PHYSICAL REVIEW D **78,** 064022 (2008)

# Hidden symmetries of higher-dimensional black holes and uniqueness of the Kerr-NUT-(A)dS spacetime

Pavel Krtouš,[1,*] Valeri P. Frolov,[2,+] and David Kubizňák[2,1,‡]

[1] *Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University in Prague, V Holešovičkách 2, Prague, Czech Republic*

[2] *Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada T6G 2G7*

(Received 29 April 2008; published 9 September 2008)

We prove that the most general solution of the Einstein equations with the cosmological constant which admits a principal conformal Killing-Yano tensor is the Kerr-NUT-(A)dS metric. Even when the Einstein equations are not imposed, any spacetime admitting such hidden symmetry can be written in a canonical form which guarantees the following properties: it is of the Petrov type D, it allows the separation of variables for the Hamilton-Jacobi, Klein-Gordon, and Dirac equations, the geodesic motion in such a spacetime is completely integrable. These results naturally generalize the results obtained earlier in four dimensions.

DOI: 10.1103/PhysRevD.78.064022 PACS numbers: 04.50.Gh, 04.20.Jb, 04.50.−h, 04.70.Bw

## I. INTRODUCTION

Higher-dimensional black holes which might play the role of natural probes of extra dimensions are being discussed intensively at present (see, e.g., the review [1] and references therein). Recently, the subject of hidden symmetries of higher-dimensional black hole metrics has become of high interest. By studying the hidden symmetries it was demonstrated that higher-dimensional black holes are in many aspects similar to their four-dimensional "cousins."

Explicit spacetime symmetries are represented by Killing vectors. Hidden symmetries are related to generalizations of this concept. One of the most important of these generalizations is the hidden symmetry encoded in the *principal conformal Killing-Yano (CKY) tensor* [2,3]. It was demonstrated that the Myers-Perry metric [4], describing the higher-dimensional rotating black hole, as well as its generalization, the Kerr-NUT-(A)dS metric [5] which includes the NUT parameters and the cosmological constant, admit such a tensor [6].

The principal CKY tensor generates hidden symmetries which are responsible for the existence of nonreducible quadratic in momenta conserved integrals of motion for geodesic motion. In consequence, it was shown that geodesic equations in the Kerr-NUT-(A)dS spacetime are completely integrable [2,7]. Moreover, the Hamilton-Jacobi, Klein-Gordon, and Dirac equations are separable in this spacetime [8–10]. This spacetime is of the algebraic type D [11,12] of the higher-dimensional generalization of the Petrov classification [13] and can be presented in the generalized Kerr-Schild form [14]. All these remarkable properties make higher-dimensional black hole solutions very similar to the 4D black holes (see, e.g., [15,16] for a review).

The study of 4D metrics admitting the principal CKY tensor showed that the corresponding class of solutions of the Einstein equations with the cosmological constant reduces to the Kerr-NUT-(A)dS spacetime [17]. The aim of this paper is to demonstrate that in the higher-dimensional gravity the situation is similar. Namely, we shall prove the following results: The metric of any higher-dimensional spacetime which admits a principal CKY tensor can be put into a special *canonical form* which guarantees the following properties: (1) it is of the algebraic type D, (2) it allows a separation of variables for the Hamilton-Jacobi, Klein-Gordon, and Dirac equations, and (3) the geodesic motion in such a spacetime is completely integrable. When the Einstein equations with the cosmological constant are imposed the canonical form becomes the Kerr-NUT-(A)dS metric [5].

Similar results were proved recently for spacetimes with a principal CKY tensor obeying special additional restrictions [18]. The conditions imposed, however, might have seemed very restrictive. In this paper we demonstrate that it is not so. Namely, we demonstrate that these restrictions are not necessary and that they in fact automatically follow from the properties of the principal CKY tensor. In particular, this means that all the results [2,7–11] (proved for the canonical form of the metric) immediately follow from the very existence of the principal CKY tensor. It also means that one cannot hope to arrive at more general spacetimes by relaxing the additional assumptions of [18].

*Pavel.Krtous@utf.mff.cuni.cz
+frolov@phys.ualberta.ca
‡kubiznak@phys.ualberta.ca

 

Here we present only a sketch of the proof of our results and for simplicity restrict to an even dimension $D = 2n$. All technical details, including the odd-dimensional case, will be discussed in [19].

## II. DEFINITIONS AND ASSUMPTIONS

A principal conformal Killing-Yano tensor is defined as a closed nondegenerate 2-form $\boldsymbol{h}$ obeying the following equation:

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a. \tag{1}$$

This equation implies

$$\nabla_{[a}h_{bc]} = 0, \qquad \xi_a = \frac{1}{D-1}\nabla^n h_{na}. \tag{2}$$

In what follows we shall assume that $\boldsymbol{\xi} \neq 0$. The case when $\boldsymbol{\xi} = 0$, and hence $\boldsymbol{h}$ is covariantly constant, requires a special consideration. The condition of nondegeneracy in an even dimension means that the skew symmetric matrix $h_{ab}$ in the $D = 2n$ dimensional spacetime has the matrix rank $2n$.

As any 2-form on a metric space, the principal CKY tensor $\boldsymbol{h}$ determines an orthonormal [20] *Darboux basis* which simultaneously diagonalizes the metric $\boldsymbol{g}$ and "skew-diagonalizes" $\boldsymbol{h}$. Namely, there exists a frame $\{\boldsymbol{e}^\mu, \hat{\boldsymbol{e}}^\mu\}$, $\mu = 1, \ldots, n$, of 1-forms in which the metric and the principal CKY tensor are

$$\boldsymbol{g} = \sum_\mu (\boldsymbol{e}^\mu\boldsymbol{e}^\mu + \hat{\boldsymbol{e}}^\mu\hat{\boldsymbol{e}}^\mu), \qquad \boldsymbol{h} = \sum_\mu x_\mu \boldsymbol{e}^\mu \wedge \hat{\boldsymbol{e}}^\mu. \tag{3}$$

(We do not use the summation convention for Greek indices—all sums over them are indicated explicitly and run in the range $1, \ldots, n$ unless stated otherwise.)

We denote by $\check{\boldsymbol{h}}$ the operator with components $h^a{}_b$ and by "$\cdot$" a symbol for contraction. For example, $\check{\boldsymbol{h}} \cdot \check{\boldsymbol{h}} \cdot \boldsymbol{v}$ denotes a vector with the components $h^a{}_b h^b{}_c v^c$. The operator $\check{\boldsymbol{h}}$ is antisymmetric with respect to the metric scalar product. This means that the operator $-\check{\boldsymbol{h}}^2 = -\check{\boldsymbol{h}} \cdot \check{\boldsymbol{h}}$ is a non-negative definite symmetric operator, and its eigenvectors are given by vectors $\{\boldsymbol{e}_\mu, \hat{\boldsymbol{e}}_\mu\}$ of the vector frame dual to the Darboux basis

$$-\check{\boldsymbol{h}}^2 \cdot \boldsymbol{e}_\mu = x_\mu^2 \boldsymbol{e}_\mu, \qquad -\check{\boldsymbol{h}}^2 \cdot \hat{\boldsymbol{e}}_\mu = x_\mu^2 \hat{\boldsymbol{e}}_\mu. \tag{4}$$

As discussed below, we assume that the eigenspaces corresponding to each eigenvalue $x_\mu^2 > 0$ are two-dimensional and we call them the *Killing-Yano (KY) 2-planes*.

It is convenient to introduce also a basis of complex null eigenvectors $\{\boldsymbol{m}_\mu, \bar{\boldsymbol{m}}_\mu\}$ which obey the relations

$$\check{\boldsymbol{h}} \cdot \boldsymbol{m}_\mu = -ix_\mu \boldsymbol{m}_\mu, \qquad \check{\boldsymbol{h}} \cdot \bar{\boldsymbol{m}}_\mu = ix_\mu \bar{\boldsymbol{m}}_\mu, \tag{5}$$

with a bar denoting the complex conjugation. These complex null vectors satisfy the normalization

$$\boldsymbol{m}_\mu \cdot \boldsymbol{m}_\nu = \bar{\boldsymbol{m}}_\mu \cdot \bar{\boldsymbol{m}}_\nu = 0, \qquad \boldsymbol{m}_\mu \cdot \bar{\boldsymbol{m}}_\nu = \delta_{\mu\nu}, \tag{6}$$

and they are connected with vectors $\{\boldsymbol{e}_\mu, \hat{\boldsymbol{e}}_\mu\}$ from Eq. (4) as follows:

$$\boldsymbol{m}_\mu = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_\mu + i\boldsymbol{e}_\mu), \qquad \bar{\boldsymbol{m}}_\mu = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_\mu - i\boldsymbol{e}_\mu). \tag{7}$$

## III. UNIQUENESS OF THE KERR-NUT-(A)DS METRIC

Passing from a local description of the principal CKY tensor $\boldsymbol{h}$ at a chosen point to the description of a spacetime properties in some domain, we include into the notion of the principal CKY tensor the following requirement: the "eigenvalues" $x_\mu$ of $\boldsymbol{h}$ are functionally independent in some spacetime domain, that is we assume that $x_\mu$ are nonconstant independent scalar functions with different values at a generic point.

This also allows us to use $x_\mu$'s as coordinates. As a part of our result we demonstrate that these $n$ coordinates can be upgraded by adding $n$ new coordinates $\psi_i$ so that the metric and the principal CKY tensor take the form

$$\boldsymbol{g} = \sum_\mu \left[\frac{1}{Q_\mu}\mathbf{d}x_\mu\mathbf{d}x_\mu + Q_\mu\left(\sum_{i=0}^{n-1}A^i_\mu\mathbf{d}\psi_i\right)\left(\sum_{j=0}^{n-1}A^j_\mu\mathbf{d}\psi_j\right)\right],$$

$$\boldsymbol{h} = \mathbf{d}\boldsymbol{b}, \qquad \boldsymbol{b} = \frac{1}{2}\sum_{j=0}^{n-1}A^{j+1}\mathbf{d}\psi_j. \tag{8}$$

Here, functions $A^i$, $A^i_\mu$, and $U_\mu$ are particular combinations of $x_\mu$'s,

$$\begin{aligned} A^i_\mu &= \sum_{\substack{\nu_1,\ldots,\nu_i \\ \nu_1<\ldots<\nu_i \\ \nu_j\neq\mu}} x^2_{\nu_1}\ldots x^2_{\nu_i}, \\ A^i &= \sum_{\substack{\nu_1,\ldots,\nu_i \\ \nu_1<\ldots<\nu_i}} x^2_{\nu_1}\ldots x^2_{\nu_i}, \\ U_\mu &= \prod_{\substack{\nu \\ \nu\neq\mu}}(x^2_\nu - x^2_\mu), \end{aligned} \tag{9}$$

and metric functions $Q_\mu$ are given by

$$Q_\mu = \frac{X_\mu}{U_\mu}, \qquad X_\mu = X_\mu(x_\mu), \tag{10}$$

with $X_\mu$ depending only on a single coordinate $x_\mu$. For odd number of dimensions the metric (8) contains few extra terms and its form can be found, e.g., in [5,11].

We call (8)–(10) the *canonical* form of the metric. It can be considered as a higher-dimensional generalization of the form of the metric constructed by Carter in four dimensions [21]. In what follows we demonstrate that coordinates $(x_\mu, \psi_i)$ used in (8) have a well-defined geometrical meaning—determined completely by the principal CKY tensor. It should be emphasized that this canonical form follows from the existence of the principal CKY tensor *off-shell*, that is without imposing the Einstein equations. When the vacuum Einstein equations with the cosmological constant are imposed, the metric (8) turns out to be the Kerr-NUT-(A)dS metric [5,11] for which one has

$$X_\mu = b_\mu x_\mu + \sum_{k=0}^{n} c_k x_\mu^{2k}. \tag{11}$$

The constants $c_k$ and $b_\mu$ are related to the cosmological constant, angular momenta, mass, and NUT charges; see, e.g., [5] for details. For $b_\mu = 0$ we obtain the constant curvature space.

As mentioned in Introduction, the uniqueness of the Kerr-NUT-(A)dS metric has been already studied in [18] where it was proved provided the following additional assumptions:

$$\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{h} = 0, \qquad \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{g} = 0. \tag{12}$$

That is, the authors of [18] explicitly required, that $\boldsymbol{\xi}$ is a Killing vector and that the principal CKY tensor $\boldsymbol{h}$ does not change along $\boldsymbol{\xi}$. We prove now that both of these conditions are superfluous since they follow from the existence of the principal CKY tensor. The proof of the first condition is presented below, the second condition is a corollary of our explicit construction of the canonical form (8) of the off-shell metric.

## IV. CONDITION ON THE PRINCIPAL CKY TENSOR

First, we concentrate on the first condition in (12). Let us denote $D_\mu = \boldsymbol{\nabla}_{\boldsymbol{m}_\mu}$ and $\bar{D}_\mu = \boldsymbol{\nabla}_{\bar{\boldsymbol{m}}_\mu}$. Using (1) one has

$$(D_\mu \check{\boldsymbol{h}}) \cdot \boldsymbol{m}_\nu = (\boldsymbol{m}_\nu \cdot \boldsymbol{\xi})\, \boldsymbol{m}_\mu. \tag{13}$$

Applying $D_\mu$ to (5) and using (13), one obtains

$$(\check{\boldsymbol{h}} + i x_\nu \boldsymbol{\delta}) \cdot D_\mu \boldsymbol{m}_\nu + i (D_\mu x_\nu)\, \boldsymbol{m}_\nu + (\boldsymbol{m}_\nu \cdot \boldsymbol{\xi})\, \boldsymbol{m}_\mu = 0. \tag{14}$$

By taking a scalar product of (14) with $\bar{\boldsymbol{m}}_\nu$, using antisymmetry of $\check{\boldsymbol{h}}$ and Eq. (5) again, the first term cancels out. Considering two cases when $\nu = \mu$ and when $\nu \neq \mu$, one gets

$$D_\mu x_\nu = 0 \quad \text{for } \nu \neq \mu, \qquad D_\mu x_\mu = i \boldsymbol{m}_\mu \cdot \boldsymbol{\xi}. \tag{15}$$

Let us define functions $Q_\mu$ in terms of magnitudes of complex quantities $D_\mu x_\mu$

$$Q_\mu = 2 |D_\mu x_\mu|^2, \qquad D_\mu x_\mu = \frac{1}{\sqrt{2}} \sqrt{Q_\mu}\, e^{i\alpha}. \tag{16}$$

The orthonormal Darboux basis is not fixed by conditions (3) uniquely. There remains a freedom of a rotation in each KY 2-plane, which in terms of the null basis (7) reads $\boldsymbol{m}_\mu \to \exp(i\varphi_\mu) \boldsymbol{m}_\mu$. We uniquely fix the Darboux basis by setting the phase factor $\alpha = \pi/2$. Then, we have

$$D_\mu x_\mu = i \frac{1}{\sqrt{2}} \sqrt{Q_\mu}. \tag{17}$$

Using (15) and (17) we find

$$\boldsymbol{\xi} = \frac{1}{\sqrt{2}} \sum_\mu \sqrt{Q_\mu} (\boldsymbol{m}_\mu + \bar{\boldsymbol{m}}_\mu) = \sum_\mu \sqrt{Q_\mu}\, \hat{\boldsymbol{e}}_\mu. \tag{18}$$

Equations (15) and (17) also give us that the gradient $\mathbf{d} x_\mu$ of the eigenvalue function $x_\mu$ is proportional to $\boldsymbol{e}^\mu$,

$$\mathbf{d}\, x_\mu = \sqrt{Q_\mu}\, \boldsymbol{e}^\mu. \tag{19}$$

A simple calculation employing Eqs. (3), (8), and (19) shows that

$$\boldsymbol{\xi} \cdot \boldsymbol{h} = -\sum_\mu x_\mu \sqrt{Q_\mu}\, \boldsymbol{e}^\mu = \mathbf{d}\Big( -\frac{1}{2} \sum_\mu x_\mu^2 \Big). \tag{20}$$

With the help of the fact that this 1-form is exact and using the closeness of $\boldsymbol{h}$, we immediately obtain the desired relation

$$\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{h} = \boldsymbol{\xi} \cdot \mathbf{d}\boldsymbol{h} + \mathbf{d}(\boldsymbol{\xi} \cdot \boldsymbol{h}) = 0. \tag{21}$$

## V. KILLING VECTOR CONDITION FOR THE EINSTEIN SPACES

The second condition in Eq. (12), which states that $\boldsymbol{\xi}$ is a Killing vector, can be easily proved if the Einstein equations are imposed. It was demonstrated in [3] that

$$\nabla_{(a} \xi_{b)} = \frac{1}{D-2} R_{n(a} h_{b)}{}^n. \tag{22}$$

For spaces satisfying the Einstein vacuum equations with the cosmological constant, we have the Ricci tensor proportional to the metric and, thanks to the antisymmetry of $\boldsymbol{h}$, we immediately get $\nabla_{(a}\xi_{b)} = 0$, that is $\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{g} = 0$. Thus, on-shell the conditions (12) are valid and using the results of [18] one can derive that the metric represents the Kerr-NUT-(A)dS spacetime.

## VI. CONSTRUCTION OF THE CANONICAL FORM OF THE METRIC

If we do not impose the Einstein equations, it is not a straightforward task to prove that $\boldsymbol{\xi}$ is a Killing vector. Therefore, instead we proceed in a different way—we prove directly the existence of coordinates $\psi_j$ and show that the metric can be written in the canonical form (8)–(10). Here we sketch only main steps, the details will be discussed in a more technical paper [19].

First, taking all projections of Eq. (14), we collect partial information about the Ricci coefficients. For example, we obtain that only those Ricci coefficients with at least two indices equal are nonvanishing. Next, using $\boldsymbol{\xi}\cdot\mathbf{d}x_\mu = 0$ we can calculate the Lie derivative of $\boldsymbol{e}^\mu$ in terms of function $\hat{q}_\mu = \boldsymbol{\xi}\cdot\mathbf{d}(\ln\sqrt{Q_\mu})$. Using duality relations and action of the principal CKY tensor, we find

$$\mathcal{L}_\xi \boldsymbol{e}_\mu = \hat{q}_\mu \boldsymbol{e}_\mu + \sum_\nu E^\nu_\mu \hat{\boldsymbol{e}}_\nu, \qquad \mathcal{L}_\xi \hat{\boldsymbol{e}}_\mu = -\hat{q}_\mu \hat{\boldsymbol{e}}_\mu, \tag{23}$$

where $E^\nu_\mu$ are yet unspecified components. Expressing these Lie derivatives using covariant derivatives gives additional information about the Ricci coefficients and determines the components $E^\nu_\mu$ in terms of the Ricci coefficients and derivatives of $Q_\mu$. It also guarantees $\hat{\boldsymbol{e}}_\nu\cdot\mathbf{d}Q_\mu = 0$ for $\mu\neq\nu$ and $\hat{q}_\mu = \hat{\boldsymbol{e}}_\mu\cdot\mathbf{d}\sqrt{Q_\mu}$. These facts allow us to calculate the Lie brackets among all vectors $\boldsymbol{e}_\mu, \hat{\boldsymbol{e}}_\mu$ of the Darboux basis. They do not commute, with the exception of "hatted" ones: $[\hat{\boldsymbol{e}}_\mu, \hat{\boldsymbol{e}}_\nu] = 0$.

Now, we introduce a new basis $\{\boldsymbol{\epsilon}_\mu, \hat{\boldsymbol{\epsilon}}_j\}$, $\mu = 1,\ldots,n$, $j = 0,\ldots,n-1$,

$$\boldsymbol{\epsilon}_\mu = \frac{1}{\sqrt{Q_\mu}}\boldsymbol{e}_\mu, \qquad \hat{\boldsymbol{\epsilon}}_j = \sum_\mu A^j_\mu \sqrt{Q_\mu}\hat{\boldsymbol{e}}_\mu, \tag{24}$$

with $A^j_\mu$ given by (9). The geometrical meaning of $\hat{\boldsymbol{\epsilon}}_j$ can be elucidated by observing that $\hat{\boldsymbol{\epsilon}}_j = \boldsymbol{K}_j\cdot\boldsymbol{\xi}$, where $\boldsymbol{K}_j$ is the $j$th Killing tensor in the tower of 2-rank Killing tensors built from the principal CKY tensor in [2].

Using the known Ricci coefficients and the Jacobi identity, we can prove that vectors of this frame do commute:

$$[\boldsymbol{\epsilon}_\mu, \boldsymbol{\epsilon}_\nu] = [\boldsymbol{\epsilon}_\mu, \hat{\boldsymbol{\epsilon}}_j] = [\hat{\boldsymbol{\epsilon}}_i, \hat{\boldsymbol{\epsilon}}_j] = 0. \tag{25}$$

Moreover, for the dual frame

$$\boldsymbol{\epsilon}^\mu = \sqrt{Q_\mu}\boldsymbol{e}^\mu = \mathbf{d}x_\mu, \qquad \hat{\boldsymbol{\epsilon}}^i = \sum_\mu \frac{(-x_\mu^2)^{n-1-i}}{U_\mu\sqrt{Q_\mu}}\hat{\boldsymbol{e}}^\mu, \tag{26}$$

we show

$$\mathbf{d}\,\boldsymbol{\epsilon}^\mu = 0, \qquad \mathbf{d}\hat{\boldsymbol{\epsilon}}^j = 0. \tag{27}$$

Both conditions (25) and (27) ensure that additionally to $x_\mu$, $\mu = 1,\ldots,n$, it is possible to introduce coordinates $\psi_j$, $j = 0,\ldots,n-1$, such that

$$\begin{aligned} \boldsymbol{\epsilon}_\mu &= \boldsymbol{\partial}_{x_\mu}, \qquad \hat{\boldsymbol{\epsilon}}_i = \boldsymbol{\partial}_{\psi_i} \quad \text{and} \\ \boldsymbol{\epsilon}^\mu &= \mathbf{d}x_\mu, \qquad \hat{\boldsymbol{\epsilon}}^i = \mathbf{d}\psi_i. \end{aligned} \tag{28}$$

Taking into account the inverse of Eqs. (26) we get

$$\boldsymbol{e}^\mu = \frac{1}{\sqrt{Q_\mu}}\mathbf{d}x_\mu, \qquad \hat{\boldsymbol{e}}^\mu = \sqrt{Q_\mu}\sum_{i=0}^{n-1} A^i_\mu \mathbf{d}\psi_i. \tag{29}$$

Substituting (29) into (3) leads to the metric (8) with unspecified metric functions $Q_\mu$. However, in the process, we also learn that metric functions $Q_\mu$ must take the form (10), particularly that $\hat{q}_\mu = 0$ and $E^\nu_\mu = 0$. This finishes the proof of our main result: we have constructed a coordinate system in which the off-shell metric takes the canonical form (8)–(10), starting only from the quantities determined by the principal CKY tensor.

As a consequence, we have also established that $\boldsymbol{\xi}$ is a Killing vector which we call the *primary* one. Thus we proved both conditions (12) without employing the Einstein equations. Actually, all vectors $\boldsymbol{\partial}_{\psi_j}$ are Killing vectors—obtained by the action of the Killing tensors $\boldsymbol{K}_j$ on the primary Killing vector $\boldsymbol{\xi} = \boldsymbol{\partial}_{\psi_0}$. They coincide with vectors used in [18].

## VII. SUMMARY

As we have already mentioned, several important results were earlier obtained for the general off-shell metric in a canonical form (8)–(10). Namely, this metric is of the type D [11,12]. It allows separation of variables for the Hamilton-Jacobi, Klein-Gordon, and Dirac equations [8,10]. Geodesic equations are completely integrable [7] and there exists a complete set of integrals of motion which are linear and quadratic in momenta [2]. Since the canonical form of the metric follows from the existence of the principal CKY tensor, all these properties are common for spacetimes which admit such a tensor.

In our consideration we have focused on a generic case when the principal CKY tensor is nondegenerate. In a degenerate case some of the eigenspaces of $\boldsymbol{h}$ may have more than 2 dimensions. We have also focused on a Euclidean form of the metric. After the Wick's rotation which transforms the Euclidean metric to the Lorentzian one, it may happen that some of the coordinates $x_\mu$ become null. Additional degeneracy may be created when the primary Killing vector $\boldsymbol{\xi}$ vanishes. All these special degenerate cases require additional consideration. See [22] for a recent discussion of some of these cases.

## ACKNOWLEDGMENTS

P. K. is supported by the Grants No. GAČR 202/08/0187 and No. MŠMT ČR LC06014, and appreciates the hospitality of the University of Alberta. V. F. thanks the Natural Sciences and Engineering Research Council of Canada and the Killam Trust for financial support. D. K. acknowledges the University of Alberta for financial support. We have benefited from discussions with Don N. Page.

[1] R. Emparan and H. S. Reall, arXiv:0801.3471 [Living Rev. Relativity (to be published)].
[2] P. Krtouš, D. Kubizňák, D. N. Page, and V. P. Frolov, J. High Energy Phys. 02 (2007) 004.
[3] T. Kashiwada, Nat. Sci. Rep. Ochanomizu Univ. **19**, 67 (1968); S. Tachibana, Tohoku Math. J. **21**, 56 (1969).
[4] R. C. Myers and M. J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).
[5] W. Chen, H. Lü, and C. N. Pope, Classical Quantum Gravity **23**, 5323 (2006).
[6] V. P. Frolov and D. Kubizňák, Phys. Rev. Lett. **98**, 011101 (2007); D. Kubizňák and V. P. Frolov, Classical Quantum Gravity **24**, F1 (2007).
[7] D. N. Page, D. Kubizňák, M. Vasudevan, and P. Krtouš, Phys. Rev. Lett. **98**, 061102 (2007); P. Krtouš, D. Kubizňák, D. N. Page, and M. Vasudevan, Phys. Rev. D **76**, 084034 (2007).
[8] V. P. Frolov, P. Krtouš, and D. Kubizňák, J. High Energy Phys. 02 (2007) 005.
[9] A. Sergyeyev and P. Krtouš, Phys. Rev. D **77**, 044033 (2008).
[10] T. Oota and Y. Yasui, Phys. Lett. B **659**, 688 (2008).
[11] N. Hamamoto, T. Houri, T. Oota, and Y. Yasui, J. Phys. A **40**, F177 (2007).
[12] V. Pravda, A. Pravdová, and M. Ortaggio, Classical Quantum Gravity **24**, 4407 (2007).
[13] A. Coley, R. Milson, V. Pravda, and A. Pravdová, Classical Quantum Gravity **21**, L35 (2004); R. Milson, A. Coley, V. Pravda, and A. Pravdová, Int. J. Geom. Methods Mod. Phys. **2**, 41 (2005); A. Coley, Classical Quantum Gravity **25**, 033001 (2008).
[14] W. Chen and H. Lü, Phys. Lett. B **658**, 158 (2008).
[15] V. P. Frolov and D. Kubizňák, Classical Quantum Gravity **25**, 154005 (2008).
[16] V. P. Frolov, Prog. Theor. Phys. Suppl. **172**, 210 (2008).
[17] W. Dietz and R. Rüdiger, Proc. R. Soc. A **375**, 361 (1981); P. Taxiarchis, Gen. Relativ. Gravit. **17**, 149 (1985).
[18] T. Houri, T. Oota, and Y. Yasui, Phys. Lett. B **656**, 214 (2007); T. Houri, T. Oota, and Y. Yasui, J. Phys. A **41**, 025204 (2008).
[19] P. Krtouš, V. P. Frolov, and D. Kubizňák (unpublished).
[20] The orthonormality condition is tailored for a Euclidian metric. For a Lorentzian metric we can use the same normalization assuming one of the frame vectors to be imaginary. For simplicity, we restrict in our exposition to such a formal Euclidian signature, which is customary in the context of the higher-dimensional Kerr-NUT-(A)dS metric, cf., Refs. [2,5,7,11].
[21] B. Carter, Phys. Lett. **26A**, 399 (1968); B. Carter, Commun. Math. Phys. **10**, 280 (1968).
[22] T. Houri, T. Oota and Y. Yasui, arXiv:0805.3877; arXiv:0805.0838.