mapsto \delta g'_{ab} = \delta g_{ab} + \mathcal{L}_\xi g_{ab} = \delta g_{ab} + 2\nabla_{(a}\xi_{b)}, \quad (4.1)$$

where $\mathcal{L}_\xi$ stands for the Lie derivative with respect to ξ^a and $\xi_a = g_{ab}\xi^b$. Consider the generator of an infinitesimal diffeomorphism of the form

$$\xi^a = \zeta_\perp n^a + \zeta^a \quad (4.2)$$

where $\zeta_\perp$ is a scalar field and ζ^a satisfies $\zeta^a = \xi^b\gamma^a_b$, both $\mathcal{O}(1)$ with respect to Ω . For linear fields δg_{ab} admitting a conformal extension, $\delta\tilde{g}_{ab} = \Omega^2\delta g_{ab}$, adding a gauge transformation

may spoil the conformal extendability property if the normal component of ξ^a does not vanish at $\mathcal{S}$. As we shall see next, this is precisely the behavior of ξ^a as prescribed. Note that although we ask $\zeta_\perp = \mathcal{O}(1)$, this is the normal component wrt n^a , which is not defined at $\{\Omega = 0\}$, so we must first express ξ^a in terms of objects extendable to $\{\Omega = 0\}$.

Since diffeomorphisms are metric independent transformations and its generators are naturally the vector fields ξ^a , we do not rescale them when considering unphysical gauge transformations. Namely, we shall consider unphysical gauge transformations generated by $\tilde{\xi}^a = \xi^a$. In order to write $\tilde{\xi}$ in terms of unphysical quantities observe

$$\tilde{\zeta}_\perp = \tilde{\xi}^a \tilde{n}_a = \xi^a \tilde{n}_a = \zeta_\perp n^a \tilde{n}_a = \Omega \zeta_\perp, \quad \tilde{\zeta}^a = \tilde{\xi}^b \tilde{\gamma}^a_b = \xi^b \gamma^a_b = \zeta^a. \quad (4.3)$$

Therefore,

$$\tilde{\xi}^a = \Omega \zeta_\perp \tilde{n}^a + \tilde{\zeta}^a. \quad (4.4)$$

Note that $\tilde{n}^a$ is a regular vector at $\mathcal{S}$ and by hypothesis $\zeta_\perp = \mathcal{O}(1)$ and $\tilde{\zeta}^a = \zeta^a = \mathcal{O}(1)$. Hence $\tilde{\xi}^a$ is extendable to $\mathcal{S}$. Since its normal component vanishes for $\Omega = 0$, this means that $\tilde{\xi}$ becomes tangent at $\mathcal{S}$. Thus, the diffeomorphisms generated by vectors of the form (4.4) preserve the locus of $\mathcal{S}$ ¹¹.

We may now write the gauge transformation (4.1) for the unphysical linear fields

$$\begin{aligned} \delta \tilde{g}'_{ab} &= \Omega^2 \delta g'_{ab} = \Omega^2 (\delta g_{ab} + \mathcal{L}_\xi g_{ab}) = \delta \tilde{g}_{ab} + \Omega^2 \mathcal{L}_{\tilde{\xi}} (\Omega^{-2} \tilde{g}_{ab}) = \delta \tilde{g}_{ab} - 2\Omega^{-1} \tilde{\xi}(\Omega) \tilde{g}_{ab} + 2\tilde{\nabla}_{(a} \tilde{\xi}_{b)} \\ &= \delta \tilde{g}_{ab} + 2\tilde{N}^{-1} \zeta_\perp \tilde{g}_{ab} + 2\zeta_\perp \tilde{N}^{-1} \tilde{n}_{(a} \tilde{n}_{b)} + 2\Omega \zeta_\perp \tilde{\nabla}_{(a} \tilde{n}_{b)} + 2\Omega \tilde{n}_{(a} \tilde{\nabla}_{b)} \zeta_\perp + 2\tilde{\nabla}_{(a} \tilde{\xi}_{b)} \\ &= \delta \tilde{g}_{ab} + 2\tilde{N}^{-1} \zeta_\perp \tilde{g}_{ab} + 2\Omega \zeta_\perp \tilde{\nabla}_{(a} \tilde{n}_{b)} + 2\Omega \tilde{n}_{(a} \tilde{\nabla}_{b)} \zeta_\perp + 2\tilde{\nabla}_{(a} \tilde{\xi}_{b)}. \end{aligned} \quad (4.5)$$

Now it is a matter of direct calculation to check that (4.5) respects the ‘minimal’ fall-off behavior of the linearized field. For $\delta \tilde{g}'_{ab}$ and $\tilde{n}^b \delta \tilde{g}'_{ab}$ it is immediate that

$$\delta \tilde{g}'_{ab} = \delta \tilde{g}_{ab} + 2\tilde{N}^{-1} \zeta_\perp \tilde{g}_{ab} + 2\Omega \zeta_\perp \tilde{\nabla}_{(a} \tilde{n}_{b)} + 2\Omega \tilde{n}_{(a} \tilde{\nabla}_{b)} \zeta_\perp + 2\tilde{\nabla}_{(a} \tilde{\xi}_{b)} = \mathcal{O}(1), \quad (4.6)$$

$$\tilde{n}^b \delta \tilde{g}'_{ab} = \tilde{n}^b \delta \tilde{g}_{ab} + \Omega \zeta_\perp \tilde{n}^b \tilde{\nabla}_b \tilde{n}_a - \Omega \tilde{\nabla}_a \zeta_\perp + \Omega \tilde{n}_a \tilde{n}^b \tilde{\nabla}_b \zeta_\perp + 2\tilde{n}^b \tilde{\nabla}_{(a} \tilde{\xi}_{b)} = \mathcal{O}(1). \quad (4.7)$$

For $\tilde{n}^a \tilde{n}^b \delta \tilde{g}'_{ab}$ we have

$$\begin{aligned} \tilde{n}^a \tilde{n}^b \delta \tilde{g}'_{ab} &= \tilde{n}^a \tilde{n}^b \delta \tilde{g}_{ab} - 2\Omega \tilde{n}^a \tilde{\nabla}_a \zeta_\perp + 2\tilde{n}^a \tilde{n}^b \tilde{\nabla}_{(a} \tilde{\xi}_{b)} \\ &= \tilde{n}^a \tilde{n}^b \delta \tilde{g}_{ab} - 2\Omega \tilde{n}^a \tilde{\nabla}_a \zeta_\perp - 2\tilde{n}^a \tilde{\zeta}^b \tilde{\nabla}_a \tilde{n}_b, \end{aligned} \quad (4.8)$$

where we have used that $\tilde{n}^a \tilde{\zeta}_a = 0$ and $\tilde{n}^a \tilde{\nabla}_b \tilde{n}_a = 0$. It follows from the Einstein equations (cf (6.5)) that $\tilde{\nabla}_a \tilde{n}_b$ has $\mathcal{O}(1)$ trace terms plus $\mathcal{O}(\Omega)$ terms. Thus $\tilde{n}^a \tilde{\zeta}^b \tilde{\nabla}_a \tilde{n}_b = \mathcal{O}(\Omega)$ and

$$\tilde{n}^a \tilde{n}^b \delta \tilde{g}'_{ab} = \tilde{n}^a \tilde{n}^b \delta \tilde{g}_{ab} - 2\Omega \tilde{n}^a \tilde{\nabla}_a \zeta_\perp - 2\tilde{n}^a \tilde{\zeta}^b \tilde{\nabla}_a \tilde{n}_b = \mathcal{O}(\Omega) \quad (4.9)$$

Taking into account that $\tilde{n}^a = \tilde{N} \tilde{\nu}^a$ and that $\tilde{N} = \mathcal{O}(1)$ (cf equation (2.6), also section 6 below), it is ready to translate (4.6), (4.7) and (4.9) into conditions on the fields $\{\tau_{ab}, \tau_a, \tau\}$ under change of gauge

¹¹ We note that for vanishing cosmological constant, when splitting the vector field ξ^a tangential to null $\mathcal{S}$ in similar way as in (4.4) we have to use null (non-normalized) normal $\tilde{\nu}^a := \tilde{g}^{ab} d_b \Omega$, namely $\xi^a = \zeta \tilde{\nu}^a + \tilde{\zeta}^a$. Therefore, we do not need any special behavior of $\tilde{\zeta}$.

$$\tau'_{ab} = \delta g'_{ab} = \mathcal{O}(1), \quad (4.10)$$

$$\tau'_a = \tilde{\nu}^b \delta g'_{ab} = N^{-1} \tilde{n}^b \delta g'_{ab} = \mathcal{O}(1), \quad (4.11)$$

$$\tau' = \tilde{\nu}^a \tilde{\nu}^b \delta g'_{ab} = N^{-2} \tilde{n}^a \tilde{n}^b \delta g'_{ab} = \mathcal{O}(\Omega), \quad (4.12)$$

which match the analysis in section 2.

Therefore, we have considered the gauge fields of the form (4.4). These vector fields generate rather general class of diffeomorphisms of $\tilde{M}$, where the only restriction that they must preserve the locus of $\mathcal{I}$. In addition, it turns out that this kind of gauge transformations respect the fall-off conditions of the linear fields obtained directly from the Einstein equation in section 2.

5. Gauge degrees of freedom

As mentioned in the introduction, several recent literature [14, 16–19] analyze gravitational waves in presence of positive cosmological constant in Fefferman–Graham gauge. The unphysical metric in the Fefferman–Graham gauge is written as

$$\tilde{g}_{ab} = -\ell^2 \tilde{\nu}_a \tilde{\nu}_b + \tilde{\gamma}_{ab}, \quad (5.1)$$

where $\tilde{\gamma}$ is an object intrinsic to the leaves with $\Omega = \text{const}$. If one assumes a one-parameter family of metrics $\tilde{g}(\epsilon)$ sharing this gauge, the linear fields that one obtains are $\delta \tilde{g}_{ab} = \delta \tilde{\gamma}_{ab}$ thus satisfying $\delta \tilde{g}_{ab} \tilde{n}^a = 0$. In the notation of section 2, this amounts to set $\tau = 0$ and $\tau_a = 0$. Our goal in this section is to verify that this indeed corresponds to a choice of linear gauge, which, moreover, we will show it is independent of the gauge one uses for the background fields. The strategy consists in splitting $\delta g'_{ab}$ into components that yield an evolution problem (via equation (4.5)) for gauge fields $\{\zeta_\perp, \zeta^a\}$ which we can control. Then we check that there always exists a choice of gauge fields which yield $\tau = 0$ and $\tau_a = 0$. This result makes explicit that the gauge degrees of freedom are localized into the normal-tangent and normal-normal components of the linear fields.

First, consider the gauge transformations discussed in the previous section in the following form:

$$\delta \tilde{g}'_{ab} = \delta \tilde{g}_{ab} - 2\Omega^{-1} \tilde{\xi}(\Omega) \tilde{g}_{ab} + 2\tilde{\nabla}_{(a} \tilde{\xi}_{b)} = \delta \tilde{g}_{ab} + 2\tilde{N}^{-1} \zeta_\perp \tilde{g}_{ab} + 2\tilde{\nabla}_{(a} \tilde{\xi}_{b)} \quad (5.2)$$

The normal-tangent terms of above equation, with definition $v_c = \tilde{n}^a \tilde{\gamma}^b_c \delta \tilde{g}_{ab}$, are

$$\begin{aligned} \tilde{n}^a \tilde{\gamma}^b_c \delta \tilde{g}'_{ab} &= \tilde{n}^a \tilde{\gamma}^b_c \left(\tilde{\nabla}_a \tilde{\xi}_b + \tilde{\nabla}_b \tilde{\xi}_a \right) + \tilde{n}^a \tilde{\gamma}^b_c \delta \tilde{g}_{ab} \\ &= \tilde{n}^a \left(\tilde{\nabla}_a \left(\tilde{\gamma}^b_c \tilde{\xi}_b \right) - \left(\tilde{\nabla}_a \tilde{\gamma}^b_c \right) \tilde{\xi}_b \right) + \tilde{\gamma}^b_c \left(\tilde{\nabla}_b \left(\tilde{n}^a \tilde{\xi}_a \right) - \left(\tilde{\nabla}_b \tilde{n}^a \right) \tilde{\xi}_a \right) + v_c \\ &= \tilde{n}^a \tilde{\nabla}_a \tilde{\xi}_c - \tilde{n}^a \tilde{\nabla}_a \tilde{\gamma}^b_c \left(\Omega \zeta_\perp \tilde{n}_b + \tilde{\xi}_b \right) - \Omega \tilde{\gamma}^b_c \tilde{\nabla}_b \zeta_\perp - \tilde{\gamma}^b_c \tilde{\nabla}_b \tilde{n}^a \left(\Omega \zeta_\perp \tilde{n}_a + \tilde{\xi}_a \right) + v_c \\ &= \tilde{n}^a \tilde{\nabla}_a \tilde{\xi}_c - \left(\tilde{n}^a \tilde{\nabla}_a \tilde{\gamma}^b_c + \tilde{\gamma}^a_c \tilde{\nabla}_a \tilde{n}^b \right) \tilde{\xi}_b \\ &\quad - \Omega \left(\tilde{\gamma}^b_c \tilde{\nabla}_b \zeta_\perp + \left(\tilde{n}^a \tilde{n}^b \tilde{\nabla}_a \tilde{\gamma}_{bc} + \tilde{\gamma}^b_c \tilde{n}_a \tilde{\nabla}_b \tilde{n}^a \right) \zeta_\perp \right) + v_c \\ &= \tilde{n}^a \tilde{\nabla}_a \tilde{\xi}_c - \left(\tilde{n}^a \tilde{\nabla}_a \tilde{\gamma}^b_c + \tilde{\gamma}^a_c \tilde{\nabla}_a \tilde{n}^b \right) \tilde{\xi}_b - \Omega \left(\tilde{\gamma}^b_c \tilde{\nabla}_b \zeta_\perp + \left(\tilde{n}^a \tilde{n}^b \tilde{\nabla}_a \tilde{\gamma}_{bc} \right) \zeta_\perp \right) + v_c, \end{aligned}$$

where for the last equality we have used that $\tilde{n}_a \tilde{\nabla}_b \tilde{n}^a = 0$. Using $\tilde{K}_{ab} = \tilde{\nabla}_a \tilde{n}_b$ we may write

$$\tilde{n}^a \tilde{\gamma}^b{}_c \delta \tilde{g}'_{ab} = \tilde{n}^a \tilde{\nabla}_a \tilde{\zeta}_c - \left(\tilde{n}^a \tilde{\nabla}_a \tilde{\gamma}^b{}_c + \tilde{\gamma}^a{}_c \tilde{K}_a{}^b \right) \tilde{\zeta}_b - \Omega \left(\tilde{\gamma}^b{}_c \tilde{\nabla}_b \tilde{\zeta}_\perp - \tilde{\gamma}_{bc} \tilde{n}^a \tilde{K}_a{}^b \tilde{\zeta}_\perp \right) + v_c, \quad (5.3)$$

For the normal-normal components, we use (4.5)

$$\begin{aligned} \tilde{n}^a \tilde{n}^b \delta \tilde{g}'_{ab} &= \tilde{n}^a \tilde{n}^b \left(\delta \tilde{g}_{ab} + 2\tilde{N}^{-1} \tilde{\zeta}_\perp \tilde{\gamma}_{ab} + 2\Omega \tilde{n}_{(a} \tilde{\nabla}_{b)} \tilde{\zeta}_\perp + 2\tilde{\nabla}_{(a} \tilde{\zeta}_{b)} \right) \\ &= -2\Omega \tilde{n}^b \tilde{\nabla}_b \tilde{\zeta}_\perp + 2\tilde{n}^a \tilde{n}^b \tilde{\nabla}_a \tilde{\zeta}_b + \tilde{n}^a \tilde{n}^b \delta \tilde{g}_{ab} \end{aligned} \quad (5.4)$$

$$= -2\Omega \tilde{n}^b \tilde{\nabla}_b \tilde{\zeta}_\perp - 2\tilde{n}^a \tilde{\zeta}_b \tilde{\nabla}_a \tilde{n}^b + \tilde{n}^a \tilde{n}^b \delta \tilde{g}_{ab} \quad (5.5)$$

Using the unphysical Einstein equation (2.8) we have

$$\tilde{n}^a \tilde{\zeta}^b \tilde{\nabla}_a \tilde{n}^b = -\Omega \tilde{N} \frac{\tilde{R}_{ab}}{2} \tilde{n}^a \tilde{\zeta}^b.$$

In addition, since $\tilde{n}^a \tilde{n}^b \delta \tilde{g}_{ab} = O(\Omega)$ (cf section 2), we may write $\tilde{n}^a \tilde{n}^b \delta \tilde{g}_{ab} = \Omega u$ for a field u regular at $\mathcal{S}$ and thus equation (5.5) yields

$$\tilde{n}^a \tilde{n}^b \delta \tilde{g}'_{ab} = -2\Omega \tilde{n}^b \tilde{\nabla}_b \tilde{\zeta}_\perp + \Omega \tilde{N} \tilde{R}_{ab} \tilde{n}^a \tilde{\zeta}^b + \Omega u. \quad (5.6)$$

We will next show that there always exists a gauge choice such that

$$\tilde{n}^a \tilde{\gamma}^b{}_c \delta \tilde{g}'_{ab} = 0, \quad \tilde{n}^a \tilde{n}^b \delta \tilde{g}'_{ab} = 0. \quad (5.7)$$

From equations (5.3) and (5.5) this amounts to finding gauge fields $\{\tilde{\zeta}_\perp, \tilde{\zeta}^a\}$ solving the following PDE system

$$\tilde{n}^a \tilde{\nabla}_a \tilde{\zeta}_c - \Omega \tilde{\gamma}^b{}_c \tilde{\nabla}_b \tilde{\zeta}_\perp - \left(\tilde{n}^a \tilde{\nabla}_a \tilde{\gamma}^b{}_c + \tilde{\gamma}^a{}_c \tilde{K}_a{}^b \right) \tilde{\zeta}_b + \Omega \tilde{\gamma}_{bc} \tilde{n}^a \tilde{K}_a{}^b \tilde{\zeta}_\perp + v_c = 0, \quad (5.8)$$

$$-2\tilde{n}^b \tilde{\nabla}_b \tilde{\zeta}_\perp + \tilde{N} \tilde{R}_{ab} \tilde{n}^a \tilde{\zeta}^b + u = 0, \quad (5.9)$$

where $\{v_c, u\}$ are given fields determined by the original components $\tilde{n}^a \tilde{\gamma}^b{}_c \delta \tilde{g}_{ab}, \tilde{n}^a \tilde{n}^b \delta \tilde{g}_{ab}$ respectively. Equations (5.8) and (5.9) form a symmetric-hyperbolic system of linear PDEs (see section A.1 of appendix A for details). This means that it admits a well-posed initial value problem and therefore we can always find a solution for given initial data, which moreover, happen to be unconstrained (cf appendix A).

Note that in terms of the fields defined in section 2, equation (5.7) means that there exists a choice of gauge $\{\tau'_{ab}, \tau'_a, \tau'\}$ such that $\tau'_a = 0$ and $\tau' = 0$. Actually the inhomogeneous terms v_c and u in the system (5.3)–(5.8) coincide, respectively, with terms τ_c and τ of the original field configuration. A similar analysis can be carried out in terms of the fields $\{\tau_{ab}, \tau_a, \tau\}$ just by multiplying by the adequate power of an $\tilde{N}$ factor (cf (4.10)–(4.12)).

6. Fall-off analysis of background fields in presence of cosmological constant

In this section, we discuss the asymptotic behavior of the background gravitational fields. We will use this result and the fall-off behavior of the linearized field from section 2 to compute symplectic flux in section 7.

Assume that (2.8) holds at a neighborhood of the boundary $\mathcal{S} = \{\Omega = 0\}$. We now examine the fall-off conditions that arise from this assumption, which will later be applied in section 7

to obtain a regular action functional up to the boundary, and consequently, a presymplectic potential. To this end, we consider the 3 + 1 decomposition of equation (2.8) as outlined in the introduction, specifically, associated to the $\Omega = \text{const}$ foliation, while using the notation from expressions (1.8)–(1.10).

The lowest order term in (2.8) falls-off with Ω^{-2} . Hence regularizing higher order divergent term, we obtain

$$\tilde{g}^{cd}\tilde{\nabla}_c\Omega\tilde{\nabla}_d\Omega + \ell^{-2} = -\tilde{N}^{-2} + \ell^{-2} = \mathcal{O}(\Omega). \quad (6.1)$$

We have discussed the regularization of this term by introducing f in (2.5). Alternatively, we can introduce an auxiliary field $\tilde{a}$ such that

$$\tilde{N}^{-1} =: \ell^{-1} - \Omega\tilde{a}, \quad (6.2)$$

and therefore

$$\tilde{g}^{cd}\tilde{\nabla}_c\Omega\tilde{\nabla}_d\Omega + \ell^{-2} = -\tilde{N}^{-2} + \ell^{-2} = \Omega\tilde{a}(2\ell^{-1} - \Omega\tilde{a}). \quad (6.3)$$

Note that $\tilde{a}$ and $\tilde{f}$ are related by

$$\tilde{f} = 2\ell^{-1}\tilde{a} - \Omega\tilde{a}^2. \quad (6.4)$$

For convenience, in this section, we will use $\tilde{a}$ instead of $\tilde{f}$.

Now (2.8) reads

$$\tilde{R}_{ab} = \Omega^{-1} \left\{ -2\tilde{n}_b\tilde{\nabla}_a\tilde{N}^{-1} - \frac{2}{\tilde{N}}\tilde{K}_{ab} - \tilde{g}_{ab}\tilde{n}^c\tilde{\nabla}_c\tilde{N}^{-1} - \frac{\tilde{g}_{ab}}{\tilde{N}}\tilde{k} + 3\tilde{a}(2\ell^{-1} - \Omega\tilde{a})\tilde{g}_{ab} \right\}, \quad (6.5)$$

where we have used the definition of $\tilde{\nu}_a, \tilde{K}_{ab}, \tilde{k}_{ab}$ from equations (1.8) and (1.10).

Next, we write the tangent components of (6.5) in terms of the intrinsic geometry of the constant Ω leaves. Recall that the Gauss identity reads

$$\tilde{R}_{abcd}\tilde{\gamma}^a{}_p\tilde{\gamma}^b{}_q\tilde{\gamma}^c{}_r\tilde{\gamma}^d{}_s = \tilde{r}_{pqrs} - \left(\tilde{k}_{ps}\tilde{k}_{qr} - \tilde{k}_{pr}\tilde{k}_{qs} \right), \quad (6.6)$$

where $\tilde{r}_{pqrs}$ is the Riemann tensor of $\tilde{\gamma}$. Taking the $\tilde{\gamma}$ -trace of (6.6), we obtain

$$\tilde{\gamma}^q{}_b\tilde{\gamma}^s{}_d\tilde{R}_{qs} + \tilde{R}_{pbrd}\tilde{n}^p\tilde{n}^r = \tilde{r}_{bd} - \tilde{k}_{bd}^2 + \tilde{k}\tilde{k}_{bd}. \quad (6.7)$$

Here $\tilde{r}_{bd}$ is the Ricci tensor of $\tilde{\gamma}$, $\tilde{k}_{bd}^2 = \tilde{k}_{ba}\tilde{k}^a{}_d$ and $\tilde{k} = \tilde{\gamma}^{ab}\tilde{k}_{ab}$. The term $\tilde{R}_{abcd}\tilde{n}^a\tilde{n}^c$ can be written in terms of the electric part of the Weyl tensor $\tilde{E}_{bd} := \tilde{C}_{abcd}\tilde{n}^a\tilde{n}^c$ as

$$\tilde{R}_{abcd}\tilde{n}^a\tilde{n}^c = \tilde{E}_{bd} - \frac{1}{2}\tilde{\gamma}^q{}_b\tilde{\gamma}^s{}_d\tilde{R}_{qs} + \frac{1}{2}\tilde{\gamma}_{bd}\tilde{R}_{ac}\tilde{n}^a\tilde{n}^c + \frac{\tilde{R}}{6}\tilde{\gamma}_{bd}. \quad (6.8)$$

Now recall the well-known fact that the rescaled Weyl tensor $\tilde{e}_{ab} = \Omega^{-1}\tilde{E}_{ab}$ is a regular object at $\mathcal{I}$. In a nutshell, this can be understood as a consequence of the contracted Bianchi identity for the Weyl tensor. In the absence of the source, i.e. $T_{ab} = 0$, contracted Bianchi identity reduces to

$$\nabla_a C^a{}_{bcd} = 0. \quad (6.9)$$

Therefore, from the properties of conformal rescaling, it follows that

$$\tilde{\nabla}_a(\Omega^{-1}C^a{}_{bcd}) = \Omega^{-1}\nabla_a C^a{}_{bcd} = 0. \quad (6.10)$$

Equation $\tilde{\nabla}_a(\Omega^{-1}C^a{}_{bcd}) = 0$ is part of the hyperbolic system of Friedrich's conformal field equations (see the review [29]) which is smoothly extendable to $\mathcal{I}$. Thus, all components of

$\Omega^{-1}C^a_{bcd}$ regularly extend to $\mathcal{I}$ so we can assume regularity of $\tilde{e}_{ab} = \Omega^{-1}\tilde{E}_{ab}$. In terms of this new variable, equation (6.8) becomes

$$\tilde{R}_{abcd}\tilde{n}^a\tilde{n}^c = \Omega\tilde{e}_{bd} - \frac{1}{2}\tilde{\gamma}^q_b\tilde{\gamma}^s_d\tilde{R}_{qs} + \frac{1}{2}\tilde{\gamma}_{bd}\tilde{R}_{ac}\tilde{n}^a\tilde{n}^c + \frac{\tilde{R}}{6}\tilde{\gamma}_{bd}. \quad (6.11)$$

Inserting this back into (6.7), after some algebra, we get the following identity

$$\tilde{\gamma}^q_b\tilde{\gamma}^s_d\tilde{R}_{qs} = -2\Omega\tilde{e}_{bd} - \left(\tilde{R}_{ac}\tilde{n}^a\tilde{n}^c + \frac{\tilde{R}}{3}\right)\tilde{\gamma}_{bd} + 2\left(\tilde{r}_{bd} - \tilde{k}_{bd}^2 + \tilde{k}\tilde{k}_{bd}\right). \quad (6.12)$$

Now taking the normal projection of the conformal Einstein equation (6.5), we get

$$\tilde{R}_{ab}\tilde{n}^a\tilde{n}^b + \frac{\tilde{R}}{3} = \Omega^{-1}\left\{\tilde{n}^c\tilde{\nabla}_c\tilde{N}^{-1} - \frac{\tilde{k}}{\tilde{N}} + \tilde{a}(2\ell^{-1} - \Omega a)\right\}. \quad (6.13)$$

Similarly, taking the spatial projection of the conformal Einstein equation (6.5), we have

$$\tilde{\gamma}^q_b\tilde{\gamma}^s_d\tilde{R}_{qs} = \Omega^{-1}\left\{-\frac{2}{\tilde{N}}\tilde{k}_{bd} - \tilde{\gamma}_{bd}\tilde{n}^c\tilde{\nabla}_c\tilde{N}^{-1} - \frac{\tilde{\gamma}_{bd}\tilde{k}}{\tilde{N}} + 3\tilde{a}(2\ell^{-1} - \Omega a)\tilde{\gamma}_{bd}\right\}. \quad (6.14)$$

Putting equations (6.13), (6.14) back into (6.12) yields

$$\Omega\tilde{e}_{bd} - \tilde{r}_{bd} + \tilde{k}_{bd}^2 - \tilde{k}\tilde{k}_{bd} = \Omega^{-1}\left\{\frac{1}{\tilde{N}}\left(\tilde{k}_{bd} + \tilde{k}\tilde{\gamma}_{bd}\right) - 2\tilde{a}(2\ell^{-1} - \Omega a)\tilde{\gamma}_{bd}\right\}. \quad (6.15)$$

Thus, using (6.2), we obtain

$$\begin{aligned} \Omega\tilde{e}_{bd} - \tilde{r}_{bd} + \tilde{k}_{bd}^2 - \tilde{k}\tilde{k}_{bd} &= \Omega^{-1}\ell^{-1}\left\{\tilde{k}_{bd} + \tilde{k}\tilde{\gamma}_{bd} - 4\tilde{a}\tilde{\gamma}_{bd}\right\} \\ &\quad - \left\{\tilde{a}\left(\tilde{k}_{bd} + \tilde{k}\tilde{\gamma}_{bd}\right) - 2\tilde{a}^2\tilde{\gamma}_{bd}\right\}. \end{aligned} \quad (6.16)$$

Let us denote by the superscript ^{tf} the trace-free part of 2-tensors X in spacelike slices,

$$X_{ab}^{\text{tf}} := X_{ab} - \frac{1}{3}\tilde{\gamma}^{cd}X_{cd}\tilde{\gamma}_{ab}. \quad (6.17)$$

With this notation, the regularization of trace-free and trace parts of right hand side of the (6.15) require the fall-off conditions for k_{ab}

$$\tilde{k}_{bd}^{\text{tf}} = \Omega\tilde{\chi}_{bd}, \quad (6.18)$$

$$\tilde{a} = \frac{\tilde{k}}{3} + \Omega\tilde{b}, \quad (6.19)$$

where $\tilde{\chi}_{bd}$ is an auxiliary trace-free tensor and $\tilde{b}$ an auxiliary scalar, both regular at $\Omega = 0$. Then, inserting (6.18) and (6.19) back into (6.16) yields, after canceling terms,

$$\Omega\tilde{e}_{bd} - \tilde{r}_{bd} + \Omega^2\tilde{\chi}_{bd}^2 = \left(\ell^{-1} - \tilde{b}\Omega^2\right)\tilde{\chi}_{bd} + 2\tilde{b}\left(-2\ell^{-1} + \tilde{b}\Omega^2\right)\tilde{\gamma}_{bd}. \quad (6.20)$$

From the trace-free and trace parts of (6.20) we obtain, respectively,

$$\ell^{-1}\tilde{\chi}_{bd} = -\tilde{r}_{bd}^{\text{tf}} + \Omega\tilde{e}_{bd} + \Omega^2\left(\tilde{b}\tilde{\chi}_{bd} + \tilde{\chi}_{bd}^2\right), \quad (6.21)$$

$$4\ell^{-1}\tilde{b} = \frac{\tilde{r}}{3} + \Omega^2\left(2\tilde{b}^2 - \frac{1}{3}\tilde{\chi}^{ac}\tilde{\chi}_{ac}\right). \quad (6.22)$$

In summary, inserting (6.18) into (6.21), as well as (6.22) into (6.19) and then the result into (6.2), we obtain the following fall-off conditions for the fields

$$\tilde{k}_{bd}^{\text{tr}} = -\tilde{\ell} r_{bd}^{\text{tr}} \Omega + \tilde{\ell} \tilde{e}_{bd} \Omega^2 + \mathcal{O}(\Omega^3), \quad (6.23)$$

$$\tilde{N}^{-1} = \ell^{-1} - \frac{\tilde{k}}{3} \Omega - \ell \frac{\tilde{r}}{12} \Omega^2 + \mathcal{O}(\Omega^4). \quad (6.24)$$

We remark that it is possible to keep generating further terms in (6.23) and (6.24) by introducing additional auxiliary fields and evaluating the conformal Einstein equation. However, for our purposes, it suffices with the orders appearing in (6.23) and (6.24). Additionally, note that these are not Taylor series expansions of field variables. Each term multiplying a power of Ω is not a coefficient of an expansion in Ω since they generally also depend on Ω . This formulation is preferable for our purposes, as it allows for the exact cancellation of certain terms that arise in the computation of the presymplectic potential. One could choose to carry out the same analysis by generating a Taylor series expansion. However, it is convenient to introduce auxiliary variables in a gauge-free approach, because a Taylor series expansion in an arbitrary gauge has a rather involved form. Taylor series expansion of field variables in the Fefferman–Graham gauge for asymptotically AdS and dS spacetimes can be found in [60] and [61] respectively¹².

Note that (6.23) provides insight into the structure of the asymptotic phase space. Specifically, the lowest non-trivial order Ω is determined by the boundary metric, which corresponds to the initial configuration state, while $\tilde{e}$ appears at the subleading order in the expansion, effectively acting as a normal derivative and thus representing a momentum-like quantity. On the other hand, due to the gauge nature of $\tilde{N}$, the sub-leading terms in (6.24) are gauge-dependent quantities and do not essentially contribute to the dynamical evolution of the system.

7. Symplectic flux

In this section, we calculate the presymplectic potential and presymplectic current leading to the formula of the symplectic flux. We will compute the presymplectic potential from the first principle starting from the action. We define the following action terms

$$\begin{aligned} S_{\text{EH}} &:= \frac{1}{16\pi G} \int_M |g|^{1/2} (R - 6\ell^{-2}), & s_{\text{GH}} &:= -\frac{1}{16\pi G} \int_{\partial M} |\gamma|^{1/2} 2k, \\ s_{\text{ct}} &:= \frac{1}{16\pi G} \int_{\partial M} |\gamma|^{1/2} (4\ell^{-1} - \ell r), \end{aligned} \quad (7.1)$$

respectively known as Einstein–Hilbert action and Gibbons–Hawking and counterterms. The total action then reads

$$\begin{aligned} S &= S_{\text{EH}} + s_{\text{GH}} + s_{\text{ct}} \\ &= \frac{1}{16\pi G} \left(\int_M |g|^{1/2} (R - 6\ell^{-2}) - \int_{\partial M} |\gamma|^{1/2} 2k + \int_{\partial M} |\gamma|^{1/2} (4\ell^{-1} - \ell r) \right). \end{aligned} \quad (7.2)$$

In order to calculate the boundary terms at $\mathcal{I}$, we shall first consider integration within the domain $\mathcal{D}_\epsilon$ with boundary $\partial\mathcal{D}_\epsilon = \{\Omega = \epsilon > 0\}$. Later we shall take the limit $\epsilon \rightarrow 0$ so that

¹² Note that the ‘Dirichlet’ or ‘reflective’ boundary condition, used the [60, 61], contributes to the vanishing gravitational flux across the boundary.

$\lim_{\epsilon \rightarrow 0} \mathcal{D}_\epsilon = M$. For simplicity, we will first assume that ∂M , as well as $\partial \mathcal{D}_\epsilon$, are compact, without a boundary. However, for clarity, we keep track of all the boundary terms. Only at the end of the computation, we will use the absence of the boundary of $\partial \mathcal{D}_\epsilon$.

In the Gibbons–Hawking term, the sign depends on the convention chosen for k and the causal character of $\partial \mathcal{D}_\epsilon$. This, in turn, fixes a unique sign for the counterterm, since this is meant to cancel divergences of the action. In our case this is fixed by $\partial \mathcal{D}_\epsilon$ being spacelike and $k_{ab} = \gamma^c{}_a \gamma^d{}_b \nabla_c n_b$ constructed with $n^a = N g^{ab} \nabla_b \Omega$, pointing outward at $\partial \mathcal{D}_\epsilon$ (see comment below equation (7.9)). Our sign convention matches with that of [18], but other conventions may be also used [14, 17, 62].

Now consider a linear variation of the metric $g \rightarrow g + \delta g$. We keep Ω as an invariant foliating function, namely $\delta \Omega = 0$. Note that under this assumption

$$\delta n_a = \delta (N \nabla_a \Omega) = \frac{n_a}{N} \delta N. \quad (7.3)$$

From the 3 + 1 decomposition, we also note that

$$0 = \delta (n^a n^b \gamma_{ab}) = \delta n^a n^b \gamma_{ab} + n^a \delta n^b \gamma_{ab} + n^a n^b \delta \gamma_{ab} = n^a n^b \delta \gamma_{ab}, \quad (7.4)$$

which in turn also implies

$$\gamma^c{}_a \gamma^d{}_b \delta g_{cd} = \gamma^c{}_a \gamma^d{}_b (-\delta n_c n_d - n_c \delta n_d + \delta \gamma_{cd}) = \gamma^c{}_a \gamma^d{}_b \delta \gamma_{cd}. \quad (7.5)$$

We calculate the contribution to the presymplectic potential of $S_{\text{EH}}, S_{\text{GH}}$ and s_{ct} separately by obtaining variations of their respective Lagrangian densities $\mathcal{L}_{\text{EH}}, \mathcal{L}_{\text{GH}}$ and $\mathcal{L}_{\text{ct}}$. Starting with $\mathcal{L}_{\text{EH}}$

$$16\pi G \delta \mathcal{L}_{\text{EH}} = |g|^{1/2} \delta R_{ab} g^{ab} + |g|^{1/2} R_{ab} \delta g^{ab} + \delta |g|^{1/2} (R - 6\ell^{-2}), \quad (7.6)$$

$$\begin{aligned} &= |g|^{1/2} \delta R_{ab} g^{ab} + |g|^{1/2} \left(R_{ab} - \frac{1}{2} R g_{ab} + 3\ell^{-2} g_{ab} \right) \delta g^{ab} \\ &= |g|^{1/2} \delta R_{ab} g^{ab}, \end{aligned} \quad (7.7)$$

where the last equality holds on-shell. Taking into account that

$$\delta R_{ab} = -\frac{1}{2} \nabla_a \nabla_b \delta g - \frac{1}{2} \square \delta g_{ab} + \nabla^c \nabla_{(a} \delta g_{b)c}, \quad (7.8)$$

we obtain

$$|g|^{1/2} \delta R_{ab} g^{ab} = |g|^{1/2} \nabla_a g^{ac} (\nabla^b \delta g_{bc} - g^{de} \nabla_a \delta g_{de}). \quad (7.9)$$

Now observe that, in the unphysical picture, the unit vector $\tilde{n}^a$ is outward pointing at $\mathcal{I}$, because $\tilde{n}^a \nabla_a \Omega = \tilde{N}^{-1} \tilde{n}_a \tilde{g}^{ab} \tilde{n}_b = -\tilde{N}^{-1} < 0$ and Ω grows inward from $\mathcal{I}$. Similarly, because $\Omega > 0$ in M , the physical unit vector $n^a = \Omega \tilde{n}^a$ is the outward oriented at $\partial \mathcal{D}_\epsilon$. Then, by the Gauss theorem (cf appendix B),

$$\begin{aligned} \int_{\mathcal{D}_\epsilon} |g|^{1/2} \nabla_a g^{ac} (\nabla^b \delta g_{bc} - g^{de} \nabla_c \delta g_{de}) &= \int_{\partial \mathcal{D}_\epsilon} |\gamma|^{1/2} (-n^c) (\nabla^b \delta g_{bc} - g^{de} \nabla_c \delta g_{de}) \\ &= 16\pi G \int_{\partial \mathcal{D}_\epsilon} \theta_{\text{EH}}, \end{aligned} \quad (7.10)$$

where θ_{EH} denotes the contribution of the Einstein–Hilbert term to the presymplectic potential

$$16\pi G \theta_{\text{EH}} = -|\gamma|^{1/2} n^a (\nabla^b \delta g_{ab} - g^{cd} \nabla_a \delta g_{cd}). \quad (7.11)$$

Note that, following notation (1.10), $k = \gamma^{ab} k_{ab} = g^{ab} K_{ab} = K$, where recall that both k_{ab} and K_{ab} are defined using the outward-oriented normal. Then we make the following observation

$$\delta k = \delta K = \delta g^{ab} K_{ab} + g^{ab} \delta K_{ab} = -\delta g_{ab} K^{ab} + g^{ab} \delta K_{ab}, \quad (7.12)$$

where we can write

$$g^{ab} \delta K_{ab} = g^{ab} \left(\nabla_a \delta n_b - \frac{n^c}{2} (\nabla_a \delta g_{bc} + \nabla_b \delta g_{ac} - \nabla_c \delta g_{ab}) \right), \quad (7.13)$$

$$= g^{ab} \nabla_a \delta n_b - \frac{1}{2} g^{ab} \nabla_a (n^c \delta g_{bc}) + \frac{1}{2} g^{ab} \delta g_{bc} \nabla_a n^c - \frac{n^c}{2} g^{ab} (\nabla_b \delta g_{ac} - \nabla_c \delta g_{ab}), \quad (7.14)$$

$$= g^{ab} \nabla_a \delta n_b - \frac{1}{2} g^{ab} \nabla_a (n^c \delta g_{bc}) + \frac{1}{2} K^{ac} \delta g_{ac} - \frac{n^c}{2} g^{ab} (\nabla_b \delta g_{ac} - \nabla_c \delta g_{ab}), \quad (7.15)$$

and hence

$$\delta k = -\frac{1}{2} \delta g_{ab} K^{ab} + g^{ab} \nabla_a \delta n_b - \frac{1}{2} g^{ab} \nabla_a (n^c \delta g_{bc}) - \frac{n^c}{2} g^{ab} (\nabla_b \delta g_{ac} - \nabla_c \delta g_{ab}). \quad (7.16)$$

Putting this back into (7.11) yields

$$16\pi G\theta_{\text{EH}} = |\gamma|^{1/2} (2\delta k + \delta g_{ab} K^{ab} - 2g^{ab} \nabla_a \delta n_b + g^{ab} \nabla_a (n^c \delta g_{bc})), \quad (7.17)$$

which after noticing

$$n^c \delta g_{bc} = \delta n_b - n_b n^c \delta n_c + n^c \delta \gamma_{bc} = 2\delta n_b + n^c \delta \gamma_{bc}, \quad (7.18)$$

we obtain

$$16\pi G\theta_{\text{EH}} = |\gamma|^{1/2} (2\delta k + \delta g_{ab} K^{ab} + g^{ab} \nabla_a (n^c \delta \gamma_{bc})). \quad (7.19)$$

Now, taking into account $K^{ab} n_b = 0$ and (7.3), we obtain,

$$K^{ab} \delta g_{ab} = K^{ab} (-\delta n_a n_b - n_a \delta n_b + \delta \gamma_{ab}) = K^{ab} \delta \gamma_{ab} = k^{ab} \delta \gamma_{ab} - n_c K^{cb} n^a \delta \gamma_{ab}. \quad (7.20)$$

From (7.4), we also have

$$n_c K^{cb} n^a \delta \gamma_{ab} = n_c (\nabla^c n^b) n^a \delta \gamma_{ab} = -n^c n^b (\nabla_c n^a \delta \gamma_{ab}). \quad (7.21)$$

Using (7.20), (7.21), we finally find

$$16\pi G\theta_{\text{EH}} = |\gamma|^{1/2} (2\delta k + k^{ab} \delta \gamma_{ab} + \gamma^{ab} \nabla_a (n^c \delta \gamma_{bc})). \quad (7.22)$$

Note that $n^c \delta \gamma_{bc}$ is a tangent vector of $\partial \mathcal{D}_\epsilon$ (cf (7.4)), thus $\gamma^{ab} \nabla_a (n^c \delta \gamma_{bc}) = \gamma^{ab} D_a (n^c \delta \gamma_{bc})$ is a total derivative of $\partial \mathcal{D}_\epsilon$ and therefore it will not contribute to the integral (as long as $\partial \mathcal{D}_\epsilon$ is compact.) We however keep track of this term by defining $u^a := \gamma^{ab} \delta \gamma_{bc} n^c$. Hence

$$16\pi G\theta_{\text{EH}} = |\gamma|^{1/2} (2\delta k + k^{ab} \delta \gamma_{ab} + D_a u^a). \quad (7.23)$$

The contribution of the Gibbons–Hawking and counter-terms is straightforwardly computable, since they both are boundary integrals

$$16\pi G\theta_{\text{GH}} = -2|\gamma|^{1/2}\delta k - |\gamma|^{1/2}k\gamma^{ab}\delta\gamma_{ab}, \quad (7.24)$$

$$16\pi G\theta_{\text{ct}} = 2\ell^{-1}|\gamma|^{1/2}\gamma^{ab}\delta\gamma_{ab} - \ell\delta\left(|\gamma|^{1/2}r\right). \quad (7.25)$$

Combining (7.22), (7.24) and (7.25) gives the following expression for the total presymplectic potential

$$16\pi G\theta = 16\pi G(\theta_{\text{EH}} + \theta_{\text{GH}} + \theta_{\text{ct}}) \quad (7.26)$$

$$= |\gamma|^{1/2}(k^{ab} - k\gamma^{ab})\delta\gamma_{ab} + 2\ell^{-1}|\gamma|^{1/2}\gamma^{ab}\delta\gamma_{ab} - \ell\delta\left(|\gamma|^{1/2}r\right) + |\gamma|^{1/2}D_a u^a. \quad (7.27)$$

Note that the exact cancelation between δk term from θ_{EH} , and θ_{GH} guarantees a well-defined variation principle [63, 64] and it is reason for introducing the Gibbons–Hawking term into the action.

At this point, we can identify the de Sitter holographic stress tensor [14, 65] analogical to that widely discussed in the anti-de Sitter case [64]. Neglecting the total derivative term $D_a u^a$ and defining $T^{ab} := \frac{2}{|\gamma|^{1/2}} \frac{\delta S}{\delta \gamma_{ab}}$, from (7.27) we have

$$T^{ab} = \frac{1}{8\pi G} \left(k^{ab} - k\gamma^{ab} + 2\ell^{-1}\gamma^{ab} + \ell \left(r^{ab} - \frac{1}{2}r\gamma^{ab} \right) \right). \quad (7.28)$$

This is also reminiscent of analogous boundary stress tensor in anti-de Sitter case (see equation (10) of [64] for AdS_4 .)

It will be useful to write (7.27) in terms of infinitesimal variations of the contravariant metric $(\gamma^{-1})^{ab} = \gamma^{ab}$. Abusing the notation we shall denote

$$\delta(\gamma^{-1})^{ab} = \delta\gamma^{ab}, \quad \delta(g^{-1})^{ab} = \delta g^{ab}, \quad (7.29)$$

Beware, that in this convention $\delta\gamma^{ab} = -\gamma^{ak}\gamma^{bl}\delta\gamma_{kl}$ and similarly for g^{ab} . Thus, (7.27) can be easily expressed in terms of $\delta\gamma^{ab}$

$$16\pi G\theta = -|\gamma|^{1/2}(k_{ab} - k\gamma_{ab})\delta\gamma^{ab} - 2\ell^{-1}|\gamma|^{1/2}\gamma_{ab}\delta\gamma^{ab} - \ell\delta\left(|\gamma|^{1/2}r\right) + |\gamma|^{1/2}D_a u^a. \quad (7.30)$$

We now transform the above expression to unphysical variables, i.e. those corresponding to $\tilde{g}_{ab} = \Omega^2 g_{ab}$ and $\tilde{\gamma}_{ab} = \Omega^2 \gamma_{ab}$. Recall that the change of connection is given by the following tensor

$$\nabla_a v_b - \tilde{\nabla}_a v_b = -S_{ab}^c v_c, \quad S_{ab}^c = -\frac{1}{\Omega} (\tilde{v}_b \delta_a^c + \tilde{v}_a \delta_b^c - \tilde{v}^c \tilde{g}_{ab}) \quad (7.31)$$

and the physical and unphysical unit normals n_a and $\tilde{n}_a$ satisfy

$$n_a = \Omega^{-1} \tilde{n}_a.$$

Then we have the following relation between extrinsic curvatures

$$k_{ab} = \gamma^c_a \gamma^d_b \nabla_a n_b = \frac{\tilde{k}_{ab}}{\Omega} + \frac{1}{\Omega^2} \frac{\tilde{\gamma}_{ab}}{\tilde{N}} \implies k = \gamma^{ab} k_{ab} = \Omega \tilde{k} + \frac{3}{\tilde{N}}. \quad (7.32)$$

To relate the mean intrinsic curvatures, observe that the Ricci tensors satisfy $r_{ab} = \tilde{r}_{ab}$. This is because they are respectively constructed out of the metrics γ_{ab} and $\tilde{\gamma}_{ab} = \Omega^2 \gamma_{ab}$. Since they

are both intrinsic to hypersurfaces with $\Omega = \text{const}$, we can consider Ω as a constant for the calculation of r_{ab} and $\tilde{r}_{ab}$. Then

$$r = \gamma^{ab} r_{ab} = \Omega^2 \tilde{\gamma}^{ab} \tilde{r}_{ab} = \Omega^2 \tilde{r},$$

and therefore

$$\delta \left(|\gamma|^{1/2} r \right) = \frac{1}{\Omega} \delta \left(|\tilde{\gamma}|^{1/2} \tilde{r} \right) = \frac{|\tilde{\gamma}|^{1/2}}{\Omega} \left(\tilde{r}_{ab} - \frac{1}{2} \tilde{r} \tilde{\gamma}_{ab} \right) \delta \tilde{\gamma}^{ab} + \frac{|\tilde{\gamma}|^{1/2}}{\Omega} \tilde{D}_a \tilde{v}^a \quad (7.33)$$

where

$$\tilde{v}^a := \tilde{\gamma}^{ad} \tilde{\gamma}^{bc} \left(\tilde{D}_b \delta \tilde{\gamma}_{cd} - \tilde{D}_d \delta \tilde{\gamma}_{bc} \right),$$

and the second equality in (7.33) is a standard result (see [66]), parallel to the variation of the spacetime curvature performed above.

Within the integral, the term $|\gamma|^{1/2} D_a u^a$ is independent from the connection employed (see appendix B, equation (B.4)), so defining

$$\tilde{u}_a = \Omega^{-1} u_a, \quad \tilde{u}^a = \Omega u^a,$$

we may substitute

$$D_a |\gamma|^{1/2} u^a \rightarrow \tilde{D}_a |\tilde{\gamma}|^{1/2} \tilde{u}^a = \frac{|\tilde{\gamma}|^{1/2}}{\Omega} \tilde{D}_a \tilde{u}^a.$$

With these definitions, inserting (7.32) and (7.33) into (7.30), we now have

$$\begin{aligned} 16\pi G\theta &= 2 \frac{|\tilde{\gamma}|^{1/2}}{\Omega^3} \left(-\ell^{-1} + \tilde{N}^{-1} \right) \tilde{\gamma}_{ab} \delta \tilde{\gamma}^{ab} - \frac{|\tilde{\gamma}|^{1/2}}{\Omega^2} \left(\tilde{k}_{ab} - \tilde{k} \tilde{\gamma}_{ab} \right) \delta \tilde{\gamma}^{ab} \\ &\quad - \frac{|\tilde{\gamma}|^{1/2}}{\Omega} \ell \left(\tilde{r}_{ab} - \frac{\tilde{r}}{2} \tilde{\gamma}_{ab} \right) \delta \tilde{\gamma}^{ab} + \frac{|\tilde{\gamma}|^{1/2}}{\Omega} \tilde{D}_a \tilde{u}^a - \frac{1}{\Omega} |\tilde{\gamma}|^{1/2} \ell \tilde{D}_a \tilde{v}^a. \end{aligned} \quad (7.34)$$

We can gather the total derivative terms, which recall do not contribute to the integral, by defining

$$\tilde{\beta}^a := \tilde{u}^a - \ell \tilde{v}^a.$$

Then

$$\begin{aligned} 16\pi G\theta &= 2 \frac{|\tilde{\gamma}|^{1/2}}{\Omega^3} \left(-\ell^{-1} + \tilde{N}^{-1} \right) \tilde{\gamma}_{ab} \delta \tilde{\gamma}^{ab} - \frac{|\tilde{\gamma}|^{1/2}}{\Omega^2} \left(\tilde{k}_{ab} - \tilde{k} \tilde{\gamma}_{ab} \right) \delta \tilde{\gamma}^{ab} \\ &\quad - \frac{|\tilde{\gamma}|^{1/2}}{\Omega} \ell \left(\tilde{r}_{ab} - \frac{\tilde{r}}{2} \tilde{\gamma}_{ab} \right) \delta \tilde{\gamma}^{ab} + \frac{|\tilde{\gamma}|^{1/2}}{\Omega} \tilde{D}_a \tilde{\beta}^a. \end{aligned} \quad (7.35)$$

We now introduce (6.24) into (7.35) and find

$$\begin{aligned} 16\pi G\theta &= -\frac{|\tilde{\gamma}|^{1/2}}{\Omega^2} \left(\tilde{k}_{ab} - \frac{\tilde{k}}{3} \tilde{\gamma}_{ab} \right) \delta \tilde{\gamma}^{ab} - \frac{|\tilde{\gamma}|^{1/2}}{\Omega} \ell \left(\tilde{r}_{ab} - \frac{\tilde{r}}{3} \tilde{\gamma}_{ab} \right) \delta \tilde{\gamma}^{ab} + \frac{1}{\Omega} |\tilde{\gamma}|^{1/2} \tilde{D}_a \tilde{\beta}^a + \mathcal{O}(\Omega) \\ &= -\frac{|\tilde{\gamma}|^{1/2}}{\Omega^2} \left(\tilde{k}_{ab} - \frac{\tilde{k}}{3} \tilde{\gamma}_{ab} \right) \delta \tilde{\gamma}^{ab} - \frac{|\tilde{\gamma}|^{1/2}}{\Omega} \ell \left(\tilde{r}_{ab} - \frac{\tilde{r}}{3} \tilde{\gamma}_{ab} \right) \delta \tilde{\gamma}^{ab} + \frac{1}{\Omega} |\tilde{\gamma}|^{1/2} \tilde{D}_a \tilde{\beta}^a + \mathcal{O}(\Omega) \\ &= -\frac{|\tilde{\gamma}|^{1/2}}{\Omega^2} \tilde{k}_{ab}^{\text{tr}} \delta \tilde{\gamma}^{ab} - \frac{|\tilde{\gamma}|^{1/2}}{\Omega} \ell \tilde{r}_{ab}^{\text{tr}} \delta \tilde{\gamma}^{ab} + \frac{1}{\Omega} |\tilde{\gamma}|^{1/2} \tilde{D}_a \tilde{\beta}^a + \mathcal{O}(\Omega). \end{aligned}$$

Introducing now (6.23), we are left with

$$16\pi G\theta = -|\tilde{\gamma}|^{1/2}\tilde{\ell}\tilde{e}_{ab}\delta\tilde{\gamma}^{ab} + \frac{1}{\Omega}|\tilde{\gamma}|^{1/2}\tilde{D}_a\tilde{\beta}^a + \mathcal{O}(\Omega). \quad (7.36)$$

In summary, we have shown that, on shell,

$$\delta S = \int_{\partial\mathcal{D}_\epsilon} \theta = -\frac{\ell}{16\pi G} \int_{\partial\mathcal{D}_\epsilon} |\tilde{\gamma}|^{1/2}\tilde{\ell}\tilde{e}_{ab}\delta\tilde{\gamma}^{ab} + \frac{\ell}{16\pi G} \frac{1}{\Omega} \int_{\partial\mathcal{D}_\epsilon} |\tilde{\gamma}|^{1/2}\tilde{D}_a\tilde{\beta}^a + \mathcal{O}(\Omega). \quad (7.37)$$

The boundary $\partial\mathcal{D}_\epsilon$ splits into two disconnected parts, $\partial\mathcal{D}_\epsilon = \partial\mathcal{D}_\epsilon^+ \cup \partial\mathcal{D}_\epsilon^-$, one representing the past boundary and the other the future one. The symplectic potential and the symplectic form is obtained by integrating over one of them, say the future one $\Sigma^+ = \partial\mathcal{D}_\epsilon^+$, and we are especially interested in the limit $\Omega \rightarrow 0$ when $\Sigma^+ \rightarrow \mathcal{I}^+$.

Typically, Σ^+ is compact without boundary (diffeomorphic to S^3 in the de Sitter-like context). Under this assumption, the total derivative term $\tilde{D}_a\tilde{\beta}^a$ in (7.37) does not contribute to the symplectic potential,

$$\Theta[\Sigma^+] = \int_{\Sigma^+} \theta = -\frac{\ell}{16\pi G} \int_{\Sigma^+} |\tilde{\gamma}|^{1/2}\tilde{\ell}\tilde{e}_{ab}\delta\tilde{\gamma}^{ab} + \mathcal{O}(\Omega), \quad (7.38)$$

so we can take the limit $\Omega \rightarrow 0$

$$\Theta[\mathcal{I}^+] = -\frac{\ell}{16\pi G} \int_{\mathcal{I}^+} |\tilde{\gamma}|^{1/2}\tilde{e}_{ab}\delta\tilde{\gamma}^{ab}. \quad (7.39)$$

One might wish to study behavior of the symplectic potential $\Theta[\Sigma]$ on an arbitrary sub-domain Σ of the future boundary $\partial\mathcal{D}_\epsilon^+$ ¹³. Observe, however, that allowing non-empty $\partial\Sigma$ is not straightforward. One would have to study the fall-off behavior of the boundary terms $\frac{1}{\Omega} \int_{\partial\Sigma} \tilde{\beta}_a e^a |\tilde{h}|^{1/2}$ arising from $\tilde{D}_a\tilde{\beta}^a$, which leads to a non-trivial discussion going beyond the scope of this paper. This is an interesting question that we shall address elsewhere.

We emphasize that the presence of the counterterms in the action and the fall-off conditions of the field in the conformal Einstein equation play an important role to make the presymplectic potential finite at the boundary. Additionally, note that this result matches standard calculations available in the literature of the symplectic potential that are carried out in the Fefferman Graham gauge.

Finally, we compute the presymplectic current, given by,

$$\omega(\delta_1\tilde{\gamma}, \delta_2\tilde{\gamma}) = \delta_1\theta(\delta_2\tilde{\gamma}) - \delta_2\theta(\delta_1\tilde{\gamma}). \quad (7.40)$$

Assuming the variations δ_1, δ_2 commute with each other, we have

$$\omega(\delta_1\tilde{\gamma}, \delta_2\tilde{\gamma}) = \frac{\ell}{16\pi G} \left(\delta_1 \left(|\tilde{\gamma}|^{1/2}\tilde{e}^{ab} \right) \delta_2\tilde{\gamma}_{ab} - \delta_2 \left(|\tilde{\gamma}|^{1/2}\tilde{e}^{ab} \right) \delta_1\tilde{\gamma}_{ab} \right). \quad (7.41)$$

Note that from the regularization of the linearized conformal Einstein equation, we have $\delta\tilde{g}_{ab} = \delta\tilde{\gamma}_{ab} = \mathcal{O}(1)$. Therefore, the presymplectic current is non-vanishing on the boundary. Physically, this is attributed to the non-zero gravitational radiation on the boundary $\mathcal{I}^+$ [9, 19]. It is also well known that in leaky boundary condition, namely where the variation of the field does not vanish, the energy flux is given by the presymplectic potential on the $\mathcal{I}^+$.

¹³ This may be useful when approaching the future de Sitter infinity using spatially flat or hyperbolic cosmological models..

In the context of linearized theory, presymplectic potential represents flux formula for isometries of de Sitter background. Note that background Killing vectors become conformal Killing vector of the background metric on $\mathcal{I}^+$, and ξ is the conformal Killing vector of the de Sitter background boundary. The expression of the flux in linearized case becomes

$$F_\xi = \int_{\mathcal{I}^+} \theta \left(\delta \tilde{\gamma}_{ab}, \mathcal{L}_{\tilde{\xi}} \delta \tilde{\gamma}_{ab} \right) = \frac{\ell}{16\pi G} \int_{\mathcal{I}^+} |\tilde{\gamma}|^{1/2} \delta \tilde{e}^{ab} \mathcal{L}_{\tilde{\xi}} \delta \tilde{\gamma}_{ab}, \quad (7.42)$$

where $\delta \tilde{e}^{ab}$ linearized rescaled electric part of the Weyl tensor. This formula has been employed in several literature [37, 43, 44, 67] to compute energy flux of linearized fields around de Sitter background. We note that under the conformal transformation $\tilde{\gamma}_{ab} = \Omega^2 \gamma_{ab}$, $\tilde{e}^{ab} \rightarrow \Omega^{-5} e^{ab}$, $\tilde{\xi}^a \rightarrow \xi^a$, therefore, the flux formula in equation (7.42) is conformally invariant, i.e. we can express the flux formula in terms of physical variables. It is well-known that linearized field is defined upto a gauge transformation, $\delta g_{ab} \rightarrow \delta g_{ab} + \mathcal{L}_\xi \bar{g}_{ab}$, where $\bar{g}_{ab}$, in our case de Sitter background metric. One can show that linearized Weyl tensor is gauge invariant on de Sitter background [68, 69]. Since the flux expression is conformally invariant and linearized Weyl tensor is gauge invariant on background de Sitter space-time, the flux is therefore insensitive to linearized diffeomorphism. This confirms the gauge invariant nature of the flux formula.

8. Conclusion

We analyze conformal Einstein equations to extract fall-off conditions of the gravitational fields. These fall-off condition are consistent to reproduce gravitational flux at the future infinity of de Sitter. We obtain presymplectic structure at the boundary in terms of rescaled electric part of the Weyl tensor. Though this result is well-known in the context of Fefferman-Graham gauge in de Sitter, our approach is unique, in particular, we did not impose any gauge condition to extract the presymplectic current at $\mathcal{I}^+$. This highlights the gauge covariant nature of the quantity, enhancing its relevance as a candidate for describing gravitational energy flux. Our result also emphasizes that a gauge covariant conformal extension of holographic stress tensor on the compact boundary naturally selects electric part of the Weyl tensor in the presymplectic potential computation.

We have found important differences in regularization of the linearized conformal Einstein equation in comparison with asymptotically flat case. The presence of cosmological constant introduces a higher order divergence term in the linearized equation. The leading order term in the metric perturbation does not vanish at $\mathcal{I}^+$, rather the non vanishing perturbation produces non-zero symplectic flux. The non-vanishing nature of the perturbation at $\mathcal{I}^+$ is also qualitatively consistent with the analysis of gravitational waves in Bondi gauge [17, 21, 41, 42] or generalized harmonic gauge [34, 37, 38]. Our result is also consistent with the linear displacement memory effect in de Sitter space-times. Note that the displacement memory in de Sitter space-times depends on the difference between the final and initial non-vanishing fields at the boundary [34]. It will also be interesting to explore the gravitational memory effect in terms of the perturbed electric part of the Weyl tensor [70] with the choice of our ‘minimal’ fall-off behavior of the linearized fields. To us, the constraint in equation (2.18) is interesting and non-trivial for non-zero cosmological constant setting. Our results do not rigorously show that the field variables we have introduced in the linearized conformal Einstein equation form a hyperbolic PDE system. We wish to return to this problem in future.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Well-posedness of symmetric-hyperbolic quasilinear PDEs

In this appendix we summarize existence, uniqueness and stability results for quasilinear, symmetric-hyperbolic systems of PDEs. This is based on the results in chapter 12 of [31], in turn based on Kato [71] and Friedrich's [28] theorems for existence of solutions of quasilinear hyperbolic systems.

Consider a domain $\mathcal{D}$ with topology $\mathbb{R} \times \Sigma$ and coordinates $\{\Omega, x^i\}$ adapted to a foliation with $\mathcal{D}_\Omega := \{\Omega = \text{const.}\}$ leaves, each one of which is homeomorphic to Σ . Let $\mathbf{U} : \mathcal{D} \rightarrow \mathbb{R}^n$ be an $\mathbb{R}^n$ -valued field and for each fixed value of Ω define $\mathbf{u}_\Omega := \mathbf{U}(\Omega, \cdot)$ as a field $\mathbf{u}_\Omega : \mathcal{D}_\Omega \rightarrow \mathbb{R}^n$. We introduce the Sobolev norm

$$\|\mathbf{u}_\Omega\|_{\mathcal{D}_\Omega, m} = \left(\sum_{k=0}^m \sum_{\alpha_1, \dots, \alpha_k=0}^3 \int_{\mathcal{D}_\Omega} |\partial_{x^{\alpha_k}} \cdots \partial_{x^{\alpha_1}} \mathbf{u}_\Omega|^2 \right)^{1/2} \quad (\text{A.1})$$

where $|\mathbf{u}_\Omega|^2 = \langle \mathbf{u}_\Omega, \mathbf{u}_\Omega \rangle$ stands for the usual $\mathbb{R}^n$ norm. The Sobolev space $H^m(\mathcal{D}_\Omega, \mathbb{R}^n)$ is the Banach space of $\mathbb{R}^n$ -valued fields of $\mathcal{D}_\Omega$ with finite norm $\|\cdot\|_{\mathcal{D}_\Omega, m}$ completed with the limit points of its Cauchy sequences. Recall that it is well-known (see e.g [72]) that $H^m(\mathcal{D}_\Omega, \mathbb{R}^n) \subset C^{m-2}(\mathcal{D}_\Omega, \mathbb{R}^n)$.

Now consider the quasilinear system of PDEs

$$A^\Omega(\Omega, x; \mathbf{U}) \partial_\Omega \mathbf{U} + A^i(\Omega, x; \mathbf{U}) \partial_i \mathbf{U} + B(\Omega, x; \mathbf{U}) = 0, \quad (\text{A.2})$$

where $A^\Omega(\Omega, x; \mathbf{U})$, $A^i(\Omega, x; \mathbf{U})$ are matrices and $B(\Omega, x; \mathbf{U})$ is a vector. Their entries may depend nonlinearly on the coordinates and the field $\mathbf{U}$, but not on its derivatives. We assume that (A.2) is symmetric, i.e. A, A^i are symmetric matrices, and hyperbolic, i.e. that we can find scalars $(\sigma_\Omega, \sigma_i)$ such that the combination $A^\Omega \sigma_\Omega + A^i \sigma_i$ is a positive definite matrix. We aim to establish existence of the Cauchy problem of (A.2) with initial data

$$\mathbf{u}_{\Omega_0} = \mathbf{u}_0(x^i) \in H^m(D_{\Omega_0}, \mathbb{R}^n) \quad (\text{A.3})$$

at some initial surface $D_{\Omega_0} = \{\Omega = \Omega_0\}$. We assume that $\det A^\Omega(\Omega_0, x; \mathbf{u}_{\Omega_0}) \neq 0$, so that all transversal derivatives $\partial_\Omega \mathbf{U}|_{\Omega_0}$ can be obtained from the initial data via (A.2). This implies that no constraint equations need to be imposed on the initial data. This assumption can be removed if suitable constraints are introduced.

The existence and uniqueness result is as follows. Consider the Cauchy problem (A.2) and (A.3) with $m \geq 4$. Assume that $A^\Omega(\Omega_0, x; \mathbf{u}_{\Omega_0})$ is bounded away from zero for some $\delta > 0$,

namely, there exists a $\delta > 0$ such that $\langle \mathbf{Z}, A^\Omega(\Omega_0, x; \mathbf{u}_{\Omega_0}) \mathbf{Z} \rangle > \delta \langle \mathbf{Z}, \mathbf{Z} \rangle$ for all $\mathbf{Z} \in \mathbb{R}^n$. Then there exists some $\Omega_1 \in \mathbb{R}$ and unique solution $\mathbf{U}(\Omega, x^i)$ to the Cauchy problem (A.2) and (A.3) such that $\mathbf{U} \in C^{m-2}([\Omega_0, \Omega_1] \times \Sigma, \mathbb{R}^n)$. Moreover, $A^\Omega(\Omega, x^i, \mathbf{U})$ remains bounded away from zero in $[\Omega_0, \Omega_1] \times \Sigma$.

To state the stability results, denote $B_r(\mathbf{u}_{\Omega_0})$ to the ball of $H^m(\mathcal{D}_{\Omega_0}, \mathbb{R}^n)$ centered at $\mathbf{u}_{\Omega_0}$ with radius r wrt to the norm $\|\cdot\|_{\mathcal{D}_{\Omega_0}, m}$, intersected with the subspace of functions $\mathbf{u}_{\Omega_0} \in H^m(\mathcal{D}_{\Omega_0}, \mathbb{R}^n)$ for which $A^\Omega(\Omega_0, x; \mathbf{u}_{\Omega_0})$ is bounded away from zero by some $\delta > 0$. Then

1. There exists some $r > 0$ such that $\Omega_1 \in \mathbb{R}$ can be chosen so that all solutions with initial conditions in $B_r(\mathbf{u}_{\Omega_0})$ exist for the same ‘time’ interval $[\Omega_0, \Omega_1] \subset \mathbb{R}$.
2. For any Cauchy sequence $\{\mathbf{u}_{\Omega_0}^n\} \subset B_r(\mathbf{u}_{\Omega_0})$ converging to $\mathbf{u}_{\Omega_0}$ on $\mathcal{D}_{\Omega_0}$, the corresponding solutions $\mathbf{U}^n$ satisfy that $\mathbf{u}_{\Omega}^n = \mathbf{U}^n(\Omega, \cdot)$ converge uniformly to $\mathbf{u}_\Omega = \mathbf{U}(\Omega, \cdot)$ on each Cauchy slice $\mathcal{D}_\Omega$ for all $\Omega \in [\Omega_0, \Omega_1]$.
3. Given a solution $\mathbf{U}$ that exists for some interval $[\Omega_0, \Omega_1] \subset \mathbb{R}$, then for $r > 0$ sufficiently small, all solutions with initial data on $B_r(\mathbf{u}_{\Omega_0})$ exist on $[\Omega_0, \Omega_1]$.

A.1. Hyperbolicity of (5.8) and (5.9)

We can now apply the existence and uniqueness results stated above to the PDE system (5.8) and (5.9). First, we note that a particular type of quasilinear PDE (A.2) is the linear case, namely,

$$A^\Omega(\Omega, x) \partial_\Omega \mathbf{U} + A^i(\Omega, x) \partial_i \mathbf{U} + B(\Omega, x) \mathbf{U} + C(\Omega, x) = 0, \quad (\text{A.4})$$

where $A^\Omega(\Omega, x)$, $A^i(\Omega, x)$ and $B(\Omega, x)$ are matrices and $C(\Omega, x)$ is a vector. Their entries in this case do not depend on $\mathbf{U}$, but they may depend smoothly on the coordinates. Recall that we have selected our foliation to have zero shift vector, so that in the adapted coordinates $\{\Omega, x^i\}$ we have $\tilde{n}^a = -\tilde{N}^{-1}(\partial_\Omega)^a$ and $\tilde{\gamma}^a_b = \delta^i_j(\partial_{x^i})^a(dx^j)_b$. This simplifies the analysis because it makes the fields ζ^i to be a linear combination of the fields ζ_i , not involving $\zeta_\perp$. Comparing (A.4) and (5.8), (5.9), we find that the latter is a PDE system like the former for the field $\mathbf{U} = (\zeta_\perp, \tilde{\zeta}^i)$ and the matrices

$$A^\Omega = \tilde{N}^{-1} \text{diag}(1, 1, 1, 1), \quad A^i = \text{diag}(\Omega, 0, 0, 0), \quad (\text{A.5})$$

where a global sign has been omitted and the matrix B and vector C are not specified because neither of them affect the hyperbolicity properties. The system is symmetric because (A^Ω, A^i) are symmetric matrices. Moreover, it is hyperbolic because for sufficiently small Ω we can find scalars $(\sigma_\Omega, \sigma_i)$ such that the combination $A^\Omega \sigma_\Omega + A^i \sigma_i$ is a positive definite matrix.

Now consider a prescribed initial field configuration $\mathbf{u}_{\Omega_0}$ at some $\{\Omega = \Omega_0 = \text{const}\}$ initial slice. There is obviously not constraint equations on the initial data because $\det A^{\Omega_0} \neq 0$. The condition of A^{Ω_0} being bounded away from zero is equivalent to finding some $\delta > 0$ such that $\tilde{N}^{-1} > \delta$. This holds for sufficiently small Ω , because by (6.24) we have $\tilde{N}^{-1} = \ell^{-1} + \mathcal{O}(\Omega)$, thus it suffices choosing $\delta = \ell^{-1}/2$ as a lower bound for $\tilde{N}^{-1}$.

A.2. Stability of Friedrich equations

Friedrich’s strategy to prove that his conformal equations form a well-posed system consists in showing that they can be reduced to a symmetric hyperbolic quasilinear system of PDEs (see [28]). This in turn entails a stability result like the one we have discussed at the beginning of this appendix, which Friedrich applies to solutions close to de Sitter (see theorem 3.3 of [28]). We note that the paper states that same stability holds for all (weakly) asymptotically simple

(i.e. admitting a conformal extension) solutions with positive Λ (see remark 3.4 (ii) of [28]). The initial data on the initial surface $D_{\Omega_0} = \Sigma$ are required to be $H^m(\Sigma, \mathbb{R}^n)$ with $m \geq 4$, which implies regularity of the data at least $C^{m-2}(\Sigma, \mathbb{R}^n)$ and solutions $C^{m-2}([\Omega_0, \Omega_1] \times \Sigma, \mathbb{R}^n)$.

Given a solution $\tilde{g}_0$ of Friedrich's equations, stability holds all over the unphysical manifold $\tilde{M}$, up to and including $\mathcal{I}$. Thus, let us consider $\tilde{g}_0$ such that it extends from some initial Cauchy slice D_{Ω_0} to $\mathcal{I}^+ = \{\Omega = 0\}$. Let u_0 be the initial data of $\tilde{g}_0$ at D_{Ω_0} . Then there is a ball $B_r(u_0)$ (as defined above in this appendix) of sufficiently small radius r such that all metrics with initial data in $B_r(u_0)$ also extend to $\mathcal{I}^+ = \{\Omega = 0\}$. Consider a family of data $\{u_\epsilon\} \subset B_r(u_0)$ depending smoothly on a parameter ϵ and converging to u_0 for $\epsilon = 0$. Then, the corresponding family of solutions $\tilde{g}_\epsilon$ converges to $\tilde{g}_0$ and extends to $\mathcal{I}^+$. From this fact it follows that the linear fields $\delta\tilde{g} = \frac{d\tilde{g}_\epsilon}{d\epsilon}|_{\epsilon=0}$, which are solutions of the linearized equations with background field $\tilde{g}_0$, must also extend to $\mathcal{I}^+$.

Appendix B. Gauss theorem

In this appendix we make explicit the Gauss theorem in the context we are working at. Namely,

$$\int_D |g|^{1/2} \nabla_a v^a = \int_{\partial D} |\gamma|^{1/2} n_a^{\text{out}} v^a, \quad (\text{B.1})$$

where n_a^{out} is outward pointing one-form normal to the boundary ∂D of a domain D .

In our specific case, we consider domain $\mathcal{D}_\epsilon$ defined by $\Omega > \epsilon$, where ϵ is a sufficiently small positive constant. The domain of integration is endowed with a metric g_{ab} . The boundary manifold $\partial\mathcal{D}_\epsilon$, given by $\Omega = \epsilon$ surface, has an induced spacelike metric γ_{ab} . Therefore, $n^a = N\nabla^a\Omega$ is the unit timelike normal, which is outward pointing (see comment below equation (7.9)) and $-n_a = -N\nabla_a\Omega$ is outward pointing normal one-form. Then we get the Gauss theorem in the form

$$\int_{\mathcal{D}_\epsilon} |g|^{1/2} \nabla_a v^a = - \int_{\partial\mathcal{D}_\epsilon} |\gamma|^{1/2} n_a v^a. \quad (\text{B.2})$$

Furthermore, recall that the Gauss theorem is a topological result, thus independent from the metric. Indeed, the LHS of (B.2) is a total derivative and can be written in terms of the unphysical metric $\tilde{g}_{ab}$ and connection $\tilde{\nabla}$ simply as

$$\int_{\mathcal{D}_\epsilon} |g|^{1/2} \nabla_a v^a = \int_{\mathcal{D}_\epsilon} |\tilde{g}|^{1/2} \tilde{\nabla}_a v^a. \quad (\text{B.3})$$

This is a general identity for integrals of total derivatives, so in particular, it also holds in the boundary manifold that

$$\int_{\partial\mathcal{D}_\epsilon} |\gamma|^{1/2} D_a u^a = \int_{\partial\mathcal{D}_\epsilon} |\tilde{\gamma}|^{1/2} \tilde{D}_a u^a, \quad (\text{B.4})$$

for tangent vectors u^a to $\partial\mathcal{D}_\epsilon$.

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