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Equal-spin limit of the Kerr–NUT–(A)dS spacetime

Eliška Polášková and Pavel Krtouš

Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic
**E-mail: eli.polaskova@email.cz, Pavel.Krtous@utf.mff.cuni.cz*

We summarize properties of the Kerr–NUT–(A)dS spacetime in a general dimension. We also investigate a limit when several rotational parameters in the metric are set equal and study geometry of the resulting spacetime. In dimension $D = 6$, we found a suitable Killing vector basis, whose algebraic structure proves that symmetries of the spacetime after the limit are further enhanced.

Keywords: Black holes; higher dimensions; rotation; NUT charges; symmetries.

1. Introduction

In four dimensions, the Plebański–Demiański metric [1] represents a large family of algebraic type D vacuum spacetimes. For example, it includes the rotating Kerr black hole, the Taub–NUT solution with one NUT parameter as well as accelerating black holes described by the C-metric.

In higher dimensions, the most general spacetime metric known so far is called the *Kerr–NUT–(A)dS metric* [2]. It includes rotational and NUT parameters as well as the cosmological constant, however, a solution with acceleration and charge is yet to be discovered.

The Kerr–NUT–(A)dS metric can describe various geometries, such as maximally symmetric spaces, a so-called Euclidean instanton and black hole solutions [3]. It also admits explicit and hidden symmetries that can be generated from a single object — a closed conformal Killing–Yano tensor, which we call the principal tensor [4,5]. Moreover, the principal tensor uniquely determines canonical coordinates in which the Hamilton–Jacobi, Klein–Gordon [6] and Dirac [7] equations are fully separable, and therefore the geodesic motion is completely integrable [8].

The main focus of our work is a limit of the metric when several rotational parameters are set equal. Other limit cases have already been studied, such as the limit with some of the black hole's rotations switched off [9] or the limit when particular roots of the metric function X_μ degenerate [10]. These papers have demonstrated that not only can performing various limits of the general metric shed light on the role of NUT charges, but it can also result in new interesting geometries.

2. Kerr–NUT–(A)dS Spacetime

This section introduces the general Kerr–NUT–(A)dS spacetime and describes some of its interesting properties. Firstly, the metric in its general form is discussed and subsequently, we show how to obtain a black hole geometry from the general metric as a special case.

For simplicity, we restrict ourselves to even dimensions $D = 2N$. Note that the generalization to odd dimensions is straightforward — a corresponding term is added to the metric.

2.1. *Metric*

A metric describing the Kerr–NUT–(A)dS geometry can be written in the form[2]

$$\boldsymbol{g} = \sum_{\mu} \left[\frac{U_\mu}{X_\mu} \mathbf{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_k A_\mu^{(k)} \mathbf{d}\psi_k \right)^2 \right] , \tag{1}$$

with greek and latin indices going over slightly different ranges

$$\begin{aligned} \mu, \nu &= 1, \ldots, N \,, \\ k, l &= 0, \ldots, N-1 \,, \end{aligned}$$

unless stated otherwise explicitly in the sum.

The functions U_μ and $A_\mu^{(k)}$ that appear in the metric are defined as polynomials in coordinates x_μ

$$U_\mu = \prod_{\substack{\nu \\ \nu \neq \mu}} \left(x_\nu^2 - x_\mu^2 \right) , \qquad A_\mu^{(k)} = \sum_{\substack{\nu_1, \ldots, \nu_k \\ \nu_1 < \cdots < \nu_k \\ \nu_i \neq \mu}} x_{\nu_1}^2 \ldots x_{\nu_k}^2 \,. \tag{2}$$

Each function $X_\mu = X_\mu(x_\mu)$ is dependent on a single coordinate x_μ and if it is left unspecified, the metric is then referred to as *off-shell*. On the other hand, if the function X_μ is also defined as a polynomial, namely

$$X_\mu = \lambda \mathcal{J}\left(x_\mu^2\right) - 2b_\mu x_\mu \,, \tag{3}$$

where $\mathcal{J}\left(x_\mu^2\right)$ reads

$$\mathcal{J}\left(x_\mu^2\right) = \prod_{\nu} \left(a_\nu^2 - x_\mu^2 \right) , \tag{4}$$

then the metric (1) satisfies the Einstein equations in vacuum and we refer to it as *on-shell*.

The coordinates we used are divided into two sets. In the black hole case, which is discussed in detail in Sec. 2.2, the coordinates x_μ represent radius and latitudinal angles whereas the coordinates ψ_k represent time and longitudinal angles. Moreover, since the metric functions are independent of ψ_k, they are also the *Killing coordinates*.

The on-shell metric is described by several parameters, but only λ can be interpreted easily — it is related to the cosmological constant Λ as

$$\Lambda = (2N-1)(N-1)\lambda \,. \tag{5}$$

Regarding the others, in general we can say that the parameters a_μ somehow describe rotations and the parameters b_μ encode mass and NUT charges. However, their interpretation strongly depends on specific other choices that can be made. This will be demonstrated in the next subsection where a black hole geometry that can be obtained from the general metric is discussed.

Instead of ψ_k, we can also use a more convenient set of angular coordinates ϕ_α defined as

$$\phi_\alpha = \lambda a_\alpha \sum_k \mathcal{A}_\alpha^{(k)} \psi_k \,, \qquad \psi_k = \sum_\alpha \frac{\left(-a_\alpha^2\right)^{N-1-k}}{\mathcal{U}_\alpha} \frac{\phi_\alpha}{\lambda a_\alpha} \,, \tag{6}$$

where the functions $\mathcal{U}_\mu$ and $\mathcal{A}_\mu^{(k)}$ are defined similarly to U_μ and $A_\mu^{(k)}$ in (2), only x_μ is replaced with a_μ

$$\mathcal{U}_\mu = \prod_{\substack{\nu \\ \nu \neq \mu}} \left(a_\nu^2 - a_\mu^2\right) \,, \qquad \mathcal{A}_\mu^{(k)} = \sum_{\substack{\nu_1, \dots, \nu_k \\ \nu_1 < \dots < \nu_k \\ \nu_i \neq \mu}} a_{\nu_1}^2 \dots a_{\nu_k}^2 \,. \tag{7}$$

The metric (1) then obtains the form

$$\boldsymbol{g} = \sum_\mu \left[\frac{U_\mu}{X_\mu} \mathbf{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_\alpha \frac{J_\mu\left(a_\alpha^2\right)}{\mathcal{U}_\alpha} \frac{1}{\lambda a_\alpha} \mathbf{d}\phi_\alpha \right)^2 \right] \,, \tag{8}$$

where $J_\mu\left(a_\alpha^2\right)$ reads

$$J_\mu\left(a_\alpha^2\right) = \prod_{\substack{\nu \\ \nu \neq \mu}} \left(x_\nu^2 - a_\alpha^2\right) \,. \tag{9}$$

Since ϕ_α are simply linear combinations of ψ_k, they are the Killing coordinates as well.

2.2. *Black hole*

The metric (8) can have both Euclidean and Lorentzian signatures, depending on the coordinate ranges and values of the parameters. Detailed discussion of geometries with the Euclidean signature, such as maximally symmetric spaces and Euclidean instantons, can be found in Ref. 3.

In order to obtain the Lorentzian signature, some of the coordinates and parameters need to be Wick-rotated, namely

$$x_N = ir \,, \qquad \phi_N = \lambda a_N t \,, \qquad b_N = iM \,. \tag{10}$$

Moreover, we use a one-parametric gauge freedom in rescaling parameters and set

$$a_N^2 = -\frac{1}{\lambda} \,. \tag{11}$$

We also assume that $a_{\bar{\mu}}$ are ordered

$$0 < a_1 < \dots < a_{\bar{N}-1} < a_{\bar{N}} \,, \tag{12}$$

where $\bar{N} = N - 1$ and barred indices go over the ranges

$$\bar{\mu}, \bar{\nu} = 1, \ldots, \bar{N}\,,$$
$$\bar{k}, \bar{l} = 0, \ldots, \bar{N} - 1\,.$$

This notation is used to separate the temporal and radial coordinates from the angular ones.

The metric (8) then becomes

$$\begin{aligned}
\boldsymbol{g} = &-\frac{\Delta_r}{\Sigma}\left(\prod_{\bar{\nu}} \frac{1+\lambda x_{\bar{\nu}}^2}{1+\lambda a_{\bar{\nu}}^2}\mathbf{d}t - \sum_{\bar{\nu}} \frac{\bar{J}\big(a_{\bar{\nu}}^2\big)}{a_{\bar{\nu}}\,(1+\lambda a_{\bar{\nu}}^2)\,\bar{\mathcal{U}}_{\bar{\nu}}}\mathbf{d}\phi_{\bar{\nu}}\right)^2 \\
&+\frac{\Sigma}{\Delta_r}\mathbf{d}r^2 + \sum_{\bar{\mu}} \frac{\big(r^2 + x_{\bar{\mu}}^2\big)\,\bar{U}_{\bar{\mu}}}{\Delta_{\bar{\mu}}}\mathbf{d}x_{\bar{\mu}}^2 \\
&+\sum_{\bar{\mu}} \frac{\Delta_{\bar{\mu}}}{\big(r^2 + x_{\bar{\mu}}^2\big)\,\bar{U}_{\bar{\mu}}}\left[\frac{1-\lambda r^2}{1+\lambda x_{\bar{\mu}}^2}\prod_{\bar{\nu}}\frac{1+\lambda x_{\bar{\nu}}^2}{1+\lambda a_{\bar{\nu}}^2}\mathbf{d}t + \sum_{\bar{\nu}} \frac{\big(r^2 + a_{\bar{\nu}}^2\big)\,\bar{J}_{\bar{\mu}}\big(a_{\bar{\nu}}^2\big)}{a_{\bar{\nu}}\,(1+\lambda a_{\bar{\nu}}^2)\,\bar{\mathcal{U}}_{\bar{\nu}}}\mathbf{d}\phi_{\bar{\nu}}\right]^2\,,
\end{aligned} \tag{13}$$

with the metric functions Δ_r, $\Delta_{\bar{\mu}}$, Σ and $\bar{J}\big(a_{\bar{\mu}}^2\big)$ defined as

$$\begin{aligned}
\Delta_r &= -X_N = \big(1 - \lambda r^2\big)\prod_{\bar{\nu}}\big(r^2 + a_{\bar{\nu}}^2\big) - 2Mr\,, & \Sigma &= U_N = \prod_{\bar{\nu}}\big(r^2 + x_{\bar{\nu}}^2\big)\,, \\
\Delta_{\bar{\mu}} &= -X_{\bar{\mu}} = \big(1 + \lambda x_{\bar{\mu}}^2\big)\,\bar{\mathcal{J}}\big(x_{\bar{\mu}}^2\big) + 2b_{\bar{\mu}}x_{\bar{\mu}}\,, & \bar{J}\big(a_{\bar{\mu}}^2\big) &= \prod_{\bar{\nu}}\big(x_{\bar{\nu}}^2 - a_{\bar{\mu}}^2\big)
\end{aligned} \tag{14}$$

and the functions $\bar{U}_{\bar{\mu}}$, $\bar{\mathcal{U}}_{\bar{\mu}}$, $\bar{J}_{\bar{\mu}}\big(a_{\bar{\nu}}^2\big)$ and $\bar{\mathcal{J}}\big(x_{\bar{\mu}}^2\big)$ defined in the same way as their unbarred counterparts in Sec. 2.1, only with modified sets of coordinates and parameters (i.e. without x_N and a_N).

Let us now discuss suitable ranges of the coordinates $x_{\bar{\mu}}$. For vanishing NUT parameters (and non-zero mass), i.e. $b_{\bar{\mu}} = 0$, the metric is non-singular and has the desired signature if we assume that the coordinates are ordered as

$$-a_1 < x_1 < a_1 < x_2 < a_2 < \ldots < x_{\bar{N}} < a_{\bar{N}}\,. \tag{15}$$

For non-vanishing NUT charges, the coordinate ranges change only slightly — we assume that they are restricted by

$$a_{\bar{\mu}-1} < {}^{-}x_{\bar{\mu}} < x_{\bar{\mu}} < {}^{+}x_{\bar{\mu}} < a_{\bar{\mu}} \tag{16}$$

with the only exception being ${}^{-}x_1$ since ${}^{-}x_1 < -a_1$. Figure 1 shows the discussed ranges.

The horizon structure is given by the roots of the metric function Δ_r defined in Eq. (14). For $\lambda \leq 0$, there exist at most two horizons — an inner and an outer one. The two horizons can also coincide, thus forming a single extremal horizon, or we can obtain a naked singularity in case there are no horizons. For $\lambda > 0$, there exists one additional cosmological horizon.

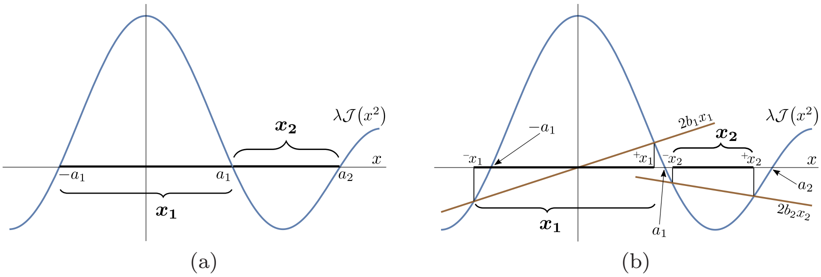

Fig. 1. Ranges of the coordinates $x_{\bar\mu}$ for (a) zero and (b) non-zero NUT parameters $b_{\bar\mu}$.

As for the interpretation of the metric parameters, for zero NUT charges $b_{\bar\mu} = 0$, $a_{\bar\mu}$ can be identified with the rotational parameters of the black hole. On the other hand, when the NUT parameters are non-trivial, then both $a_{\bar\mu}$ and $b_{\bar\mu}$ deform the geometry, however, their exact role remains elusive. This is where studying various limit cases might prove useful as they can help clarify this issue.

3. Equal-spin limit in $D = 6$

In general, performing the limiting procedure is not trivial — since some of the coordinate ranges and metric parameters degenerate, it is necessary to rescale them using an appropriate parametrization. Moreover, we expect the symmetry of the resulting spacetime to be enhanced, therefore we also need to find new Killing vectors or another proof of such enhanced symmetry.

For simplicity, we assume that the cosmological constant and NUT charges vanish, i.e. $\lambda = 0 = b_{\bar\mu}$. We are interested in the black hole form of the metric, which has only two rotational parameters in $D = 6$. Therefore, we will perform the limit $a_2 \to a_1$ using the parametrization

$$\begin{aligned} a_2 &= a_1 + 2\delta a_2 \varepsilon\,, \\ x_2 &= a_1 + (1+\xi_2)\,\delta a_2 \varepsilon\,, \end{aligned} \tag{17}$$

where $\varepsilon \ll 1$ is a small parameter and $\delta a_2 > 0$ controls how fast a_2 approaches a_1. Instead of $x_2 \in (a_1, a_2)$ whose range becomes degenerate, we have introduced a rescaled coordinate $\xi_2 \in (-1, 1)$.

Using a more convenient set of angular coordinates $\phi_\pm = \phi_1 \pm \phi_2$, the resulting metric is

$$\begin{aligned} \boldsymbol{g} \approx -\mathbf{d}t^2 &+ \frac{2Mr}{\Sigma}\left[\mathbf{d}t - \frac{x_1^2 - a_1^2}{2a_1}\left(\mathbf{d}\phi_+ + \xi_2\,\mathbf{d}\phi_-\right)\right]^2 + \frac{\Sigma}{\Delta_r}\mathbf{d}r^2 - \frac{r^2 + x_1^2}{x_1^2 - a_1^2}\mathbf{d}x_1^2 \\ &- \frac{\left(r^2 + a_1^2\right)\left(x_1^2 - a_1^2\right)}{4a_1^2}\left(\frac{\mathbf{d}\xi_2^2}{1 - \xi_2^2} + \mathbf{d}\phi_+^2 + \mathbf{d}\phi_-^2 + 2\xi_2\,\mathbf{d}\phi_+\mathbf{d}\phi_-\right) \end{aligned} \tag{18}$$

with the metric functions in the form

$$\begin{aligned} \Delta_r &\approx \left(r^2 + a_1^2\right)^2 - 2Mr\,, \\ \Sigma &\approx \left(r^2 + a_1^2\right)\left(r^2 + x_1^2\right)\,. \end{aligned} \tag{19}$$

Most importantly, in addition to the original Killing vectors $\boldsymbol{l}_+$ and $\boldsymbol{l}_-$, we have obtained two new ones after the limit — here denoted by $\boldsymbol{m}$ and $\boldsymbol{n}$. These new vectors are independent of the original Killing vectors and they Lie-preserve the full spacetime metric (18). The complete set of Killing vectors regarding the spatial part of the metric is therefore

$$\begin{aligned} \boldsymbol{l}_+ &= \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\phi_+}\,, \qquad \boldsymbol{m} = \cos\phi_- \sqrt{1-\xi_2^2}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\xi_2} - \frac{\sin\phi_-}{\sqrt{1-\xi_2^2}}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\phi_+} - \xi_2\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\phi_-}\right)\,, \\ \boldsymbol{l}_- &= \frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\phi_-}\,, \qquad \boldsymbol{n} = -\sin\phi_- \sqrt{1-\xi_2^2}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\xi_2} - \frac{\cos\phi_-}{\sqrt{1-\xi_2^2}}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\phi_+} - \xi_2\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\phi_-}\right)\,. \end{aligned} \tag{20}$$

Their Lie brackets suggest that the symmetry of the metric (18) is indeed further enhanced as three of the Killing vectors above generate the algebra of an $SO(3)$ group

$$[\boldsymbol{l}_-, \boldsymbol{m}] = \boldsymbol{n}\,, \qquad [\boldsymbol{m}, \boldsymbol{n}] = \boldsymbol{l}_-\,, \qquad [\boldsymbol{n}, \boldsymbol{l}_-] = \boldsymbol{m}\,. \tag{21}$$

The vector $\boldsymbol{l}_+$ commutes with all the other vectors. Therefore, the symmetry group in the spatial part of the spacetime decouples from the original $U(1) \times U(1)$ into $SO(3) \times U(1)$ after the limit.

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