

Hidden symmetries and integrability in higher dimensional rotating black hole spacetimes

Marco Cariglia^{1,*}, Pavel Krtouš^{2,**}, and David Kubizňák^{3,***}

¹ Universidade Federal de Ouro Preto, ICEB, Departamento de Física, Campus Morro do Cruzeiro, Morro do Cruzeiro, 35400-000 – Ouro Preto, MG – Brasil

² Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, Prague, Czech Republic

³ Perimeter Institute, 31 Caroline St. N. Waterloo Ontario, N2L 2Y5, Canada

Received 13 January 2012, revised 13 January 2012, accepted 14 January 2012

Published online 13 February 2012

Key words Killing–Yano tensors, integrability, Kerr-NUT-(A)dS black holes.

This is a short pedagogical introduction to the subject of Killing–Stäckel and Killing–Yano tensors and their role in the integrability of various types of equations that are of physical interest in curved space-time, the main application being higher dimensional rotating black holes with cosmological constant.

© 2012 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

1 Introduction

Much of recent years' research in theoretical physics has been devoted to understanding the link between quantum and gravitational physics. Different themes occur in this line of work, to mention some: string theory and the AdS/CFT correspondence, supergravity and supersymmetric theories, black holes and the study of their microscopic degrees of freedom and their stability.

In this brief report we are going to sketch the role of Killing–Stäckel and Killing–Yano (KY) tensors in the study of (quantum) physics in curved space-time, and provide some useful references. On one hand these objects have a direct relationship with the idea of symmetry of motion in a curved spacetime: the seminal work [1] of Carter in 1968 showed that geodesic trajectories in the Kerr geometry possess an unexpected constant of motion that is not related to isometries. It is related to a symmetry of dynamics that acts upon the full phase space but has no direct action on configuration space alone: a so called 'hidden symmetry'. Hidden symmetries are in correspondence with Killing–Stäckel tensors.

On the other hand, Floyd [2] and Penrose [3] showed that the Kerr Killing–Stäckel tensor admits a tensorial square root, a KY tensor, and Carter and McLenaghan [4] showed it can be used to build a differential operator that commutes with the Dirac operator in this metric, thus proving how special tensors can be used to study quantum mechanical problems for fields of non-zero spin. There is also an interesting connection with supersymmetric theories, as an appropriate semi-classical limit of the Dirac equation is the theory of the point particle with $\mathcal{N} = 1$ worldline supersymmetry. In this theory rank two KY tensors are associated to generators of additional supersymmetries [5].

In recent years interest in higher dimensions coming from string theory, braneworld models, study of black holes' hair, motivated the study of hidden symmetries in dimension greater than four. The Kerr-NUT-(A)dS black hole spacetime [6] which is a generalization of the Kerr metric to higher dimensions,

* Corresponding author E-mail: mc355@cantab.net, marco@iceb.ufop.br

** E-mail: Pavel.Krtous@utf.mff.cuni.cz

*** E-mail: dkubiznak@perimeterinstitute.ca

has horizon with spherical topology and is parametrised by spin, NUT charges and a cosmological constant. Therefore it lends itself to an analysis first as a black hole in its own merit, then in terms of a possible Kerr/CFT correspondence [7], and also in terms of the AdS/CFT duality. It is remarkable that such geometry possesses many of the important properties of the original Kerr metric: it has a KY tensor [8] and the Hamilton–Jacobi, Klein–Gordon and Dirac equations are separable [9, 10]. It is also possible to construct conserved quantities in this metric for the theory of the spinning particle that generalize the original one found by Carter to the case of non-zero spin [11].

The study of KY tensors can be generalized by looking for these special tensors in different types of metrics, see for example [12] for a connection between KY tensors and G-structures, or by generalizing the KY equation to metrics with fluxes, see [13–16] for applications.

Thus it is manifest how KY tensors link diverse topics such as black hole physics, special geometry, integrability, supersymmetry and supergravity, CFT type dualities; see also recent extended review [17].

2 Phase space and hidden symmetries

Let a manifold \mathcal{M} be a configuration space of dimension n , its co-tangent bundle $T^*(\mathcal{M})$ be the phase space with coordinates $y^a = \{x^\mu, p_\nu\}$, and $\omega = \omega_{ab} dy^a \wedge dy^b = dx^\mu \wedge dp_\mu$ be the natural symplectic form, with inverse components ω^{ab} . Given a scalar Hamiltonian function $H : T^*(\mathcal{M}) \mapsto \mathcal{R}$, the equations of motion are written as

$$\frac{dy^a}{d\tau} = \omega^{ab} \partial_b H := X_H^a. \quad (1)$$

The vector X_H is the *symplectic gradient* of the function H . Symplectic gradients are in one-to-one correspondence with (local) canonical transformations, which are (local) continuous transformations of $T^*(\mathcal{M})$ generated by a vector field X such that the symplectic form is left invariant, $\mathcal{L}_X \omega = 0$. So in particular the form of Hamilton's equations is invariant and volume in phase space is invariant.

Now consider a Killing vector K^μ : the phase-space quantity $C_K = K^\mu p_\mu$ is conserved as it can be checked that $\dot{C}_K = \{C_K, H\} = 0$ due to the Killing equation. Its corresponding symplectic gradient is

$$X_{C_K} = K^\mu \frac{\partial}{\partial x^\mu} - \frac{\partial K^\lambda}{\partial x^\mu} p_\lambda \frac{\partial}{\partial p_\mu}. \quad (2)$$

The pushforward of X_{C_K} under the canonical projection $\pi : T^*(\mathcal{M}) \rightarrow \mathcal{M}$ is given by $\pi_*(X_{C_K}) = K^\mu \frac{\partial}{\partial x^\mu} = K$. The geometrical interpretation is that the integral curves of X_{C_K} in phase space have a projection on configuration space, and this is given by the integral curves of K which are the isometries.

Next, a Killing–Stäckel tensor is totally symmetric tensor $K^{\mu_1 \dots \mu_p} = K^{(\mu_1 \dots \mu_p)}$ such that $\nabla^{(\lambda} K^{\mu_1 \dots \mu_p)} = 0$, which implies that $C_K = K^{\mu_1 \dots \mu_p} p_{\mu_1} \dots p_{\mu_p}$ is conserved. The symplectic gradient of C_K is

$$X_{C_K} = p K^{\mu_1 \dots \mu_{p-1} \nu} p_{\mu_1} \dots p_{\mu_{p-1}} \frac{\partial}{\partial x^\nu} - \frac{\partial K^{\mu_1 \dots \mu_p}}{\partial x^\nu} p_{\mu_1} \dots p_{\mu_p} \frac{\partial}{\partial p_\nu}. \quad (3)$$

It still generates a symmetry of phase space but this time the push forward to configuration space vanishes: $\pi_*(X_{C_K}) = 0$. So there is no visible action of the flux of X_{C_K} in configuration space. It is said that K generates a *hidden symmetry* of the dynamics. New non-trivial Killing–Stäckel tensors of order ≥ 3 have recently been built in [18, 19].

3 Killing–Yano tensors

The Killing equation can be generalized alternatively using a totally symmetric tensor $f_{\mu_1 \dots \mu_p} = f_{[\mu_1 \dots \mu_p]}$. The differential equation satisfied by f is

$$\nabla_\lambda f_{\mu_1 \dots \mu_p} + \nabla_{\mu_1} f_{\lambda \dots \mu_p} = 0. \quad (4)$$

When $p = 1$ this reduces to the Killing equation. Such a tensor is called a *Killing–Yano tensor*, and it was first introduced by Yano [20] from a purely mathematical point of view. The tensor

$$K_{\mu\nu} = f_{\mu\lambda_1\dots\lambda_{p-1}} f_{\nu}^{\lambda_1\dots\lambda_{p-1}}, \tag{5}$$

can be seen to be Killing–Stäckel, using Eq. (4). In order to be able to generate a conserved quantity using f we cannot contract f with factors of the momenta p_μ , we need appropriate antisymmetric variables: this can be done in the theory of the spinning particle and for the Dirac equation. For the latter it is possible to show that a KY tensor always allows to construct an operator that commutes with the Dirac operator, with no quantum anomaly [21]. KY tensors can also be used to construct conserved gravitational charges [22].

There exists a natural way to generalize the KY equation to an equation that is invariant under Hodge duality. A tensor satisfying this latter equation is called a *conformal Killing–Yano tensor*, and its square is a conformal Killing–Stäckel tensor. Since a KY tensor has zero divergence, then its Hodge dual is a closed form, called *closed conformal Killing–Yano tensor*. Such tensors satisfy a very useful property: they form an algebra under the wedge product. In particular, closed conformal Killing–Yano tensors of rank 2 that are non-degenerate are called *Principal conformal Killing–Yano (PCKY) tensors*. They are crucial for the integrability of various systems in four and higher dimensional black hole spacetimes.

4 Kerr-NUT-(A)dS black holes

While a classification of Lorentzian metrics with a PCKY tensor is not available, the analogue problem in Riemannian signature has been solved [23, 24]. The most general *canonical metric* admitting a PCKY tensor in $n = 2N + \varepsilon$ dimensions, $\varepsilon = 0, 1$, is given by

$$ds^2 = \sum_{\mu=1}^N \left[\frac{dx_\mu^2}{Q_\mu} + Q_\mu \left(\sum_{j=0}^{N-1} A_\mu^{(j)} d\psi_j \right)^2 \right] + \varepsilon S \left(\sum_{j=0}^N A^{(j)} d\psi_j \right)^2. \tag{6}$$

Here, coordinates x_μ ($\mu = 1, \dots, N$) stand for the (Wick rotated) radial coordinate and longitudinal angles, and Killing coordinates ψ_k ($k = 0, \dots, N-1+\varepsilon$) denote time and azimuthal angles associated with Killing vectors $\xi^{(k)} = \partial_{\psi_k}$. We have further defined the functions

$$Q_\mu = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\nu \neq \mu} (x_\nu^2 - x_\mu^2), \quad S = \frac{-c}{A^{(N)}}, \tag{7}$$

$$A_\mu^{(k)} = \sum_{\substack{\nu_1, \dots, \nu_k \\ \nu_1 < \dots < \nu_k, \nu_i \neq \mu}} x_{\nu_1}^2 \cdots x_{\nu_k}^2, \quad A^{(j)} = \sum_{\substack{\nu_1, \dots, \nu_k \\ \nu_1 < \dots < \nu_k}} x_{\nu_1}^2 \cdots x_{\nu_k}^2. \tag{8}$$

The quantities X_μ are functions of a single variable x_μ , and c is an arbitrary constant. The vacuum (with a cosmological constant) black hole geometry is recovered by setting

$$X_\mu = \sum_{k=\varepsilon}^N c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\varepsilon} + \frac{\varepsilon c}{x_\mu^2}. \tag{9}$$

This choice of X_μ describes the most general known Kerr-NUT-(A)dS spacetimes in all dimensions [6]. The constant c_N is proportional to the cosmological constant and the remaining constants are related to angular momenta, mass and NUT parameters.

The PCKY tensor reads [8]

$$h = db, \quad b = \frac{1}{2} \sum_{j=0}^{N-1} A^{(j+1)} d\psi_j. \tag{10}$$

$2j$ -forms $h^{(j)}$, which are the j -th wedge power of the PCKY tensor h , $h^{(j)} = h \wedge \dots \wedge h$, form the tower of associated closed conformal Killing–Yano tensors, and can be ‘squared’ to rank 2 Killing–Stäckel tensors.

5 Results on integrability for the canonical spacetimes

The canonical metric has a tower of $N + \varepsilon$ Killing vectors $\xi^{(k)}$ and N rank 2 Killing–Stäckel tensors $K^{(i)ab}$. It can be shown that these satisfy the geometric compatibility conditions to form a *separability structure* as discussed in [25, 26], and as a consequence both the Hamilton–Jacobi and Klein–Gordon equations are separable. Separability for both equations was obtained with a direct calculation in [9], then in the case of Hamilton–Jacobi it was associated to the separability structure in [27], and last for the Klein–Gordon equation it was shown to be associated to a set of commuting differential operators of first and second order associated to, respectively, Killing vectors and Killing tensors, that are simultaneously diagonalizable [28]:

$$\mathcal{L}^{(k)} = -i\xi^{(k)a}\nabla_a, \quad \mathcal{K}^{(i)} = -\nabla_a \left[K^{(i)ab}\nabla_b \right]. \quad (11)$$

The Nambu–Goto equations for a stationary string were integrated in [29].

When looking at the Dirac equation, a theory of necessary and sufficient conditions for separation of variables is not available. Separation of variables was obtained by explicit calculation in [10]. In [30] it was shown that the canonical metric admits a set of n mutually commuting operators defined on the Clifford bundle, one being the Dirac operator D . The other ones divide into $N + \varepsilon$ operators constructed from the Killing vectors $\xi^{(k)}$ and $N - 1$ operators constructed from $2j$ -forms $h^{(j)}$, according to

$$K_k = \xi^{(k)a}\nabla_a + \frac{1}{4}\gamma^{a_1 a_2}\nabla_{[a_1}\xi^{(k)}]_{a_2}, \quad (12)$$

$$M_j = \frac{1}{(2j)!} \left[\gamma^{ab_1 \dots b_{2j}} (h^{(j)})_{b_1 \dots b_{2j}} \nabla_a - \frac{2j(n-2j)}{2(n-2j+1)} \gamma^{b_1 \dots b_{2j-1}} (\delta h^{(j)})_{b_1 \dots b_{2j-1}} \right]. \quad (13)$$

These operators admit a common eigenfunction which displays separation of variables [31], and coincides with that found in [10].

Recently in [11] conserved quantities of order two in the momenta have been constructed for the theory of the spinning particle: these generalize the Killing–Stäckel charges of Sect. 2 to the case of non-zero spin.

Acknowledgements M.C. would like to thank the organizers of the “XVII European Workshop on String Theory”, held in September 2011 in Padova, Italy, and acknowledges partial financial support from Fundação de Amparo à Pesquisa de Minas Gerais – FAPEMIG.

References

- [1] B. Carter, *Phys. Rev.* **174**, 1559 (1968).
- [2] R. Floyd, *The Dynamics of Kerr Fields*, Ph.D. thesis (London University, London, 1973).
- [3] R. Penrose, *Ann. N.Y. Acad. Sci.* **224**, 125 (1973).
- [4] B. Carter and R. G. McLenaghan, *Phys. Rev. D* **19**, 1093 (1979).
- [5] G. W. Gibbons, R. H. Rietdijk and J. W. van Holten, *Nucl. Phys. B* **404**, 42 (1993), arXiv:hep-th/9303112.
- [6] W. Chen, H. Lü, and C. Pope, *Class. Quantum Gravity* **23**, 5323 (2006), hep-th/0604125.
- [7] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger, *Nucl. Phys. Proc. Suppl.* **216**, 194 (2011), arXiv:1103.2355.
- [8] D. Kubizňák and V. P. Frolov, *Class. Quantum Gravity* **24**, F1 (2007), arXiv:gr-qc/0610144.
- [9] V. P. Frolov, P. Krtouš, and D. Kubizňák, *J. High Energy Phys.* **02**, 005 (2007), arXiv:hep-th/0611245.
- [10] T. Oota and Y. Yasui, *Phys. Lett. B* **659**, 688 (2008) arXiv:0711.0078.
- [11] D. Kubizňák and M. Cariglia, to appear in *Phys. Rev. Lett.*, arXiv:1110.0495.
- [12] G. Papadopoulos, *Class. Quantum Gravity* **25**, 105016 (2008), arXiv:0712.0542.
- [13] D. Kubizňák, H. K. Kunduri, and Y. Yasui, *Phys. Lett. B* **678**, 240 (2009), arXiv:0905.0722.

- [14] T. Houri, D. Kubizňák, C. M. Warnick, and Y. Yasui, *Class. Quantum Gravity* **27**, 185019 (2010), arXiv:1002.3616.
- [15] T. Houri, D. Kubizňák, C. M. Warnick and Y. Yasui, *J. High Energy Phys.* **1007**, 055 (2010), arXiv:1004.1032.
- [16] G. Papadopoulos, arXiv:1111.6744.
- [17] T. Houri, and Y. Yasui, *Prog. Theor. Phys. Suppl.* **189**, 126 (2011), arXiv:1104.0852.
- [18] G. W. Gibbons, T. Houri, D. Kubizňák, and C. M. Warnick, *Phys. Lett. B* **700**, 68 (2011), arXiv:1103.5366.
- [19] G. W. Gibbons and C. Rugina, to appear in *JMP*, arXiv:1107.5987.
- [20] K. Yano, *Ann. Math.* **55**, 328 (1952).
- [21] M. Cariglia, *Class. Quantum Gravity* **21**, 1051 (2004), arXiv:hep-th/0305153.
- [22] D. Kastor and J. Traschen, *J. High Energy Phys.* **0408**, 045 (2004), arXiv:hep-th/0406052.
- [23] T. Houri, T. Oota, and Y. Yasui, *Phys. Lett. B* **656**, 214 (2007), arXiv:0708.1368.
- [24] V. P. Frolov, P. Krtouš, and D. Kubizňák, *Phys. Rev. D* **78**, 064022 (2008), arXiv:0804.4705.
- [25] S. Benenti and M. Francaviglia, *Gen. Relativ. Gravit.* **10**, 79 (1979).
- [26] E. G. Kalnins and W. Miller, *Adv. Math.* **51**, 91 (1984).
Available online at <http://www.ima.umn.edu/~miller/Rsep.pdf>.
- [27] T. Houri, T. Oota, and Y. Yasui, *J. Phys. A.* **41**, 025204 (2008), arXiv:0707.4039.
- [28] P. Krtouš and A. Sergyeyev, *Phys. Rev. D* **77**, 044033 (2008), arXiv:0711.4623.
- [29] V. P. Frolov and D. Kubizňák, *J. High Energy Phys.* **02**, 007 (2008), arXiv:0711.2300.
- [30] M. Cariglia, P. Krtouš, and D. Kubizňák, *Phys. Rev. D* **84**, 024004 (2011), arXiv:1102.4501.
- [31] M. Cariglia, P. Krtouš, and D. Kubizňák, *Phys. Rev. D* **84**, 024008 (2011), arXiv:1104.4123.