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On a lower-dimensional Killing vector origin of irreducible Killing tensors

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Abstract: Considering a spacetime foliated by co-dimension-2 hypersurfaces, we find the conditions under which lower-dimensional symmetries of a base space can be lifted up to irreducible Killing tensors of the full spacetime. In this construction, the key ingredient for irreducibility is the non-commutativity of the underlying Killing vectors. It gives rise to a tower of growing rank Killing tensors determined by the structure constants of the corresponding Lie algebra. A canonical example of a metric with such emergent non-trivial hidden symmetries in all dimensions is provided by rotating (off-shell) generalized Lense-Thirring spacetimes, where the irreducible Killing tensors arise from the underlying spherical symmetry of the base space. A physical on-shell realization of this construction in four dimensions is embodied by a rotating black hole in the Einstein-Maxwell-Dilaton-Axion theory. Further examples of equal spinning Myers-Perry spacetimes and spacetimes built on planar and Taub-NUT base metrics are also discussed.

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Contents

1 Introduction **1**

2 Lifting symmetries from codimension-2 hypersurfaces **3**
- 2.1 Metric ansatz 3
- 2.2 Lifting the SN bracket 4
- 2.3 Rank-2 irreducible Killing tensors from lower-dimensional Killing vectors 6
- 2.4 Axisymmetric spacetimes 7
- 2.5 Growing tower of higher rank Killing tensors 10

3 Generalized Lense-Thirring spacetimes **11**
- 3.1 Introducing the metric 12
- 3.2 Rank-2 Killing tensors and their brackets 13
- 3.3 Comparison with previous results 16
- 3.4 Exact on-shell realization 19

4 More examples **20**
- 4.1 Kerr-(A)dS metrics with equal rotation parameters 20
- 4.2 Planar symmetry 23
- 4.3 Lifting the Taub-NUT metric 26

5 Discussion and summary **26**

A Foliation of spacetime by codimension-2 hypersurfaces **28**
- A.1 Construction 28
- A.2 Gauge fixing 29

B Alternative construction **30**
- B.1 Dual metric ansatz 30
- B.2 Comparison with Taub-NUT metrics 31

C Schouten-Nijenhuis brackets **32**

1 Introduction

Symmetries play an important role in the study of dynamical systems. In general relativity, *explicit* spacetime symmetries are characterized by Killing vector fields, which generate conserved quantities and simplify the analysis of geodesic motion and field equations. However, beyond these explicit symmetries, certain spacetimes exhibit *hidden symmetries*, encoded in higher-rank tensor fields known as (symmetric) *Killing tensors* [1] and (antisymmetric) *Killing-Yano tensors* [2]. Such symmetries are known to underlie remarkable properties of rotating black holes described by Kerr geometry [3] and its higher-dimensional generalizations [4–6], see e.g. [7–11] for details and references. In this work we focus on hidden symmetries encoded in Killing tensors.

A Killing tensor of rank-p is a completely symmetric tensor, $K^{\mu_1\cdots\mu_p} = K^{(\mu_1\cdots\mu_p)}$, obeying the following Killing tensor equation:

$$\nabla^{(\nu} K^{\mu_1\cdots\mu_p)} = 0\,. \tag{1.1}$$

Incorporating Killing vectors as $p = 1$ subcase, Killing tensors are in one-to-one correspondence with monomial constants of geodesic motion — obtained by contracting all indices of the Killing tensor with the corresponding geodesic momenta p_μ, $p^\nu \nabla_\nu p^\mu = 0$, namely

$$C_p = K^{\mu_1\cdots\mu_p} p_{\mu_1} \dots p_{\mu_p}\,. \tag{1.2}$$

The Poisson bracket on these constants induces the so called *Schouten-Nijenhuis (SN) bracket* [12, 13] on Killing tensors (see appendix C). This gives rise to an algebra of Killing tensors (see refs. [14–21] for details on the algebraic structure), and in principle allows one to generate new (higher rank) Killing tensors via the SN bracket. If, for a geodesic motion in d spacetime dimensions, one can find d functionally *independent* and mutually Poisson commuting constants of motion (1.2), the motion is *completely integrable*.

To capture the 'independence' of the corresponding Killing tensors, the following notions have often been used. Killing tensors are called *reducible*, if they can be decomposed as a linear combination of a symmetrized product of lower rank Killing tensors. Otherwise they are called *irreducible* [1, 14]. Although this terminology is not particularly useful when it comes to the integrability purposes[1] we will keep using it in this work.

Recently, in a series of papers [23–25], it was argued that 'magic square' *Lense-Thirring spacetimes*, first introduced in [26, 27] (see also [28]), admit a growing tower of higher-rank Killing tensors. These tensors, constructed via the SN bracket, form a growing (quadratically with the number of spacetime dimensions) sequence of Killing tensors, giving rise to a spacetime with more hidden symmetries than explicit ones. While these results suggest a deep underlying algebraic structure, a clear geometric and physical explanation of this phenomenon remains an open question.

In this work, we aim to demystify these results by proposing a systematic construction of irreducible Killing tensors from lower-dimensional Killing vectors. Similar such ideas have appeared earlier. We refer the reader to [29, 30] for work on conditions under which Killing initial data (i.e. Killing vectors on a hypersurface-spacelike or characteristic) extend to the full spacetime using the Einstein equations. On the other hand, the works [31, 32] have considered a foliation of co-dimension-1 hypersurfaces and a geometric procedure to lift symmetries of the hypersurface to the full spacetime. Similarly, in [33] the Killing tensors

[1]Reducible Killing tensors obviously do not imply the existence of new independent constants of geodesic motion and thence can be discarded. However, the above notion of irreducibility itself does not imply that the corresponding constants of geodesic motion are automatically functionally independent (as complicated non-polynomial relations may exist among such constants). Note also that, whereas the upper bound on independent constants of motion in d dimensions is $2d - 1$ (corresponding to a *maximally superintegrable system*, see e.g. [22]), the number of rank-p Killing tensors can be larger, see (2.42) below.

of the off shell Kerr-NUT-AdS class of spacetimes have been shown to be Killing tensors of the induced metric on the t =const. co-dimension-1 foliation. See also [34, 35] for similar ideas (the latter with an application to Killing-Yano tensors), [36, 37] for studies of the lift of hidden symmetries on warped spacetimes, and [38] for the Eisenhart lift of lower-dimensional hidden symmetries to higher-rank Killing tensors.

In contrast, we consider a spacetime in which an off-shell geometry is foliated by *codimension-2 hypersurfaces* and demonstrate that the *non-commutativity* of Killing vectors in the lower-dimensional base space comes into play when generating a hierarchy of Killing tensors. The structure constants of the underlying Lie algebra dictate the emergence of a growing tower of Killing tensors in the full spacetime. To illustrate this general construction, we explore its realization in the aforementioned generalized Lense-Thirring spacetimes. Furthermore, we identify an explicit on-shell realization of our framework in the context of *Einstein-Maxwell-Dilaton-Axion (EMDA)* black hole solutions [39]. We also show that the well-known $\mathrm{U}([d-1]/2)$ symmetry of the Myers-Perry black holes in d dimensions with equal rotation parameters [40] can be understood as arising from the Killing vector symmetries of the base space. Finally, to further illustrate the construction we consider two (mathematical) examples: a lifting of a planar metric and the Euclidean Taub-NUT instanton to a Lorentzian spacetime in two dimensions higher.

The structure of this paper is as follows. In section 2, we introduce our metric ansatz and outline the formalism for lifting Killing vectors to Killing tensors. In section 3, we apply our framework to generalized Lense-Thirring spacetimes — demonstrating the construction of Killing tensors in arbitrary number of spacetime dimensions and presenting an explicit on-shell realization in the EMDA theory in 4D. Further examples are discussed in section 4. We conclude in section 5. Additional material has been moved to the appendices. Namely, appendix A reviews the construction of the codimension-2 foliation, appendix B presents an alternative construction with 'dual metric ansatz', and appendix C reviews the properties of the SN brackets.

2 Lifting symmetries from codimension-2 hypersurfaces

Let us consider a d-dimensional spacetime $(\mathcal{M}, g)$, foliated by a (Riemannian) codimension-2 hypersurface $(\mathcal{S}, \gamma)$, which we will sometimes refer to as the 'base space'. Our aim is to investigate the conditions under which the full spacetime 'inherits' interesting symmetry properties from $\mathcal{S}$.

2.1 Metric ansatz

In what follows, we will use the Greek alphabet $\mu, \nu, \rho, \ldots$ to label coordinates x^μ and the indices of tensors in the full spacetime $\mathcal{M}$ and the uppercase Latin script $A, B, C, \ldots$ to label coordinates x^A and the indices of tensors intrinsic to codimension-2 hypersurface $\mathcal{S}$. We further assume the spacetime metric takes the following form:

$$g = g_{\mu\nu}dx^\mu dx^\nu = -Nf dt^2 + \frac{dr^2}{f} + \gamma_{AB}(dx^A + \nu^A dt)(dx^B + \nu^B dt)\,. \tag{2.1}$$

Here, ∂_t and ∂_r are (decoupled) timelike and radial directions, $N = N(t,r)$ and $f = f(t,r)$ are arbitrary functions of t and r, and $\gamma_{AB} = \gamma_{AB}(t,r,x^C)$ and $\nu^A = \nu^A(t,r,x^C)$ are the metric and an arbitrary vector on $\mathcal{S}$.[2] The inverse metric is given by

$$g^{-1} = g^{\mu\nu}\partial_\mu\partial_\nu = -\frac{1}{fN}(\partial_t - \nu^A\partial_A)^2 + f\partial_r^2 + \gamma^{AB}\partial_A\partial_B\,, \tag{2.2}$$

where γ^{AB} is the inverse of γ. For completeness we discuss the gauge freedom of such foliations in appendix A and an alternative anasatz in appendix B.

2.2 Lifting the SN bracket

In what follows, we will employ a '*natural lift*' of tensors on $\mathcal{S}$ to $\mathcal{M}$. Namely, for a vector $\xi \in T\mathcal{S}$, $\xi^A = \xi^A(t,r,x^B)$, we define the corresponding 'lifted' vector $\hat{\xi} \in T\mathcal{M}$ as:

$$\hat{\xi} \equiv \hat{\xi}^\mu\partial_\mu = 0\times\partial_t + 0\times\partial_r + \xi^A\partial_A\,. \tag{2.3}$$

More generally, the above lift naturally extends to any rank-p-tensor $X^{A_1\dots A_p} \in T^p\mathcal{S}$, namely:

$$T^p\mathcal{M} \ni \hat{X} \equiv X^{A_1\dots A_p}\partial_{A_1}\dots\partial_{A_p}\,, \tag{2.4}$$

with any component in t and r directions missing. Due to the splitting of the metric (2.1) it can be shown that this lift preserves the inner product: $< X, Y >_\gamma = < \hat{X}, \hat{Y} >_g$.

In order to lift the symmetries from $\mathcal{S}$ to $\mathcal{M}$, we shall employ the tool of SN brackets. For symmetric tensors $A^{\mu_1\cdots\mu_p}$ and $B^{\nu_1\cdots\nu_q}$ on $\mathcal{M}$, these are defined as follows [12, 13]:

$$[A,B]_{\text{SN}}^{\mu_1\cdots\mu_{p+q-1}} = pA^{\rho(\mu_1\cdots\mu_{p-1}}\partial_\rho B^{\mu_p\cdots\mu_{p+q-1})} - qB^{\rho(\mu_1\cdots\mu_{q-1}}\partial_\rho A^{\mu_q\cdots\mu_{q+p-1})}\,. \tag{2.5}$$

In particular, for a vector $A = \xi$, the above reduces to the Lie derivative. The definition also extends naturally to scalars, $B = \Phi$ by

$$[A,\Phi]_{\text{SN}}^{\mu_1\cdots\mu_{p-1}} = pA^{\rho\mu_1\cdots\mu_{p-1}}\partial_\rho\Phi\,. \tag{2.6}$$

Note further, that the rank-p Killing tensor equation (1.1) can be conveniently expressed as

$$[K, g^{-1}] = 0\,. \tag{2.7}$$

Moreover, it follows from the definition of the SN bracket (derived from the Poisson algebra of the corresponding monomial constants of geodesic motion) that the SN bracket of two Killing tensors produces again a (possibly trivial) Killing tensor; see appendix C for more details. We also refer the reader to [15–21] for the characterization of Killing tensors in spaces of constant curvature.

[2]While t, r are coordinates on the full spacetime, they are 'external parameters' for objects on $\mathcal{S}$ such as γ_{AB} and ν^A. In fact, this means that we truly consider a family of hypersurfaces $\mathcal{S}^{(r,t)}$ together with the corresponding family of intrinsic metrics $\gamma^{(r,t)}$. This subtlety does not play any significant role in what follows. Moreover, if f and N are allowed to depend on x^A as well then the metric (2.1) is valid (locally) for any generic spacetime — although some gauge fixing is used, see appendix (A) for details.

Since we also need to distinguish brackets and symmetries on $\mathcal{M}$ from symmetries on $\mathcal{S}$, we shall use the following notation for the SN bracket between tensors intrinsic to $\mathcal{S}$, $X^{A_1\dots A_p}$ and $Y^{B_1\dots B_q}$:

$$[\![X,Y]\!]_{\text{SN}}^{A_1\dots A_{p+q-1}} = pX^{C(A_1\dots A_{p-1}}\partial_C Y^{A_p\dots A_{p+q-1})} - qY^{C(A_1\dots A_{q-1}}\partial_C X^{A_q\dots A_{q+p-1})} . \tag{2.8}$$

Notice that we have the natural relationship between the bracket of the lift of an object and the lift of the intrinsically defined object:

$$[\hat{X},\hat{Y}]_{\text{SN}} = \widehat{[\![X,Y]\!]}_{\text{SN}} . \tag{2.9}$$

Next, a short calculation using (2.2) shows we can decompose the bracket of a lifted object on $\mathcal{M}$ with g^{-1} into the bracket on $\mathcal{S}$ with γ^{-1} as follows: for a given $X^{A_1\dots A_p}$ we have

$$\begin{aligned}[\hat{X},g^{-1}]_{\text{SN}}^{\mu_1\dots\mu_{p+1}}\partial_{(\mu_1\dots\mu_{p+1})} = &-[\![X,(fN)^{-1}]\!]_{\text{SN}}^{A_1\dots A_{p-1}}\ \partial_{(t}\partial_t\partial_{A_1}\dots\partial_{A_{p-1})}\\ &+[\![X,f]\!]_{\text{SN}}^{A_1\dots A_{p-1}}\ \partial_{(r}\partial_r\partial_{A_1}\dots\partial_{A_{p-1})}\\ &+2\left((fN)^{-1}\partial_t X^{A_1\dots A_p}+[\![X,(fN)^{-1}\nu]\!]_{\text{SN}}^{A_1\dots A_p}\right)\partial_{(t}\partial_{A_1}\dots\partial_{A_p)}\\ &-2f\left(\partial_r X^{A_1\dots A_p}\right)\ \partial_{(r}\partial_{A_1}\dots\partial_{A_p)}\\ &-2(fN)^{-1}\nu^{(A_1}\partial_t X^{A_2\dots A_{p+1})}\partial_{(A_1}\dots\partial_{A_{p+1})}\\ &+\left([\![X,\gamma^{-1}]\!]_{\text{SN}}-[\![X,(fN)^{-1}\nu\otimes\nu]\!]_{\text{SN}}\right)^{A_1\dots A_{p+1}}\partial_{(A_1}\dots\partial_{A_{p+1})} .\end{aligned} \tag{2.10}$$

Here, on the right hand side, f, N, and ν^A are viewed as intrinsic quantities to $\mathcal{S}$.

Let us now assume that X is a symmetry of γ, that is

$$[\![X,\gamma^{-1}]\!]_{\text{SN}} = 0 . \tag{2.11}$$

Then, $\hat{X}$ will be a symmetry of the full spacetime, i.e. $[\hat{X},g^{-1}]_{\text{SN}} = 0$, provided that i) $[\![X,f]\!]_{\text{SN}} = 0 = [\![X,N]\!]_{\text{SN}}$ (which is in our case automatically satisfied as the functions f, N depend on t and r only), and ii)

$$\partial_t X = 0 , \quad \partial_r X = 0 , \quad [\![X,\nu]\!]_{\text{SN}} = 0 . \tag{2.12}$$

Notice here that requiring X to be lifted to a symmetry of the full spacetime imposes no requirements for ν^A to be a symmetry of $(\mathcal{S},\gamma)$.[3]

In particular, the above equations imply that $\hat{\nu}$ itself will not be a Killing vector on $\mathcal{M}$ as it is (t,r)-dependent. In what follows, we assume that ν can be decomposed as

$$\nu^A \equiv p(t,r)\xi_0^A(x^B) , \tag{2.13}$$

where ξ_0 is a Killing vector on $\mathcal{S}$, in order that our full spacetime have at least one Killing vector, namely $\hat{\xi}_0$.

[3]As mentioned in Footnote 2, if f and N depend on the full set of coordinates $x^\mu = (t,r,x^A)$ the metric (2.1) actually has enough freedom to describe (locally) any spacetime. Thus, (2.10), or points i) and ii), provide generic conditions for symmetries of $(\mathcal{S},\gamma)$ to lift to $(\mathcal{M},g)$ using the mapping in (2.4).

Let ξ_1 be another Killing vector on $(\gamma, \mathcal{S})$. Then, in order that its lift $\hat{\xi}_1$ be a Killing vector of the full spacetime, the above conditions then impose that the vector ξ_1^A be (t, r)-independent

$$\partial_r \xi_1^A = 0 = \partial_t \xi_1^A \,, \tag{2.14}$$

and that the two Killing vectors ξ_0 and ξ_1 on $\mathcal{S}$ must Lie commute:[4]

$$[\![\xi_0, \xi_1]\!]^A = \frac{1}{p(t,r)} [\![\nu, \xi_1]\!]^A = 0 \,. \tag{2.15}$$

Lifting of explicit symmetries thus requires commutativity on $\mathcal{S}$. In what follows we shall concentrate now on the case where the lower-dimensional Killing vectors do not commute, but give rise to hidden symmetries in the full spacetime.

2.3 Rank-2 irreducible Killing tensors from lower-dimensional Killing vectors

Suppose now that, apart from the Killing vector ξ_0 (2.13), we have two additional Killing vectors on $T\mathcal{S}$, $\xi_1^A = \xi_1^A(t, r, x^B)$ and $\xi_2^A = \xi_2^A(t, r, x^B)$, which do not necessarily commute with ξ_0^A. Instead we have the Lie algebra

$$[\![\xi_i, \xi_j]\!] = f_{ijk} \xi_k \,, \quad \text{where } i, j, k, \in \{0, 1, 2\} \,, \tag{2.16}$$

where $f_{ijk} = f_{[ij]k}$ are some (nonzero) structure constants. This implies that individually $\hat{\xi}_1$ and $\hat{\xi}_2$ are not necessarily Killing vectors of g. However, suppose further that their 'square' (which clearly commutes with γ^{-1} due to the Leibniz rule C.13),

$$C_{12} \equiv \xi_1 \otimes \xi_1 + \xi_2 \otimes \xi_2 , \,, \tag{2.17}$$

also commutes with ξ_0 in the SN sense:

$$[\![C_{12}, \xi_0]\!]_{\mathrm{SN}} = 0 \,. \tag{2.18}$$

In fact, the latter follows automatically if the symmetry group of γ is *semi-simple*. In that case, the structure constants will be totally anti-symmetric

$$f_{ijk} = f_{[ijk]} \,, \tag{2.19}$$

provided $\{\xi_i\}$ forms a basis of the Lie algebra, see e.g. chapter 1 in [41], and C_{12} is essentially the Casimir of the algebra.[5] To show this, we use the fact that for a symmetric rank-p tensor X and q vectors ξ_j, the SN-bracket (on $\mathcal{S}$ or $\mathcal{M}$) satisfies a Leibniz type rule (see appendix C for details)

$$\Big[\bigodot_{j=1}^{q} \xi_j, X\Big]_{\mathrm{SN}} = \sum_{i=1}^{q} [\xi_i, X]_{\mathrm{SN}} \odot \Big(\bigodot_{\substack{j=1 \\ j\neq i}}^{q-1} \xi_j\Big) \,, \tag{2.20}$$

[4] We will differentiate notation for the Lie bracket of a vector field on $T\mathcal{S}$ and $T\mathcal{M}$, as above. That is, $[\![\cdot, \cdot]\!]$ and $[\cdot, \cdot]$ respectively — a bracket without a subscript SN will denote the Lie bracket (which for two vectors is the same object).

[5] In fact, in maximally symmetric spaces (constant curvature manifolds), e.g. [15–21], a certain quotient of the universal covering algebra of the symmetry group is isomorphic to the space of Killing tensors with the Lie bracket realized by the SN bracket (see Theorem 4.13 and corollary 4.14 in [21]).

where $\bigodot$ denotes the repeated symmetrized tensor product $\odot$, e.g. for two vectors $X \odot Y = 1/2(X \otimes Y + Y \otimes X)$. Thus, employing the form of C_{12} the commutation (2.18) reads

$$[\![C_{12}, \xi_0]\!]_{\rm SN}^{AB} = 2\sum_{i=1}^{2} \xi_i{}^{(A}[\![\xi_i, \xi_0]\!]^{B)} = 2\sum_{\substack{i=1\\j=0}}^{2} f_{i0j}\xi_i^{(A}\xi_j^{B)} = \sum_{i,j=0}^{2} f_{i0j}\xi_i^{(A}\xi_j^{B)} \underbrace{=0}_{\text{if } f_{ijk}=f_{[ijk]}}, \tag{2.21}$$

as anticipated.

In any case, provided ξ_1 and ξ_2 are (t,r)-independent, and condition (2.18) holds:

$$[\![C_{12}, \xi_0]\!]_{\rm SN}^{AB} = 0\,, \quad \partial_t \xi_1^A = 0 = \partial_t \xi_2^A\,, \quad \partial_r \xi_1^A = 0 = \partial_r \xi_2^A\,, \tag{2.22}$$

we also have $[\![C_{12}, \nu]\!]_{\rm SN} = p(t,r)[\![C_{12}, \xi_0]\!]_{\rm SN} = 0$. In other words, the above conditions (2.22) are sufficient to guarantee that the lift $\hat{C}_{12}$ is a Killing tensor of the full spacetime.

In summary, starting from the form of the metric (2.1) with ν given by (2.13) and assuming ξ_0^A (2.13) is a Killing vector of γ_{AB}, we have shown:

1. $\hat{\xi}_0$ is a Killing vector in $(\mathcal{M}, g)$.

2. If ξ_1 is another (t,r)-independent Killing vector of γ which commutes with ξ_0, i.e.

$$[\![\xi_0, \xi_1]\!]^A = 0\,,$$

then $\hat{\xi}_1$ is a Killing vector in $(\mathcal{M}, g)$.

3. If instead we have two (t,r)-independent Killing vectors of γ whose square SN commutes with ξ_0;

$$C_{12} = \xi_1 \otimes \xi_1 + \xi_2 \otimes \xi_2\,, \quad [\![C_{12}, \xi_0]\!]_{\rm SN}^{AB} = 0\,,$$

then $\hat{C}_{12}$ is a Killing tensor of the full spacetime $(\mathcal{M}, g)$.

Notice that the last point extends to any second order C constructed from a finite number of Killing vectors of γ. Moreover, if C is a rank-p Killing tensor of γ which is (t,r)-independent and SN commutes with ν then (2.10) shows that its lift $\hat{C}$ will be a Killing tensor of $(\mathcal{M}, g)$.

2.4 Axisymmetric spacetimes

We would now like to adapt the above discussion to metrics in d spacetime dimensions which are symmetric about $m = [\frac{d-1}{2}]$ axes of rotation (where $[A]$ denotes the integer part of A). That is, let us assume the existence of m Killing vectors ∂_{ϕ_i} of the metric γ_{AB}, where each ϕ_i is a 2π periodic coordinate on an axis of rotation of γ_{AB}. Similar to before, we further assume that the vector ν^A can be written

$$\nu^A = \sum_{i=1}^{m} p_i(t,r)(\partial_{\phi_i})^A\,, \tag{2.23}$$

where $p_i(t,r)$ are arbitrary functions of t and r. Since all the ∂_{ϕ_i}'s mutually commute on $T\mathcal{S}$ their lifts are also Killing vectors of the metric $g_{\mu\nu}$.

JHEP07(2025)098

Construction of Killing tensors. We can also now assume that γ_{AB} has some particular n-dimensional symmetry group $G(n)$ which is generated by some set of Killing vectors $(\xi_p)^A \in \mathfrak{g}(n)$ where $p \in S = \{1,\dots,n\}$ (not to be confused with the base space $\mathcal{S}$), of which the ∂_{ϕ_i}'s generate an Abelian subgroup. Let us write again the Lie algebra $\mathfrak{g}(n)$ of the symmetries as

$$[\![\xi_p, \xi_q]\!] = f_{pqr}\xi_r \,. \tag{2.24}$$

Again, if $\mathfrak{g}$ is semi-simple and the $\{\xi_p\}$ form a basis then the structure constants are totally antisymmetric

$$f_{pqr} = f_{[pqr]} \,. \tag{2.25}$$

For example, γ_{AB} could represent a metric of constant curvature, such as the $(d-2)$-sphere S^{d-2}, in which case the symmetry group would be $\mathrm{SO}(d-2)$.

Suppose now, that there exists a subset $I \subseteq S$ such that

$$C_I = \sum_{p \in I \subseteq S} \xi_p \otimes \xi_p \,, \tag{2.26}$$

commutes with ν. That is,

$$[\![C_I, \nu]\!]_{\mathrm{SN}} = \sum_i p_i(t,r)[\![C_I, \partial_{\phi_i}]\!]_{\mathrm{SN}} = 2\sum_i p_i(t,r)\sum_{q\in I}\xi_q \odot [\![\xi_q, \partial_{\phi_i}]\!] = 0 \,. \tag{2.27}$$

In the case that each of the functions p_i are independent[6] this is equivalent to

$$[\![C_I, \partial_{\phi_i}]\!]_{\mathrm{SN}} = 0 \quad \forall i \in \{1,\dots,m\} \,. \tag{2.28}$$

Now, the results of the previous section apply and we have immediately that $\hat{C}_I$ is a Killing tensor of $(\mathcal{M}, g)$ (here we again silently assume that all ξ_p are 'innate to' $\mathcal{S}$, that is t and r independent).

Moreover, if we have two such subsets $I, I' \subseteq S$, with possible overlap, i.e. $I \cap I'$ need not be empty, such that

$$[\![C_I, \partial_{\phi_i}]\!]_{\mathrm{SN}} = 0 = [\![C_{I'}, \partial_{\phi_i}]\!]_{\mathrm{SN}} \quad \forall i \in \{1,\dots,m\} \,, \tag{2.29}$$

Then if their SN bracket is non zero, by the Jacobi identity (C.9), we have a new non-trivial symmetry given by the following rank 3 Killing tensor:

$$C_{II'} \equiv [\![C_I, C_{I'}]\!]_{\mathrm{SN}} \,, \tag{2.30}$$

which lifts to the full spacetime symmetry $\hat{C}_{II'}$. On the other hand, if this bracket vanishes, C_I and $C_{I'}$ yield constants of motion on $\mathcal{S}$ and $\mathcal{M}$ which are in involution. More explicitly,

[6] In the case that the metric is asymptotically flat or (A)dS this is equivalent to independent asymptotic angular momenta corresponding to the Killing vectors ∂_{ϕ_i} introduced section 3.

using (2.20) repeatedly, the SN bracket (2.30) can be written as

$$\begin{aligned}
[\![C_I, C_{I'}]\!]^{ABC}_{\rm SN} &= 2\sum_{q\in I'}[\![C_I,\xi_q]\!]^{(AB}_{\rm SN}\,\xi_q^{C)} = 4\sum_{p\in I\,,q\in I'}\xi_p^{(A}\xi_q^{B}[\![\xi_p,\xi_q]\!]^{C)}\\
&= 4\sum_{\substack{p\in I,q\in I',\\ r\in S}} f_{pqr}\,\xi_p^{(A}\xi_q^{B}\xi_r^{C)}\\
&= 4\sum_{r\in S}\Bigg(\sum_{\substack{p\in I\setminus I'\\ q\in I'\setminus I}} + \sum_{\substack{p\in I\cap I'\\ q\in I'\setminus I}} + \sum_{\substack{p\in I\setminus I'\\ q\in I'\cap I}}\Bigg) f_{pqr}\,\xi_p^{(A}\xi_q^{B}\xi_r^{C)}\,. \qquad (2.31)
\end{aligned}$$

Notice, that due to the automatic antisymmetry of the structure constants on the first two indices, the sum on p and q must be over distinct ranges. Moreover, if we have a semi-simple Lie algebra for which ξ_p form a basis then the ranges of p, q, and r have to be all distinct. Of course, this relation continues to hold at the level of the spacetime, i.e.

$$\begin{aligned}
[\hat{C}_I, \hat{C}_{I'}]^{\mu\nu\rho}_{\rm SN}\partial_\mu\partial_\nu\partial_\rho &= 4\sum_{r\in S}\Bigg(\sum_{\substack{p\in I\setminus I'\\ q\in I'\setminus I}} + \sum_{\substack{p\in I\cap I'\\ q\in I'\setminus I}} + \sum_{\substack{p\in I\setminus I'\\ q\in I'\cap I}}\Bigg) f_{pqr}\,\hat{\xi}_p^{(\mu}\hat{\xi}_q^{\nu}\hat{\xi}_r^{\rho)}\partial_\mu\partial_\nu\partial_\rho\\
&= 4\sum_{r\in S}\Bigg(\sum_{\substack{p\in I\setminus I'\\ q\in I'\setminus I}} + \sum_{\substack{p\in I\cap I'\\ q\in I'\setminus I}} + \sum_{\substack{p\in I\setminus I'\\ q\in I'\cap I}}\Bigg) f_{pqr}\,\xi_p^{(A}\xi_q^{B}\xi_r^{C)}\partial_A\partial_B\partial_C\,. \qquad (2.32)
\end{aligned}$$

However, whereas (2.31) is a reducible Killing tensor on the base space (simply written as a linear combination of symmetrized products of various 3 Killing vectors), (2.32) is an irreducible Killing tensor of the full spacetime, provided not all Killing vectors ξ lift to Killing vectors on the spacetime.

Mutual commutativity and geodesic integrability. If we want a *completely integrable* geodesic system, then we need a set of $(d-1-m)$ mutually commuting Killing tensors to complement m axial Killing vectors, ∂_{ϕ_i} plus the inverse metric g^{-1}. If, in addition, the spacetime is also stationary, an extra timelike Killing vector ∂_t is present, and we need one less Killing tensor lifted from $\mathcal{S}$.

One way that a set of *mutually commuting* Killing tensors of $\mathcal{M}$ could arise is if we have a collection of nested subsets $I_1 \subset I_2 \subset \ldots \subset I_N$, each of which generate a Lie subalgebra, such that each

$$C_{I_i} = \sum_{p\in I_i\subseteq S}\xi_p\otimes\xi_p = \sum_{p\in I_i}\xi_p\otimes\xi_p\,, \qquad (2.33)$$

commute with all of the rotational Killing vectors,

$$[\![C_{I_i}, \partial_{\phi_k}]\!]_{\rm SN} = 0 \quad \forall k\,. \qquad (2.34)$$

From (2.31) it is clear that any two such Killing tensors C_{I_i} and C_{I_j} will not necessarily commute with each other. Now, since either $I_i \subset I_j$, or $I_j \subset I_i$, if $i<j$, or $j<i$ respectively, the first term in the sum in (2.31) will be empty. However, one of the last two terms will survive: e.g. if $I_i \subset I_j$,

$$[\![C_{I_i}, C_{I_j}]\!] = 4\sum_{\substack{p,r\in I_i\\ q\in I_j\setminus I_i}} f_{pqr}\,\xi_p\odot\xi_q\odot\xi_r\,, \qquad (2.35)$$

where the summation range of r is restricted to I_i because the set generates a Lie subalgebra and so is closed. If, in addition, the Killing vectors that these subsets label form a semi-simple Lie subalgebra, i.e. ξ_p, ξ_q, ξ_r for $p, q, r \in I_i$

$$[\![\xi_p, \xi_q]\!] = \tilde{f}_{pqr}\xi_r \,, \quad \tilde{f}_{[pqr]} = \tilde{f}_{pqr} \,, \tag{2.36}$$

then similarly to (2.21) the quadratic Killing tensor C_{I_i} commutes with each ξ_p

$$[\![C_{I_i}, \xi_p]\!]_{\mathrm{SN}} = 0 \,, \tag{2.37}$$

hence they are Casimirs of the subalgebra and we have the desired commutativity.[7]

Finally, the results of the preceding calculations (in particular (2.10)) show that the lift, $\hat{C}_{I_i}$, of the Casimirs to the full spacetime $\mathcal{M}$ form a set of mutually commuting Killing tensors of $g_{\mu\nu}$. That is,

$$[\hat{C}_{I_i}, g^{-1}]^{\mu\nu\rho}_{\mathrm{SN}} = 0 \,, \quad [\hat{C}_{I_i}, \hat{C}_{I_j}]^{\mu\nu\rho}_{\mathrm{SN}} = 0 \,. \tag{2.38}$$

Furthermore, each $\hat{C}_S$ is irreducible to Killing vectors provided one removes any set of the axial Killing vectors, ∂_{ϕ_i}, which may have appeared in the sum (2.33). In fact one could already quotient out this subgroup in (2.33) because all the ∂_{ϕ_i}'s mutually commute and we are only interested in additional symmetries.

2.5 Growing tower of higher rank Killing tensors

In the previous we have shown as to how the reducible 'Casimirs' on $\mathcal{S}$ can give rise to irreducible rank 2 Killing tensors on $\mathcal{M}$, and that, in their turn, these may generate new (possibly irreducible) rank 3 Killing tensors on $\mathcal{M}$ via the SN bracket. Starting from these objects and applying the SN bracket iteratively, one can, at least in principle, generate a tower of (possibly irreducible) growing rank Killing tensors.

For example, we can generate the following rank 4 Killing tensor (applying the Jacobi identity (C.9)):

$$\begin{aligned}\Big[\!\!\Big[[\![C_I, C_{I'}]\!], C_{I''}\Big]\!\!\Big]^{ABCD}_{\mathrm{SN}} = & \sum_{\substack{p\in I, q\in I' \\ s\in I'', r\in S}} \Big[12 f_{pqr}\xi^{(M}\xi^A_q\xi^B_r\partial_M(\xi^C_s\xi^{D)}_s) \\ & \qquad -8\xi^M_s\xi^{(A}_s f_{pqr}\partial_M(\xi^B_p\xi^C_q\xi^{D)}_r)\Big] \end{aligned} \tag{2.39}$$

$$\begin{aligned} = -8 & \sum_{\substack{p\in I, q\in I', s\in I'' \\ r,t\in S}} \Big[f_{pqr}f_{spt}\xi^{(A}_s\xi^B_t\xi^C_q\xi^{D)}_r + \\ & \qquad + f_{pqr}f_{sqt}\xi^{(A}_s\xi^B_p\xi^C_t\xi^{D)}_r + f_{pqr}f_{srt}\xi^{(A}_s\xi^B_p\xi^C_q\xi^{D)}_t\Big] \,. \end{aligned} \tag{2.40}$$

[7] Essentially this construction is related to the Noetherian property of the universal enveloping algebra (see section 7 of [42]). For example, in $\mathrm{SO}(d-2)$ we have the canonical chain [43] $\mathrm{SO}(d-2) \supset \mathrm{SO}(d-3) \cdots \supset \mathrm{SO}(2)$ and so the nested subsets correspond to labelling rotation subgroups. It may be of interest to make such statements precise, at least for the case where γ is maximally symmetric.

As before, the summation ranges will reduce when $\mathfrak{g}$ is a semi-simple Lie algebra (totally antisymmetric structure constants); however, the resulting expression is not particularly enlightening and so we do not reproduce it here.

More generally, taking repeated SN brackets we can go to arbitrary rank $k+1$ as follows:

$$\Big[\!\!\Big[[\![\cdots [\![C_{I_1}, C_{I_2}]\!], \cdots]\!], C_{I_k} \Big]\!\!\Big] = -(-2)^k \prod_{i=0}^{k-2} \sum_{\substack{s_j \in I_j \\ r_j \in S \\ t_i \in \{s_1,\dots,s_{i+1},r_1,\dots,r_i\} \setminus \\ \{t_1,\dots,t_{i-1}\}}} f_{s_{i+2} t_i r_{i+1}} \bigodot_{\substack{u \in \{s_1,\dots,s_k, \\ r_1,\dots,r_{k-1}\} \setminus \\ \{t_1,\dots,t_{k-2}\}}} \xi_u \,, \tag{2.41}$$

where again the summation range of the first two indices s_{i+2}, t_i in each structure $f_{s_{i+2}t_i r_i}$ constant must be disjoint and, in particular, when the algebra is semi-simple all three summation ranges must be disjoint (when appearing in combination with the symmetrized vector product $\xi_{s_{i+2}} \odot \xi_{t_i}$ or $\xi_{s_{i+2}} \odot \xi_{t_i} \odot \xi_{r_{i+1}}$ respectively). However, similar to the last line of (2.31) each of the terms in the product will have a different collection of (disjoint) summation ranges. Hence, it is not *obvious* that any cancellations cause this nested commutator to terminate. Furthermore, as the summation terms are distinct (2.41) is not reducible to the Killing tensors in (2.33) which involve a summation over products of two identical vectors. Together, this hints that these higher rank Killing tensors are irreducible.

On the other hand, the process in (2.41) will not generate an infinite set of functionally independent constants of geodesic motion since the commutators are determined from the structure constants of the group. In particular, for geodesic motion confined to $\mathcal{S}$ with dimension $d-2$ there can only be $2d-5$ functionally independent constants of motion (corresponding to a maximally superintegrable system) [22]. Furthermore, the maximum number of independent rank-p Killing tensors (including Killing vectors), achieved when $\mathcal{S}$ is maximally symmetric, is (see e.g. [44] for an extended discussion and the original references [15–18]):

$$k_{\max} = \frac{1}{d-2} \binom{(d-2)+p}{p+1} \binom{(d-2)+p-1}{p} \,. \tag{2.42}$$

This gives a bound on the number of Killing tensors of rank $p = k+1$ generated by k subsets $I_1, \dots, I_k$.

3 Generalized Lense-Thirring spacetimes

We will now apply our formalism to a physically motivated example, namely the class of generalized Lense-Thirring spacetimes [23–25], which are an off-shell (i.e. no field equations imposed) class of spacetimes well suited to describing slowly rotating black holes in many different theories; as discussed at the end of this section, they also represent an exact rotating black hole solution in the EMDA theory.

JHEP07(2025)098

3.1 Introducing the metric

The generalized Lense-Thirring metric is stationary and axially symmetric and reads[8]

$$ds^2 = -Nfdt^2 + \frac{dr^2}{f} + r^2\sum_{i=1}^{m}\mu_i^2\Big(d\phi_i + p_i dt\Big)^2 + r^2\sum_{i=1}^{m+\epsilon} d\mu_i^2\,. \tag{3.1}$$

Here, as before, $m = [\frac{d-1}{2}]$ and now $\epsilon = 1, 0$ for even, odd dimensions $d = 2m + \epsilon + 1$, respectively. The metric functions f, N, p_i are functions of the radial coordinate r, and the 'angular' coordinates μ_i are constrained as follows:

$$\sum_{i=1}^{m+\epsilon}\mu_i^2 = 1\,. \tag{3.2}$$

Moreover, in [24], it was useful to define

$$p_i(r) = \frac{\sum_{j=1}^{m} p_{ij}(r)a_j}{r^2}\,, \tag{3.3}$$

to explicitly introduce the corresponding rotation parameters a_i.

The above introduced spacetime is off-shell, meaning that no field equations have been imposed. However, by specifying the metric functions f, N, p_i one can obtain approximate (slowly rotating) black hole solutions in various theories. For example, the vacuum Einstein gravity slowly rotating black solution in all dimensions is recovered upon setting

$$N = 1\,,\quad f = 1 - \left(\frac{r_+}{r}\right)^{d-3}\,,\quad p_i = \frac{a_i}{r^2}(f-1)\,, \tag{3.4}$$

where the parameter r_+ is related to the black hole mass and parameters a_i are its rotation parameters (see [24] for more such examples).

More generally, let us assume that the above spacetime describes a black hole solution. In that case the black hole horizon is located at r_+, given by the due root of $f(r_+) = 0$. Such a horizon is then a Killing horizon generated by the following Killing vector:

$$\xi = \partial_t + \sum_{i=1}^{m}\Omega_+^i\partial_{\phi_i}\,,\quad \Omega_+^i \equiv -p_i(r_+)\,. \tag{3.5}$$

The spacetime then features an ergoregion, a region outside of the horizon where the Killing vector ∂_t has negative norm; Ω_i here play the role of angular velocities of the horizon. Moreover, one can impose leading order asymptotic flat or (A)dS fall-off conditions for large r:

$$N = 1 + O(r^{-k_N})\,,\; f = -\frac{2\Lambda r^2}{(d-1)(d-2)} + 1 - \frac{2m}{r^{d-3}} + O(r^{-k_f})\,,\; p_i = \frac{\sum_{j=1}^{m} p_{ij}^0 a_j}{r^{d-5}} + O(r^{-k_{p_i}})\,, \tag{3.6}$$

where Λ is the cosmological constant, p_{ij}^0 is the leading order term of $p_{ij}(r)$ in (3.3), and $\{k_N, k_f, k_{p_i}\}$ are positive integers.[9] Then, to each of the Killing vectors one can associate

[8] As discussed in the previous section, the assumption of stationarity is not necessary, and one could consider time dependent metric functions N, f and p_i. In such a case, the same symmetries except for ∂_t would arise.

[9] In principle, one could choose a more general, e.g. polyhomogeneous [45], asymptotic expansion but that is not required here.

the generalized Komar charge [46, 47]:

$$\mathcal{Q}[\zeta] \propto \int_{S^{d-2}_{\infty}} \star \left(d\zeta - \frac{4\Lambda}{d-2} \omega^{\zeta} \right) , \tag{3.7}$$

where ω^{ζ} is a Killing co-potential $\nabla^a \omega^{\zeta}_{ab} = \zeta_b$. In this case the mass and angular momenta of the spacetime are

$$M \equiv \mathcal{Q}[\partial_t] \propto m \,, \quad J_i \equiv \mathcal{Q}[\partial_{\phi_i}] \propto \sum_{j=1}^{m} p^0_{ij} a_j \,. \tag{3.8}$$

Thus, the requirement for independent asymptotic angular momenta is equivalent to the requirement for independent p_i functions.

Irrespective of its physical interpretation, it was shown in [23–28] that the above generalized Lense-Thirring spacetime (3.1) possesses many remarkable properties. Namely, it can be cast in the Painlevé-Gullstrand form and admits arapidly growing tower of (increasing rank) Killing tensors which guarantee separability of the Hamilton-Jacobi equation for geodesics and the Klein-Gordon equation for scalar fields (see [28] and [25] for their respective explicit separation). Our goal here is to demystify the appearance of such hidden symmetries.

3.2 Rank-2 Killing tensors and their brackets

To start with, observe that the metric (3.1) is of the form (2.1) with the vector ν^A given by (2.23). Clearly, we have the Killing vectors of the full metric ∂_t and ∂_{ϕ_i}, while the metric on $\mathcal{S}$ is

$$\gamma_{AB} dx^A dx^B = r^2 \left(\sum_{i=1}^{m+\epsilon} d\mu_i^2 + \sum_{i=1}^{m} \mu_i^2 d\phi_i^2 \right) , \tag{3.9}$$

which is just the metric on the $(d-2)$-sphere. Naturally, the symmetry group is simply $\mathrm{SO}(d-1) = \mathrm{SO}(2m+\epsilon)$ which becomes manifest in the coordinates on the i-th two-plane, namely [4, 48]:

$$x_i = \mu_i \cos\phi_i \,, \quad y_i = \mu_i \sin\phi_i \,, \quad 1 = \sum_{i=1}^{m+\epsilon} x_i^2 + y_i^2 \,, \quad y_{m+1} = 0 \,, \tag{3.10}$$

upon which the metric on $\mathcal{S}$ becomes

$$\gamma_{AB} dx^A dx^B = r^2 \left(\epsilon dx^2_{m+\epsilon} + \sum_{i=1}^{m} \left[dx_i^2 + dy_i^2 \right] \right) . \tag{3.11}$$

In these coordinate the symmetry algebra is generated by the following Killing vectors for $i, j \leq m + \epsilon$:[10]

$$
\begin{aligned}
L_{x_i x_j} &= x_i \partial_{x_j} - x_j \partial_{x_i} \\
&= \cos\phi_i \cos\phi_j (\mu_i \partial_{\mu_j} - \mu_j \partial_{\mu_i}) - \cos\phi_i \sin\phi_j \frac{\mu_j}{\mu_i} \partial_{\phi_i} + \cos\phi_j \sin\phi_i \frac{\mu_i}{\mu_j} \partial_{\phi_j} \,, \\
L_{x_i y_j} &= x_i \partial_{y_j} - y_j \partial_{x_i} \\
&= \cos\phi_i \sin\phi_j (\mu_i \partial_{\mu_j} - \mu_j \partial_{\mu_i}) + \cos\phi_i \cos\phi_j \frac{\mu_i}{\mu_j} \partial_{\phi_j} + \sin\phi_i \sin\phi_j \frac{\mu_j}{\mu_i} \partial_{\phi_i} \,, \\
L_{y_i y_j} &= y_i \partial_{y_j} - y_j \partial_{y_i} \\
&= \sin\phi_i \sin\phi_j (\mu_i \partial_{\mu_j} - \mu_j \partial_{\mu_i}) + \sin\phi_i \cos\phi_j \frac{\mu_j}{\mu_i} \partial_{\phi_i} - \sin\phi_j \cos\phi_i \frac{\mu_i}{\mu_j} \partial_{\phi_j} \,,
\end{aligned}
\tag{3.12}
$$

where it should be understood that $\partial_{\mu_{m+\epsilon}} = 0 = \partial_{\phi_{m+1}}$. Notice $L_{x_i y_i} = \partial_{\phi_i}$ and $L_{y_i y_{m+1}} = 0 = L_{x_i y_{m+1}}$.[11] In total this gives the required number, $\frac{1}{2}(2m+\epsilon)(2m+\epsilon-1)$, of generators.

If I, J, K, L stand for any of x_i, y_j (except $y_{m+1} = 0$) then we can express the algebra as simply

$$
[\![L_{IJ}, L_{KL}]\!] = \delta_{IL} L_{JK} + \delta_{JK} L_{IL} - \delta_{IK} L_{JL} - \delta_{JL} L_{IK} \equiv f_{IJKLMN} L_{MN} \,. \tag{3.13}
$$

This is exactly the usual algebra of $\mathfrak{so}(2m+\epsilon)$ with structure constants in this basis

$$
\begin{aligned}
f_{IJKLMN} = \frac{1}{2}(&\delta_{IL}\delta_{JM}\delta_{KN} + \delta_{JK}\delta_{IM}\delta_{LN} - \delta_{IK}\delta_{JM}\delta_{KN} - \delta_{JL}\delta_{IM}\delta_{KN} \\
&- \delta_{IL}\delta_{JN}\delta_{KM} - \delta_{JK}\delta_{IN}\delta_{LM} + \delta_{IK}\delta_{JN}\delta_{KM} + \delta_{JL}\delta_{IN}\delta_{KM}) \,.
\end{aligned}
\tag{3.14}
$$

We can then introduce the Casimirs for each plane

$$
\begin{aligned}
C_{ij} &= \frac{1}{2} \sum_{\substack{I \in \{x_i, y_i\} \\ J \in \{x_j, y_j\}}} L_{IJ} \odot L_{IJ} \\
&= \frac{1}{2} \left(L_{x_i x_j} \odot L_{x_i x_j} + L_{x_i y_j} \odot L_{x_i y_j} + L_{y_i x_j} \odot L_{y_i x_j} + L_{y_i y_j} \odot L_{y_i y_j} \right) \\
&= \frac{1}{2} \left((\mu_i \partial_{\mu_j} - \mu_j \partial_{\mu_i})^2 + \left(\frac{\mu_i}{\mu_j} \partial_{\phi_j} + \frac{\mu_j}{\mu_i} \partial_{\phi_i} \right)^2 \right) .
\end{aligned}
\tag{3.15}
$$

We call them Casimirs here because they SN commute with any Killing vector in a different two-plane to i and j, i.e. $[\![C_{ij}, L_{kl}]\!]_{\mathrm{SN}} = 0$ for $\{i, j\} \neq \{k, l\}$. This follows immediately from the Kronecker deltas in (3.13). Moreover the Casimirs (3.15) also commute with all $\partial_{\phi_k} = L_{x_k, y_k}$.

[10] In these coordinates it is clear that if $x_{m+\epsilon}, y_{m+\epsilon}$ were not constrained, we would have just a flat metric, with the usual Killing vectors of $(2m+\epsilon)$-dimensional Euclidean space, i.e. rotations and translations. However, due to the constraint we are left with only the rotations in (3.12).

[11] Thus, in all of the following expressions with sums it should be understood that y_{m+1} does not appear.

Namely, for all i, j, k we have (no sum)

$$\begin{aligned}
[\![C_{ij}, \partial_{\phi_k}]\!]_{\rm SN} &= \frac{1}{2} \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}}} [\![L_{IJ}\odot L_{IJ}, L_{x_k,y_k}]\!]_{\rm SN} = \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}}} [\![L_{IJ}, L_{x_k,y_k}]\!]\odot L_{IJ} \\
&= \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}}} (\delta_{Iy_k} L_{Jx_k} + \delta_{jx_k} L_{Iy_k} - \delta_{Ix_k} L_{Jy_k} - \delta_{Jy_k} L_{Ix_k})\odot L_{IJ} \\
&= \delta_{ik} \sum_{J\in\{x_j,y_j\}} (L_{Jx_k}\odot L_{y_kJ} - L_{Jy_k}\odot L_{x_kJ}) \\
&\quad + \delta_{jk} \sum_{I\in\{x_i,y_i\}} (L_{Iy_k}\odot L_{Ix_k} - L_{Iy_k}\odot L_{x_kI}) = 0\,.
\end{aligned} \tag{3.16}$$

Thus, from our previous discussion and in particular (2.10), we see that each C_{ij} lifts to a Killing tensor of the full spacetime

$$\hat{C}_{ij} = \frac{1}{2}\left((\mu_i\partial_{\mu_j} - \mu_j\partial_{\mu_i})^2 + \left(\frac{\mu_i}{\mu_j}\partial_{\phi_j} + \frac{\mu_j}{\mu_i}\partial_{\phi_i}\right)^2 \right). \tag{3.17}$$

It should be observed that not all of these are independent since $C_{ij} = C_{ji}$. Moreover, $C_{ii} = 2\partial_{\phi_i}\otimes\partial_{\phi_i}$ so these are only irreducible Killing tensors on $\mathcal{M}$; for $m+\epsilon \geq i > j > 1$. Therefore, there are $\frac{1}{2}(m+\epsilon)(m+\epsilon-1)$ irreducible Killing tensors. We note that these Killing tensors do not all commute with each other. Using (2.31) and (3.13) we have

$$\begin{aligned}
[\![C_{ij}, C_{kl}]\!]_{\rm SN} &= \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}}} \sum_{\substack{K\in\{x_k,y_k\}\\ L\in\{x_l,y_l\}}} (\delta_{IL}L_{JK} + \delta_{JK}L_{IL} - \delta_{IK}L_{JL} - \delta_{JL}L_{IK})\odot L_{IJ}\odot L_{KL} \\
&= \delta_{il} \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}\\ K\in\{x_k,y_k\}}} L_{JK}\odot L_{IJ}\odot L_{KI} + \delta_{jk} \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}\\ L\in\{x_l,y_l\}}} L_{IL}\odot L_{IJ}\odot L_{JL} \\
&\quad - \delta_{ik} \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}\\ L\in\{x_l,y_l\}}} L_{JL}\odot L_{IJ}\odot L_{IL} - \delta_{jl} \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}\\ K\in\{x_k,y_k\}}} L_{IK}\odot L_{IJ}\odot L_{KJ} \\
&= (\delta_{jl}-\delta_{il}) \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}\\ K\in\{x_k,y_k\}}} L_{IJ}\odot L_{IK}\odot L_{JK} + (\delta_{jk}-\delta_{ik}) \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}\\ L\in\{x_l,y_l\}}} L_{IJ}\odot L_{IL}\odot L_{JL}\,.
\end{aligned} \tag{3.18}$$

We can observe the following consequences of (3.18):

1. As expected, when $l = k$ or when $i = j$ the commutator vanishes corresponding to the situation that either $C_{ii} = 2\partial^2_{\phi_i}$ or $C_{kk} = 2\partial^2_{\phi_k}$ and hence the bracket must vanish by (3.16).

2. The expression is symmetric on the swap of indices $i \leftrightarrow j$ or $k \leftrightarrow l$ as required by (3.15).

3. Since the SN bracket is antisymmetric, the right hand side of (3.18) is antisymmetric under the swap of pairs of indices $\{ij\} \leftrightarrow \{kl\}$. This is not immediately obvious but requires relabelling using the Kronecker deltas.

4. For $i \neq j$ and $k \neq l$ this bracket does not vanish if one of the indices equals each other, as then one of the Kronecker deltas will be nonzero. Thus, when $\{i, j, k, l\}$ are all distinct (3.15) gives a set of new non-trivial (i.e. irreducible) constants of motion which are mutually commuting, i.e. in involution.

Point 4 gives a total of $m + \epsilon - 2$ Casimirs, corresponding to the distinct pairs, which are mutually commuting. We can add to this

$$C = \sum_{i,j=1}^{m+\epsilon} C_{ij} = r^2 \gamma^{-1} \,, \tag{3.19}$$

which commutes with all other C_{kl}'s. Thus giving $m + \epsilon - 1$ mutually commuting Killing tensors of $\mathcal{S}$. In addition we have the m Killing vectors ∂_{ϕ_i} and thus we have explicitly obtained the $d - 3 = 2m + \epsilon - 1$ symmetries necessary for the complete integrability of motion in the $(d-2)$-sphere. Moreover the motion in the full spacetime $\mathcal{M}$ is also completely integrable as we also have the full metric g and timelike Killing vector ∂_t.

We now make contact with the previous work on the generalized LT spacetimes [23, 24, 28].

3.3 Comparison with previous results

Alternative expressions for Killing tensors. To begin, the Killing tensors $\hat{C}_{ij}$, (3.17), were first written down in [28] in response to the works [23, 24]. However, they were described in terms of the $\mathfrak{u}(m)$ subalgebra with generators (see [40, 49])

$$\begin{aligned}
\xi_{ij} &= L_{x_i x_j} + L_{y_i y_j} = x_i \partial_{x_j} - x_j \partial_{x_i} + y_i \partial_{y_j} - y_j \partial_{y_i} \\
&= \cos(\phi_i - \phi_j)(\mu_i \partial_{\mu_j} - \mu_j \partial_{\mu_i}) + \sin(\phi_i - \phi_j) \left(\frac{\mu_i}{\mu_j} \partial_{\phi_j} + \frac{\mu_j}{\mu_i} \partial_{\phi_i} \right) ,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
\rho_{ij} &= L_{x_i y_j} + L_{x_j y_i} = x_i \partial_{y_j} - y_j \partial_{x_i} + x_j \partial_{y_i} - y_i \partial_{x_j} \\
&= -\sin(\phi_i - \phi_j)(\mu_i \partial_{\mu_j} - \mu_j \partial_{\mu_i}) + \cos(\phi_i - \phi_j) \left(\frac{\mu_i}{\mu_j} \partial_{\phi_j} + \frac{\mu_j}{\mu_i} \partial_{\phi_i} \right) ,
\end{aligned} \tag{3.21}$$

for $i, j \in \{1, \dots m\}$. Explicitly these generators satisfy the following $\mathfrak{u}(m)$ sub-algebra:

$$\begin{aligned}
[\xi_{ij}, \xi_{kl}] &= (\delta_{jk} \xi_{il} + \delta_{il} \xi_{jk} - \delta_{ik} \xi_{jl} - \delta_{jl} \xi_{ik}) \,, \\
[\rho_{ij}, \rho_{kl}] &= (-\delta_{jk} \xi_{il} - \delta_{ik} \xi_{jl} - \delta_{il} \xi_{jk} - \delta_{jl} \xi_{ik}) \,, \\
[\xi_{ij}, \rho_{kl}] &= (\delta_{jk} \rho_{il} + \delta_{jl} \rho_{ik} - \delta_{ik} \rho_{jl} - \delta_{il} \rho_{jk}) \,.
\end{aligned} \tag{3.22}$$

Moreover, the square of these generators gives the same Casimir in (3.15), i.e.

$$C_{ij} = \frac{1}{2} (\xi_{ij} \otimes \xi_{ij} + \rho_{ij} \otimes \rho_{ij}) \,, \tag{3.23}$$

but they do not cover the $C_{i\,m+\epsilon}$ cases. Furthermore, the relationship between the symmetries of the transverse space $\mathcal{S}$ and the full spacetime, as in (2.10), was not observed.

Next, as previously shown in [23, 24] up to $d = 13$, the Lense-Thirring spacetimes (3.1) admit the following Killing tensors explicitly written as:

$$K^{(I)} = \sum_{i\notin I}^{m-1+\epsilon} \left[(1 - \mu_i^2 - \sum_{j\in I} \mu_j^2)(\partial_{\mu_i})^2 - 2 \sum_{j\notin I\cup\{i\}} \mu_i\mu_j\, \partial_{\mu_i}\partial_{\mu_j} \right] + \sum_{i\notin I}^{m} \left[\frac{1 - \sum_{j\in I}\mu_j^2}{\mu_i^2} (\partial_{\phi_i})^2 \right] . \tag{3.24}$$

where we define the set $S = \{1, \ldots, m\}$ (not to be confused with the base space $\mathcal{S}$) and let $I \in P(S)$ where $P(S)$ is the power set of S, i.e. the set of all subsets $I \subset S$. Again not all of these are independent, in particular $\sum_{i=0}^{m-3} \binom{m}{i}$ of these are reducible, leaving

$$k = \sum_{i=0}^{m-2+\epsilon} \binom{m}{i} - \sum_{i=0}^{m-3} \binom{m}{i} = \frac{1}{2}(m+\epsilon)(m+\epsilon-1) \tag{3.25}$$

irreducible rank-2 Killing tensors in d dimensions.[12] The question then remains: what is the relationship between these two expressions for the Killing tensors?

Since the number of independent Killing tensors, $\hat{C}_{ij}$ and $K^{(I)}$, is the same, it is perhaps not surprising that we can express them in terms of each other. Namely, using the constraint (3.2) we can write

$$\begin{aligned} K^{(I)} &= \sum_{i\notin I}^{m+\epsilon} \sum_{j\notin I\cup\{i\}}^{m+\epsilon} \left[\mu_j^2(\partial_{\mu_i})^2 - 2\mu_i\mu_j\, \partial_{\mu_i}\partial_{\mu_j} \right] + \sum_{i\notin I}^{m+\epsilon} \sum_{j\notin I\cup\{i\}}^{m+\epsilon} \frac{\mu_j^2}{\mu_i^2}(\partial_{\phi_i})^2 \\ &= \frac{1}{2}\left(\sum_{i\notin I}^{m+\epsilon} \sum_{j\notin I\cup\{i\}}^{m+\epsilon} (\mu_i\partial_{\mu_j} - \mu_j\partial_{\mu_i})^2 + \sum_{i\notin I}^{m+\epsilon} \sum_{j\notin I\cup\{i\}}^{m+\epsilon} \left[\left(\frac{\mu_j}{\mu_i}\partial_{\phi_i} + \frac{\mu_j}{\mu_i}\partial_{\phi_i} \right)^2 - 2\partial_{\phi_i}\partial_{\phi_j} \right] \right) \\ &= \sum_{i\notin I}^{m+\epsilon} \sum_{j\notin I\cup\{i\}}^{m+\epsilon} \left(\hat{C}_{ij} - 2\partial_{\phi_i}\partial_{\phi_j} \right) , \end{aligned} \tag{3.27}$$

where we have used the symmetry on the summed indices and it should be understood that, as above, $\partial_{\phi_{m+1}} = 0 = \partial_{\mu_{m+\epsilon}}$, and $\mu_{m+\epsilon}$ is given by the constraint (3.2).

Therefore, up the reducible product of Killing vectors we see that $K^{(I)}$ is given by a sum over the Killing tensors lifted from the $\mathrm{SO}(2m+\epsilon)$ symmetry of the transverse space $\mathcal{S}$. Moreover, the previously obtained expressions (3.24) are therefore valid in every dimension.

Commutation relations. Previously it was verified in [23, 24] up to $d = 13$, that the SN bracket of any two Killing tensors (3.24) vanishes if the intersection of the two label sets equals the first. That is,

$$[K^{(I_1)}, K^{(I_2)}]_{\text{SN}} = 0\,, \qquad \text{iff } I_1 \subset I_2 \text{ or } I_2 \subset I_1\,. \tag{3.28}$$

[12] Previously the following equivalent expression was stated (for $\epsilon = 0, 1$) in [23, 24]:

$$k = \frac{1}{2}m(m-1+2\epsilon)\,. \tag{3.26}$$

However, the current form makes the structure clearer.

In the framework of this paper, such a commutation relation seems to follow from the general discussion in section 2.4. However, we now seek to prove this relation using the explicit algebra of Killing vectors of $\mathcal{S}$. From (3.27) and (3.18) we have

$$
\begin{aligned}
[K^{(I_1)},K^{(I_2)}]_{\rm SN} &= \sum_{\substack{i\notin I_1\\ j\notin I_1\cup\{i\}}}^{m+\epsilon}\ \sum_{\substack{k\notin I_2\\ l\notin I_2\cup\{k\}}}^{m+\epsilon} [\hat{C}_{ij},\hat{C}_{kl}]_{\rm SN} = \sum_{i,j\in S_\epsilon\backslash I_1}\ \sum_{k,l\in S_\epsilon\backslash I_2} \widehat{[\![C_{ij},C_{kl}]\!]}_{\rm SN} \\
&= \sum_{i,j\in S_\epsilon\backslash I_1}\ \sum_{k,l\in S_\epsilon\backslash I_2} (\delta_{jl}-\delta_{il}) \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}\\ K\in\{x_k,y_k\}}} \hat{L}_{IJ}\odot\hat{L}_{IK}\odot\hat{L}_{JK} \\
&\quad + \sum_{i,j\in S_\epsilon\backslash I_1}\ \sum_{k,l\in S_\epsilon\backslash I_2} (\delta_{jk}-\delta_{ik}) \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}\\ L\in\{x_l,y_l\}}} \hat{L}_{IJ}\odot\hat{L}_{IL}\odot\hat{L}_{JL} \\
&= 2\sum_{i,j\in S_\epsilon\backslash I_1}\ \sum_{k,l\in S_\epsilon\backslash I_2} (\delta_{jl}-\delta_{il}) \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}\\ K\in\{x_k,y_k\}}} \hat{L}_{IJ}\odot\hat{L}_{IK}\odot\hat{L}_{JK} \\
&= 4\sum_{\substack{i\in I_1^c\backslash I_2^c\\ j\in I_1^c\cap I_2^c\\ k\in I_2^c\backslash I_1^c}}\ \sum_{\substack{I\in\{x_i,y_i\}\\ J\in\{x_j,y_j\}\\ K\in\{x_k,y_k\}}} \hat{L}_{IJ}\odot\hat{L}_{IK}\odot\hat{L}_{JK}\,.
\end{aligned}
\tag{3.29}
$$

Here, we have introduced the notation $S_\epsilon=\{1,\dots,m+\epsilon\}$, $I_i^c:=S_\epsilon\backslash I_i$, and to change the index range, we have used the fact that if $i=j$ or $k=l$ then, as C_{ii} or C_{kk} commute with all other C's, their commutator is zero. So we can add this zero to the sum to get the second equality. To show the final equality, one needs to expand the sum and observe that the double sums over the Kronecker deltas will not only eliminate one sum but also restrict the range of the remaining one to the intersection of the sets. Finally one notices that the antisymmetry of the terms in the summation on any exchange of two indices will reduce the sum to be over disjoint ranges.

We also mention this can be written explicitly in terms of the $\mathfrak{u}(m)$ generators:

$$
[K^{(I_1)},K^{(I_2)}]_{\rm SN} = 4\sum_{\substack{i\in I_1^c\backslash I_2^c\\ j\in I_1^c\cap I_2^c\\ k\in I_2^c\backslash I_1^c}} \left(\hat{\rho}_{ij}\odot\hat{\rho}_{ik}\odot\hat{\xi}_{jk}-\hat{\rho}_{ij}\odot\hat{\rho}_{jk}\odot\hat{\xi}_{ik}+\hat{\xi}_{ij}\odot\hat{\xi}_{ik}\odot\hat{\xi}_{jk}+\hat{\xi}_{ij}\odot\hat{\rho}_{ik}\odot\hat{\rho}_{jk}\right). \tag{3.30}
$$

Now, to complete the proof of (3.28) we note that the final result in (3.29) or (3.30) is a sum over all three disjoint sets and so none of the terms can ever cancel. Therefore it is never zero unless the sum is over the empty set. From set algebra, we have $I_i^c\backslash I_j^c \iff I_j\backslash I_i$ and therefore $I_1^c\backslash I_2^c$ (resp. $I_2^c\backslash I_1^c$) is empty if $I_2\subset I_1$ (resp. $I_1\subset I_2$). Thus we have proven the equivalence: $[K^{(I_1)},K^{(I_2)}]=0$ iff $I_2\subset I_1$ or $I_1\subset I_2$.

Finally, we have an obvious collection of $m+\epsilon-1$ mutually commuting Killing tensors given by $I_i\in\{\emptyset,\{1\},\{1,2\},\dots,\{1,2,\dots,m+\epsilon-2\}\}$ or any other such collection of $m+\epsilon$ nested sets. Physically speaking we can understand this as follows. The choice of subsets is a choice of which rotation axes to hold fixed: i.e. ∂_{ϕ_j} for $j\in I_i$. Then $K^{(I_i)}$ becomes the Casimir of the subgroup of rotations which holds fixed these same axes of rotation ∂_{ϕ_j}. Thus, if each successive subset only fixes an additional axis, i.e. $I_{i+1} = I_i \cup \{k\}$ for some ∂_{ϕ_k}, then the Casimirs will commute with each other. The Casimirs $K^{(I_i)}$ thus represent the totally angular momentum about the axis of rotation $\partial_{\phi_j} \in I_i$.

The construction of the higher rank Killing tensors from the iterated SN brackets, as in (2.41), applies straightforwardly to this case. This is the origin of the growing tower of Killing tensors in the Lense-Thirring spacetimes.

3.4 Exact on-shell realization

The generalized Lense-Thirring spacetimes discussed above represent an off-shell example of geometry for which our construction works. It is well known, e.g. [24], that upon choosing particular metric functions these embody approximate slowly rotating black hole solutions in various theories, see e.g. (3.4) for slowly rotating black solutions in Einstein gravity. However, it is interesting to ask whether any *exact* solution of possibly modified Einstein gravity with some matter fields can be cast in this form. Fortunately, at least in four dimensions, this is the case. In what follows we present an exact rotating black hole solution of the EMDA theory which takes the form of generalized Lense-Thirring spacetimes and for which our construction straightforwardly applies.

The EMDA theory is a scalar-vector-tensor theory of gravity, which arises from the low energy effective action of superstring theories, upon compactifying the ten-dimensional heterotic string theory on a six-dimensional torus [50]. It is described by the following action:

$$S = \int d^4x\sqrt{-g}\Big[R - 2\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{4\phi}\,\partial_\mu\kappa\partial^\mu\kappa + e^{-2\phi}F_{\mu\nu}F^{\mu\nu} + \kappa F_{\mu\nu}(*F)^{\mu\nu}\Big]\,,$$

where $F_{\mu\nu}$ is the field strength of U(1) Maxwell field, and κ and ϕ are the axion and the dilaton, respectively. A generalization of the vacuum Kerr solution in this theory has been found by Sen [50], while [51] represents the most general, locally asymptotically flat, extension known (see also [52] and more recently [53]). Here, we are interested in a different exact rotating black hole solution in this theory, namely the one obtained in [39]. It reads

$$\begin{aligned} g &= -\frac{f}{r_0 r}dt^2 + r_0 r\Big[\frac{dr^2}{f} + d\theta^2 + \sin^2\!\theta\big(d\varphi - \frac{a}{r_0 r}dt\big)^2\Big]\,,\\ A &= \frac{\sqrt{2}}{2}\Big(\frac{\rho^2}{r_0 r}dt + a\sin^2\theta d\varphi\Big)\,,\\ e^{-2\phi} &= \frac{r_0 r}{\rho^2}\,,\quad \kappa = -\frac{r_0 a\cos\theta}{\rho^2}\,, \end{aligned} \tag{3.31}$$

where $\rho^2 = r^2 + a^2\cos^2\!\theta$ and $f = r^2 - 2mr + a^2$. This solution represents a rotating black hole with mass $M = \frac{m}{2}$ and angular momentum $J = \frac{ar_0}{2}$ [39].

Interestingly, upon the following coordinate transformation

$$r_0 r \to r^2\,, \tag{3.32}$$

the metric takes the form

$$\begin{aligned} g &= -N\tilde{f}dt^2 + \frac{dr^2}{\tilde{f}} + r^2\sin^2\!\theta\Big(d\varphi - \frac{a}{r^2}dt\Big)^2 + r^2d\theta^2\,,\\ \tilde{f} &= \frac{1}{4} - \frac{mr_0}{2r^2} + \frac{a^2r_0^2}{4r^4}\,,\quad N = \frac{4r^2}{r_0^2}\,, \end{aligned} \tag{3.33}$$

which is precisely of the generalized Lense-Thirring form. Thus, the above EMDA solution represents an exact on-shell example of our construction, with the irreducible Killing tensor given by [39]:

$$K = \frac{1}{\sin^2\theta}(\partial_\varphi)^2 + (\partial_\theta)^2 = L_x^2 + L_y^2 + L_z^2\,, \tag{3.34}$$

which originates from the underlying Killing vectors of the spherical transverse space $\mathcal{S}$; $L_z = \partial_\varphi,\ L_x = \cot\theta\cos\varphi\partial_\varphi + \sin\varphi\partial_\theta,\ L_y = -\cot\theta\sin\varphi\partial_\varphi + \cos\varphi\partial_\theta$.

4 More examples

We now turn to a discussion of other metrics which are of the form (2.1) starting with a physical example where the Killing vector symmetries of the base space lift to the full spacetime. We then consider some illustrative but purely mathematical examples of applications of this formalism.

4.1 Kerr-(A)dS metrics with equal rotation parameters

It is well known that the symmetry of the Kerr-(A)dS metrics in $d = 2m + 1 + \epsilon$ dimensions is enhanced from a U$(1)^m$ to a U(m) symmetry [40, 49]. We point out here that this fits within our metric ansatz (2.1). In particular, the Kerr-Ad(S) metric is given by [5, 6]:

$$\begin{aligned} ds^2 =& -W(1+r^2/\ell^2)dt^2 + \frac{2M}{U}\left(W dt + \sum_{i=1}^{m}\frac{a_i\mu_i^2 d\phi_i}{\Xi_i}\right)^2 \\ &+\sum_{i=1}^{m}\frac{r^2+a_i^2}{\Xi_i}(\mu_i^2 d\phi_i^2 + d\mu_i^2) + \frac{U dr^2}{V-2M} + \epsilon r^2 d\mu_{m+\epsilon}^2 \\ &-\frac{1}{W(\ell^2+r^2)}\left(\sum_{i=1}^{m}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i + \epsilon r^2\mu_{m+\epsilon}d\mu_{m+\epsilon}\right)^2, \end{aligned} \tag{4.1}$$

where ℓ is the AdS radius and

$$\begin{aligned} W =& \sum_{i=1}^{m}\frac{\mu_i^2}{\Xi_i} + \epsilon\mu_{m+\epsilon}^2\,, \quad V = r^{\epsilon-2}(1+r^2/\ell^2)\prod_{i=1}^{m}(r^2+a_i^2)\,, \\ U =& r^{\epsilon}\left(\sum_{i=1}^{m+\epsilon}\frac{\mu_i^2}{r^2+a_i^2}\right)\prod_{i=1}^{m}(r^2+a_i^2)\,, \quad \Xi_i = 1-\frac{a_i^2}{\ell^2}\,. \end{aligned} \tag{4.2}$$

Here, as before, $\epsilon = 1, 0$ for even, odd dimensions, $m = \left[\frac{d-1}{2}\right]$, and the coordinates μ_i obey the same constraint as before (3.2). Finally in these expressions $a_{m+1} = 0 = \phi_{m+1}$.

Next, we can use calculations similar to those detailed in section 3.8 and appendix A of [54] to put (4.1) into a form like (2.1), namely:

$$\begin{aligned} ds^2 =& -f(r,\mu_i)N(r,\mu_i)dt^2 + \frac{dr^2}{f(r,\mu_i)} \\ &+\sum_{i,j=1}^{m}h_{ij}(d\phi_i+\nu^i dt)(d\phi_j+\nu^j dt) + \sum_{i=1}^{m}\frac{r^2+a_i^2}{\Xi_i}d\mu_i^2 + \epsilon r^2 d\mu_{m+\epsilon}^2 \\ &-\frac{1}{W(\ell^2+r^2)}\left(\sum_{i=1}^{m}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i + \epsilon r^2\mu_{m+\epsilon}d\mu_{m+\epsilon}\right)^2. \end{aligned} \tag{4.3}$$

JHEP07(2025)098

Here we have introduced

$$f(r,\mu_i) = \frac{V-2M}{U}\,, \qquad N(r,\mu_i) = \frac{W\sum_{i=1}^{m+\epsilon}\frac{r^2\mu_i^2}{r^2+a_i^2}}{1+\frac{2M}{U}\sum_{i=1}^{m}\frac{a_i^2\mu_i^2}{(r^2+a_i^2)\Xi_i}}\,, \tag{4.4}$$

and

$$\begin{aligned} h_{ij}d\phi_i d\phi_j &= \sum_{i=1}^{m}\frac{r^2+a_i^2}{\Xi_i}\mu_i^2 d\phi_i^2 + \frac{2M}{U}\left(\sum_{i=1}^{m}\frac{a_i\mu_i^2 d\phi_i}{\Xi_i}\right)^2\,,\\ \nu^i\partial_{\phi_i} &= \frac{2M}{U}\frac{N(r,\mu_i)}{\sum_{i=1}^{m+\epsilon}\frac{r^2\mu_i^2}{r^2+a_i^2}}\frac{a_i}{r^2+a_i^2}\partial_{\phi_i}\,. \end{aligned} \tag{4.5}$$

Some comments are in order. Although, the form of (4.3) is similar to the generic form (2.1), it is much messier. In particular, $f(r,\mu_i)$ and $N(r,\mu_i)$ depend on the transverse coordinates μ_i, and the transverse metric γ_{AB} itself (γ_{AB} being h_{ij} and all the remaining terms in the second and third lines of (4.3)) has no obvious symmetries to consider trying to lift to the full spacetime. Thus, although it is possible that the Killing tensor symmetries of Kerr lift from Killing vector symmetries of the transverse space we consider it unlikely, at least from this construction.

However, the situation is very different when all rotation parameters are equal, i.e. $a_i = a$. The constraint implies that the terms in the bracket of the final line of (4.3) are zero in odd dimensions because they are its derivative, i.e. $0 = d(\sum_{i=1}^{m+\epsilon}\mu_i^2) = 2\sum_{i=1}^{m+\epsilon}\mu_i d\mu_i$. In even dimensions, they can be written together in terms of $\epsilon\mu_{m+\epsilon}^2 d\mu_{m+\epsilon}^2$. That is,

$$\frac{1}{W(\ell^2+r^2)}\left(\sum_{i=1}^{m}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i + \epsilon r^2\mu_{m+\epsilon}d\mu_{m+\epsilon}\right)^2 = \frac{(r^2+\ell^2)}{(1-a^2/\ell^2)}\frac{\epsilon a^4\mu_{m+\epsilon}^2/\ell^2}{(1-\epsilon a^2/\ell^2\mu_{m+\epsilon}^2)}d\mu_{m+\epsilon}^2 \tag{4.6}$$

Thus combining the $d\mu_i^2$ terms gives,

$$\begin{aligned} &\frac{r^2+a^2}{1-a^2/\ell^2}\sum_{i=1}^{m}d\mu_i^2 + \epsilon r^2 d\mu_{m+\epsilon}^2 - \frac{(r^2+\ell^2)}{(1-a^2/\ell^2)}\frac{\epsilon a^4\mu_{m+\epsilon}^2/\ell^2}{(1-\epsilon a^2/\ell^2\mu_{m+\epsilon}^2)}d\mu_{m+\epsilon}^2 d\mu_{m+\epsilon}^2\\ &= \frac{r^2+a^2}{1-a^2/\ell^2}\sum_{i=1}^{m+\epsilon}d\mu_i^2 + \epsilon\left(-\frac{r^2+a^2}{1-a^2/\ell^2} + r^2 - \frac{(r^2+\ell^2)}{(1-a^2/\ell^2)}\frac{\epsilon a^4\mu_{m+\epsilon}^2/\ell^2}{(1-\epsilon a^2/\ell^2\mu_{m+\epsilon}^2)}\right)d\mu_{m+\epsilon}^2\\ &= \frac{r^2+a^2}{1-a^2/\ell^2}\sum_{i=1}^{m+\epsilon}d\mu_i^2 - \epsilon\frac{a^2}{\ell^2}\frac{(r^2+\ell^2)}{(1-a^2/\ell^2)(1-\epsilon a^2\mu_{m+\epsilon}^2/\ell^2)}d\mu_{m+\epsilon}^2\,. \end{aligned} \tag{4.7}$$

Hence, in this case we have

$$\begin{aligned} ds^2 &= -f(r,\epsilon\mu_{m+\epsilon}^2)N(r,\epsilon\mu_{m+\epsilon}^2)dt^2 + \frac{dr^2}{f(r,\epsilon\mu_{m+\epsilon}^2)} \\ &\quad + \sum_{i,j=1}^{m}h_{ij}(d\phi_i+\nu^i dt)(d\phi_j+\nu^j dt) + \frac{r^2+a^2}{\Xi}\sum_{i=1}^{m+\epsilon}d\mu_i^2 - \epsilon\Sigma(r,\epsilon\mu_{m+\epsilon}^2)\,d\mu_{m+\epsilon}^2\,, \end{aligned} \tag{4.8}$$

where now

$$\Sigma(r,\epsilon\mu_{m+\epsilon}^2) = \frac{a^2}{\ell^2}\frac{(r^2+\ell^2)}{(1-a^2/\ell^2)(1-\epsilon a^2\mu_{m+\epsilon}^2/\ell^2)}\,, \tag{4.9}$$

and

$$W=\frac{1-\epsilon a^2\mu_{m+\epsilon}^2/\ell^2}{\Xi}\,,\quad \Xi=1-\frac{a^2}{\ell^2}\,,\quad U=r^{\epsilon-2}(r^2+a^2)^{m-1}(r^2+\epsilon a^2\mu_{m+\epsilon}^2)\,. \tag{4.10}$$

Notice that N and f are now functions of r and $\epsilon\mu_{m+\epsilon}^2=\epsilon(1-\sum_i^{m+\epsilon-1}\mu_i^2)$, i.e. $N=N(r,\epsilon\mu_{m+\epsilon}^2)$, $f=f(r,\epsilon\mu_{m+\epsilon}^2)$, whose explicit expressions are not important. Moreover, this yields

$$\begin{aligned} h_{ij}d\phi_i d\phi_j &= \frac{r^2+a^2}{\Xi}\sum_{i=1}^m \mu_i^2 d\phi_i^2+\frac{2Ma^2r^{2-\epsilon}}{(r^2+a^2)^{m-1}(r^2+\epsilon a^2\mu_{m+\epsilon}^2)\Xi^2}\left(\sum_{i=1}^m\mu_i^2 d\phi_i\right)^2\,,\\ \nu^i\partial_{\phi_i} &= \frac{2MN(r,\epsilon\mu_{m+\epsilon}^2)}{r^\epsilon(r^2+a^2)^{m-1}}\partial_{\phi_i}\,. \end{aligned} \tag{4.11}$$

One can check that this recovers the expressions in [49] in the asymptotically flat limit, i.e. when $\ell\to\infty$.

Now, in odd dimensions it is clear that the metric is in the form of (2.1) since $\epsilon=0$ the dependence on μ_i in N, f, and ν^i is gone.[13] In fact, we shall show that certain symmetries of the transverse metric lift to symmetries of $\mathcal{M}$ in both even and odd dimensions.

The question is then: what are the Killing vectors of the transverse metric? To answer this, we observe the transverse metric of (4.8) is

$$\begin{aligned} \gamma_{AB}dx^Adx^B = &\underbrace{\frac{r^2+a^2}{1-\frac{a^2}{\ell^2}}\left(\sum_{i=1}^{m+\epsilon}d\mu_i^2+\sum_{i=1}^m\mu_i^2 d\phi_i^2\right)-\epsilon\Sigma(r,\epsilon\mu_{m+\epsilon}^2)\,d\mu_{m+\epsilon}^2}_{:=\mathring{\gamma}_{AB}dx^Adx^B}\\ &+\frac{2Ma^2r^{2-\epsilon}}{(r^2+a^2)^{m-1}(r^2+\epsilon a^2\mu_{m+\epsilon}^2)\Xi^2}\left(\sum_{i=1}^m\mu_i^2 d\phi_i\right)^2\,. \end{aligned} \tag{4.12}$$

The first line of this equation is, in odd dimensions, simply the metric on the m sphere $\mathring{\gamma}_{AB}$. Thus it has the same Killing vectors (3.12) for $i,j\le m$. The extra $\epsilon d\mu_{m+\epsilon}^2$ in even dimensions breaks this symmetry. However, one can easily check, using Cartan's formula for the Lie derivative, that the sub-algebra $\mathfrak{u}(m)$ generators ((3.20) and (3.21)) are still symmetries. Clearly they are symmetries of the bracketed term in both even and odd dimensions and moreover,

$$\begin{aligned} \mathcal{L}_\xi(d\mu_{m+\epsilon}^2) &= i_\xi d^2(\mu_{m+\epsilon}^2)+d(i_\xi(\mu_{m+\epsilon}^2))=0\,,\\ \mathcal{L}_\rho(d\mu_{m+\epsilon}^2) &= i_\rho d^2(\mu_{m+\epsilon}^2)+d(i_\rho(\mu_{m+\epsilon}^2))=0\,. \end{aligned} \tag{4.13}$$

The last equality in each case follows from using the explicit expressions of ξ and ρ ((3.20) and (3.21)) to see that

$$\xi^A\partial_A(\mu_{m+\epsilon}^2)=0=\rho^A\partial_A(\mu_{m+\epsilon}^2)\,. \tag{4.14}$$

This also implies the Lie derivative along ξ or ρ of any function of $\mu_{m+\epsilon}^2$ is zero, e.g. $\Sigma(r,\epsilon\mu_{m+\epsilon}^2)$. Hence, their Lie derivatives of the final term in $\mathring{\gamma}$ vanishes. Thus, ξ and ρ are Killing vectors of $\mathring{\gamma}$.

[13] In odd dimensions the metric has also be rewritten into the dual ansatz form presented in appendix B (B.1), see e.g. [55].

Next, one can see that the off-diagonal $d\phi_i d\phi_j$ terms in (4.12) are of the form

$$h(r,\epsilon\mu_{m+\epsilon}^2)(\mathring{\gamma}_{AB}\zeta^B dx^A)^2\,,\quad \text{where } \zeta^A\partial_A=\sum_{i=1}^m \partial_{\phi_i}=\frac{1}{2}\sum_{i=1}^m \rho_{ii}\,, \tag{4.15}$$

for some function $h(r,\epsilon\mu_{m+\epsilon}^2)$ whose particular form is unimportant. Then one can immediately check, using (3.22) that the $\mathfrak{u}(m)$ generators (ξ and ρ) commute with ζ. That is,

$$\begin{aligned}
[\![\xi_{ij},\zeta]\!]&=\frac{1}{2}\sum_{k=1}^m[\![\xi_{ij},\rho_{kk}]\!]=\sum_{k=1}^m(\delta_{jk}\rho_{ik}-\delta_{ik}\rho_{jk})=\rho_{ji}-\rho_{ij}=0\,,\\
[\![\rho_{ij},\zeta]\!]&=\frac{1}{2}\sum_{k=1}^m[\![\rho_{ij},\rho_{kk}]\!]=-\sum_{k=1}^m(\delta_{jk}\xi_{ik}+\delta_{ik}\xi_{jk})=-(\xi_{ji}+\xi_{ij})=0\,.
\end{aligned}\tag{4.16}$$

As mentioned above, they also Lie commute with any function $h(\epsilon\mu_{m+\epsilon}^2)$ and hence are Killing vectors not only of $\mathring{\gamma}$ but also of the full γ_{AB} in (4.12).[14]

Furthermore, this means that ξ and ρ lift to Killing vectors of the full spacetime, since $\nu=\tilde{h}(r,\epsilon\mu_{m+\epsilon}^2)\zeta$ and so for precisely the same reasons

$$\begin{aligned}
[\![\xi,\nu]\!]=0=[\![\rho,\nu]\!]\,,\quad \xi^A\partial_A f(r,\mu_{m+\epsilon}^2)=0=\rho^A\partial_A f(r,\mu_{m+\epsilon}^2)\,,\\
\xi^A\partial_A N(r,\mu_{m+\epsilon}^2)=0=\rho^A\partial_A N(r,\mu_{m+\epsilon}^2)\,.
\end{aligned}\tag{4.17}$$

and thence making use of the decomposition (2.10), we see

$$[\hat{\xi},g^{-1}]_{\text{SN}}=0=[\hat{\rho},g^{-1}]_{\text{SN}}\,. \tag{4.18}$$

Therefore, we have shown that the $\mathrm{U}(m)$ symmetry of Myers-Perry black holes with equal rotation parameters can be understood as the lifting of the $\mathrm{U}(m)$ symmetry of the transverse metric to the full spacetime. At the same time, the corresponding Casimirs give rise to reducible Killing tensors of the full spacetime. We expect that similar conclusions can also be derived in less symmetric cases, such as those when there are two sets of equal rotation parameters [56], in which case, however, there already exists a single irreducible Killing tensor.

4.2 Planar symmetry

Four-dimensional example. To discuss our first 'purely mathematical example', we consider the following metric:

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=-Nf dt^2+\frac{dr^2}{f}+r^2d\rho^2+r^2\rho^2(d\phi+\omega dt)^2\,, \tag{4.19}$$

where $\omega=\omega(r)$ is a function of the radial coordinate.

In the special case where $\omega=0$, the metric exhibits four Killing vector fields, corresponding to the isometries ∂_t, ∂_ϕ, ∂_x, and ∂_y, where we introduce Cartesian coordinates $x=\rho\cos\phi$ and $y=\rho\sin\phi$. These symmetries reflect the underlying planar invariance of the two-dimensional

[14] In four dimensions these results apply to Kerr-(A)dS, in particular (4.12) remains true. However, the only non-zero vector is the U(1) symmetry of ∂_ϕ. So the only obvious symmetry of the base space is the usual axi-symmetry.

spatial slices parametrised by (x, y). Specifically, ∂_x and ∂_y generate translations in the Cartesian plane, while ∂_ϕ corresponds to rotational symmetry about the origin, and ∂_t represents time translation invariance.

However, when $\omega \neq 0$, the presence of additional terms in the metric explicitly breaks the full planar symmetry. In particular, the full Killing algebra no longer includes ∂_x and ∂_y as global symmetries of the spacetime. Nevertheless, the two-dimensional spatial metric, given by

$$\gamma = \gamma_{AB} dx^A dx^B = r^2(d\rho^2 + \rho^2 d\phi^2) = r^2(dx^2 + dy^2)\,, \tag{4.20}$$

remains invariant under translations in the (x, y) plane.

Using the notation from section 2, the shift vector in (4.19) is given by $\nu = \omega(r)\partial_\phi$. Killing vector ∂_ϕ does not commute with ∂_x and ∂_y separately, but it does commute with their quadratic combination given by $C \equiv \partial_x^2 + \partial_y^2 = r^2\gamma^{-1}$:

$$\begin{aligned} &[\![\partial_\phi, \partial_x]\!] = -\partial_y, \quad [\![\partial_\phi, \partial_y]\!] = \partial_x\,, \\ &[\![\partial_\phi, C]\!]_{\mathrm{SN}} = [\![\partial_\phi, \partial_x^2]\!]_{\mathrm{SN}} + [\![\partial_\phi, \partial_x^2]\!]_{\mathrm{SN}} = -2\partial_x\partial_y + 2\partial_x\partial_y = 0\,, \end{aligned} \tag{4.21}$$

with the latter obvious, as ∂_ϕ is a Killing vector of γ. Since it is a linear combination of Killing vectors, C is a rank-two reducible Killing tensor of the 2-planes. Its commutativity with ∂_ϕ ensures that this rank-two reducible Killing tensor of the 2-planes lifts to an irreducible Killing tensor of the full spacetime

$$\hat{C} = \partial_x^2 + \partial_y^2\,. \tag{4.22}$$

The implications for geodesic motion are as follows. The 2-dimensional base space admits 3 independent integrals of motion, associated with for example the following set of independent symmetries: $\{\partial_\phi, \partial_x, C\}$. Of these one can choose 2 commuting: $\{\partial_\phi, C\}$. As expected, the motion on $\mathcal{S}$ is thus maximally superintegrable. Upon the lift to $\mathcal{M}$ we now have the following set of 4 independent and mutually commuting symmetries: $\{\partial_t, \partial_\phi, \hat{C}, g^{-1}\}$, while ∂_x is no longer a Killing vector. The motion in $\mathcal{M}$ is thus 'only' completely integrable.

The spacetime (4.19) was identified in [57] in the context of broken planar symmetry, and shown to admit a hidden symmetry encoded in the rank-two Killing tensor matching our expression for $\hat{C}$. Here we have shown that this hidden symmetry stems from explicit symmetries in the two-dimensional leaves.[15]

Higher dimensions. The previous example generalizes naturally to higher dimensions. The metric now takes the form:

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + r^2 d\Sigma^2_{(n)} + r^2\rho^2(d\theta_{n-1} + \omega dt)^2\,, \tag{4.24}$$

[15] Moreover, one can check that the generalization of (4.19) introduced in [57]:

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + r^2 d\rho^2 + \frac{r^2\rho^2}{1 + s(r)\rho^2}(d\phi + \omega dt)^2\,, \tag{4.23}$$

represents the most general deformation of the transverse metric, $\gamma_{AB}dx^A dx^B = r^2(d\rho^2 + \rho^2/(1 + s(r)\rho^2)d\phi^2$, which breaks the Killing vector symmetry of ∂_x and ∂_y, such that C in (4.22) remains a Killing tensor. The lifting of C to a Killing tensor of the full metric $\hat{C}$ is immediate in our construction.

where $d\Sigma^2_{(n)}$ represents the metric of an n-dimensional Euclidean space in polar coordinates, given by

$$d\Sigma^2_{(n)} = d\rho^2 + \rho^2 d\Omega^2_{n-1}\,. \tag{4.25}$$

Here, $d\Omega^2_{n-1}$ is the standard metric on the $(n-1)$-dimensional unit sphere, explicitly written as

$$d\Omega^2_{(n-1)} = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2 + \cdots + \prod_{i=1}^{n-2} \sin^2\theta_i d\theta^2_{n-1}\,. \tag{4.26}$$

The shift vector in (4.24) is again proportional to a Killing vector,

$$\nu = \omega(r)\partial_{\theta_{n-1}}\,. \tag{4.27}$$

To analyse the isometries, we introduce Cartesian coordinates $(x_1, \ldots, x_n)$ related to the spherical coordinates by

$$\begin{aligned}
x_1 &= \rho\cos\theta_1\,,\\
x_2 &= \rho\sin\theta_1\cos\theta_2\,,\\
&\;\;\vdots\\
x_{n-1} &= \rho\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1}\,,\\
x_n &= \rho\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}\,.
\end{aligned} \tag{4.28}$$

Isometries of the n-dimensional Euclidean space correspond to the generators ∂_{x_i}. The special Killing vector $\partial_{\theta_{n-1}}$ induces a rotation in the (x_{n-1}, x_n)-plane, leading to the following commutation relations:

$$[\![\partial_{\theta_{n-1}}, \partial_{x_{n-1}}]\!] = -\partial_{x_n}\,, \quad [\![\partial_{\theta_{n-1}}, \partial_{x_n}]\!] = \partial_{x_{n-1}}\,, \quad [\![\partial_{\theta_{n-1}}, \partial_{x_i}]\!] = 0 \quad \text{for } i < n-1\,. \tag{4.29}$$

Vectors ∂_{x_i} for $i < n-1$ remain Killing vectors of the full spacetime due to their commutativity with the shift vector. However, $\partial_{x_{n-1}}$ and ∂_{x_n} do not, as they mix under the action of $\partial_{\theta_{n-1}}$. Nevertheless, their quadratic combination remains a symmetry and lifts to a rank-two Killing tensor. Specifically, defining $C \equiv \partial^2_{x_{n-1}} + \partial^2_{x_n}$, we find that it remains invariant under the action of $\partial_{\theta_{n-1}}$:

$$[\![\partial_{\theta_{n-1}}, C]\!]_{\text{SN}} = [\![\partial_{\theta_{n-1}}, \partial^2_{x_{n-1}}]\!]_{\text{SN}} + [\![\partial_{\theta_{n-1}}, \partial^2_{x_n}]\!]_{\text{SN}} = -2\partial_{x_{n-1}}\partial_{x_n} + 2\partial_{x_n}\partial_{x_{n-1}} = 0\,. \tag{4.30}$$

Again, C is a rank-two reducible Killing tensor in the n-dimensional Euclidean space. Its commutativity with $\partial_{\theta_{n-1}}$ ensures that it lifts to an irreducible Killing tensor of the full spacetime:

$$\hat{C} = \partial^2_{x_{n-1}} + \partial^2_{x_n}. \tag{4.31}$$

4.3 Lifting the Taub-NUT metric

To emphasise the utility of the formalism, let us now construct a final mathematical example. A recent analysis of symmetry reductions of the gravitational Lagrangian [58] offers a compelling starting point for selecting the symmetry group of the base space in our construction. Their classification of admissible isometry groups can be used as a guide for building spacetimes with hidden symmetries within our framework. While their approach is formulated in Lorentzian signature, it can be adapted to our setting either by implementing a timelike foliation in our construction or by extending their methods to spacetimes with Euclidean signature. One of the viable symmetry groups identified there is that of Taub-NUT spacetime.

Here we start from its Euclideanized version [59–61] discussed in appendix B, where we set $n \to in$ and $t \to i\rho$. Let us also (for simplicity) suppress the radial direction of the full Taub-NUT, that is, we consider $x^A = \{\varphi, \theta, \rho\}$. Then we have

$$\gamma = \gamma_{AB} dx^A dx^B = g(d\rho - 2n\cos\theta d\varphi)^2 + (r^2 - n^2)(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{4.32}$$

Here we have introduced an arbitrary function $g = g(t, r)$ and t, r are treated as external parameters. The metric γ admits 4 Killing vectors corresponding to $\mathrm{SU}(2) \times \mathbb{R}$ isometry, namely:

$$\begin{aligned}
\xi_1 &= -\sin\varphi\cot\theta\partial_\varphi + \cos\varphi\partial_\theta - 2n\sin\varphi\frac{1-\cos\theta}{\sin\theta}\partial_\rho\,,\\
\xi_2 &= \cos\varphi\cot\theta\partial_\varphi + \sin\varphi\partial_\theta + 2n\cos\varphi\frac{1-\cos\theta}{\sin\theta}\partial_\rho\,,\\
\xi_3 &= \partial_\varphi - 2n\partial_\rho\,, \quad \eta = \partial_\rho\,,
\end{aligned} \tag{4.33}$$

see [61–63] for a complete discussion of the symmetries of γ_{AB}. We also have

$$[\![C, \xi_3]\!]_{\mathrm{SN}} = 0\,, \quad C = \sum_i \xi_i^2\,. \tag{4.34}$$

Then, choosing $\nu = p(t, r)\xi_3$ the following spacetime:

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + \gamma_{AB}\Big(dx^A + p[\delta^A_\varphi - 2n\delta^A_\rho]dt\Big)\Big(dx^B + p[\delta^B_\varphi - 2n\delta^B_\rho]dt\Big)\,, \tag{4.35}$$

where $f = f(t, r), N = N(t, r)$, and $p = p(t, r)$ are arbitrary functions of t and r, satisfies all conditions of our construction. In particular it will admit a non-trivial Killing tensor, given by $\hat{C}$. Of course one could generalize this example to arbitrary dimensions using the $(d-2)$-dimensional version of the Euclideanized Taub-NUT, e.g. [64, 65] (suppressing potentially its radial direction which plays no role).

5 Discussion and summary

In this work we have investigated the emergence of irreducible Killing tensors from lower-dimensional Killing vectors. While explicit spacetime symmetries are described by Killing vectors, many spacetimes exhibit hidden symmetries encoded in higher-rank Killing tensors. Recent studies on Lense-Thirring spacetimes have suggested the existence of an infinite hierarchy of such tensors, but their geometric origin had remained unclear.

In contrast to previous related ideas [29–32] we have proposed a systematic construction based on a spacetime ansatz foliated by codimension-2 hypersurfaces. Our key finding is that Killing tensors, which are reducible to products of non-commuting Killing vectors in the base space, naturally lift to irreducible Killing tensors of the full spacetime. This structure is dictated by the Lie algebra of the base space symmetries, leading to a tower of higher-rank Killing tensors. We then applied this framework to generalized Lense-Thirring spacetimes and demonstrated its realization in exact four-dimensional EMDA black hole solutions. It remains to be seen, if any other irreducible Killing tensors of exact black hole spacetimes can be understood in this way. Moreover, the $\mathrm{U}([d/2]-1)$ symmetry of Kerr-(A)dS spacetimes with equal angular momenta was shown to emerge from the same such symmetry of the lower-dimensional hypersurface.

These results provide a new geometric understanding of hidden symmetries in higher-dimensional spacetimes and have several natural extensions. In particular, one could consider topological Lense-Thirring spacetimes, where the cross-sectional metric γ_{AB} on $\mathcal{S}$ has for example negative constant curvature. Such metrics solve, for example, the linearised Einstein equations (see appendix H of [66]) and will inherit in exactly the same way the symmetries of γ_{AB}. Of course, if $\mathcal{S}$ is compact and negatively curved it has no (global) Killing vectors. We expect this would also lead to new topological slowly rotating black holes in many such theories beyond Einstein gravity (e.g, those discussed in [24]) with lifted symmetries.

More generally speaking, can one construct initial data à la [29] for higher order symmetries i.e. Killing tensors? And under what conditions would Killing-Yano tensors lift to the full spacetime for our metric ansatz (2.1)? Furthermore, we could seek to adapt the geometric lifting procedure in [31, 32] to the general co-dimension-2 foliations outlined in appendix A. It would be natural to expect that geometrically nice (e.g. geodesic) foliations lead to natural lifts of symmetries of the transverse space. These would be applicable to such special coordinate systems like Gaussian null (Isenberg-Moncrief) [67] and Bondi [68, 69], which have become ubiquitous for understanding important physical situations.

Finally, continuing this theme, we could seek to understand the emergence of near horizon/near future null infinity symmetries via our geometric construction. In these asymptotic regimes, the spacetime geometry simplifies and can be effectively described by a fibration over a transverse spatial surface $\mathcal{S}$. Using our construction one could study possible extensions of asymptotic symmetry algebras, such as the BMS group at null infinity [68–70], to include higher-rank symmetry generators. It would be interesting to investigate whether lifted Killing tensors associated with the transverse space in codimension-2 foliations could give rise to a novel class of asymptotic symmetries, potentially forming a higher-spin or tensorial extension of the BMS algebra, similar to what was shown for flat space in [71], or generalizing the Killing tensor surface charges presented in [72] for constant curvature spacetimes. Such a structure could lead to new insights into conserved quantities relevant for gravitational radiation, memory effects, and soft theorems. We leave these directions for future work.

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A Foliation of spacetime by codimension-2 hypersurfaces

A.1 Construction

In this appendix we follow the conventions set up in chapter 2 of [73]. We consider a $(2+(d-2))$ foliation of spacetime induced by two scalar fields ψ^a, with $a = 0, 1$. These define two closed one-forms, $n^a = d\psi^a$, which we interpret as normal covectors to a family of codimension-2 surfaces, denoted $\mathcal{S}$.

The symmetric lapse matrix is defined by

$$N_{ab} \equiv n^a_\mu n^{b\mu}\,, \tag{A.1}$$

and assumed to be non-degenerate with $\det N_{ab} < 0$, so that the surfaces $\mathcal{S}$ are spacelike. The inverse matrix N^{ab} is used to define the dual basis of vector fields

$$n^\mu_a \equiv N_{ab} g^{\mu\nu} n^b_\nu\,, \tag{A.2}$$

where each n^μ_a is tangent to the hypersurface $\psi^b = \text{const}$ for $b \neq a$. We introduce the projector onto the codimension-2 leaves $\mathcal{S}$ as

$$\gamma^\mu_\nu \equiv \delta^\mu_\nu - N_{ab} n^a_\nu n^{b\mu}, \tag{A.3}$$

with the induced metric on S given by

$$\gamma_{\mu\nu} \equiv \gamma^\lambda_\mu \gamma^\sigma_\nu g_{\lambda\sigma}. \tag{A.4}$$

The complement of the surfaces S is a two-dimensional plane spanned by n^μ_a, which need not be integrable. The obstruction to integrability is the twist, defined through the commutator

$$[n_0, n_1]^\mu \notin \text{Span}\,\{n^\mu_a\}, \quad \text{or equivalently} \quad \gamma^\mu_\nu [n_0, n_1]^\nu \neq 0. \tag{A.5}$$

To facilitate coordinate expressions, we introduce a set of coordinates adapted to the foliation, (ψ^a, x^A), where x^A parameterize the leaves $\mathcal{S}$, and ψ^a serve as foliation parameters. The coordinate vectors ∂^a_ψ need not be orthogonal to $\mathcal{S}$; this motivates the introduction of shift vectors

$$b^\mu_a \equiv (\partial_{\psi^a})^\mu - n^\mu_a. \tag{A.6}$$

In this coordinate system, the spacetime metric takes the form

$$g_{\mu\nu} = \begin{pmatrix} N_{ab} + \gamma_{AB} b^A_a b^B_b & \gamma_{AC} b^C_b \\ \gamma_{BC} b^C_a & \gamma_{AB} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} N^{ab} & -N^{a\mu} b^B_\mu \\ -N^{b\mu} b^A_\mu & \gamma^{AB} + N^{\mu\nu} b^A_\mu b^B_\nu \end{pmatrix}, \tag{A.7}$$

JHEP07(2025)098

with the determinant

$$g = \det N_{ab} \det \gamma_{AB}. \tag{A.8}$$

We also note the expression for the inverse relation

$$n_a^\mu = N_{ab} g^{\mu\nu} n_\nu^b = N_{ab} g^{\mu b}, \tag{A.9}$$

and the expression for the commutator of the normal vectors

$$[n_a, n_b]^\mu = -2\partial_{[a} b_{b]}^\mu + [b_a, b_b]^\mu, \tag{A.10}$$

with $b_a^\mu = (0, b_a^A)$ in adapted coordinates.

A.2 Gauge fixing

Up to this point, the formalism is fully general and no gauge fixing has been imposed. To connect with the metric ansatz discussed in section 2, we now impose a specific choice of coordinates (t, r, x^A) and gauge conditions:

1. We fix the off-diagonal components of the lapse matrix $N_{tr} = N_{rt} = 0$.
2. We choose the radial shift vector to vanish $b_r^\mu = 0$, which amounts to taking the coordinates x^A constant along the radial direction.

With this, we identify:

$$b_t^A = \nu^A, \tag{A.11}$$

$$N_{tt} = -Nf, \tag{A.12}$$

$$N_{rr} = \frac{1}{f}. \tag{A.13}$$

The resulting parametrized metric becomes

$$g_{\mu\nu} = \begin{pmatrix} -Nf + r^2\gamma_{CD}\nu^C\nu^D & 0 & \gamma_{AC}\nu^C \\ 0 & \frac{1}{f} & 0 \\ \gamma_{BC}\nu^C & 0 & \gamma_{AB} \end{pmatrix}, \tag{A.14}$$

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{Nf} & 0 & \frac{1}{Nf}\nu^B \\ 0 & f & 0 \\ \frac{1}{Nf}\nu^A & 0 & \gamma^{AB} - \frac{1}{Nf}\nu^A\nu^B \end{pmatrix}, \tag{A.15}$$

which agrees with (2.1) and (2.2) in the main text. Here, however for full generality the functions f and N depend on all the coordinates $x^\mu = (t, r, x^A)$.

From the general integrability condition in eq. (A.10), we observe that the normal 2-planes are integrable if and only if ν^A is independent of the radial coordinate r. However, if ν^A has the form

$$\nu^A = p(t, r)\xi_0^A(x^B), \tag{A.16}$$

JHEP07(2025)098

as assumed in the main text, then the twist is non-vanishing and proportional to the Killing vector ξ_0^A. Explicitly,

$$[n_r, n_t]^\mu = -\partial_r \hat{\nu}^\mu = -\frac{\partial p(t,r)}{\partial r}\hat{\xi}_0^\mu . \tag{A.17}$$

B Alternative construction

B.1 Dual metric ansatz

In this appendix we briefly comment on a possible dual formulation of our general ansatz (2.1). Namely, let us consider the following metric:

$$\tilde{g} = -\tilde{f}\tilde{N}(dt + \tilde{\nu}_A dx^A)^2 + \frac{dr^2}{\tilde{f}} + \tilde{\gamma}_{AB} dx^A dx^B , \tag{B.1}$$

cf. the inverse metric (2.2). Here, we again assume $\tilde{f}$ and $\tilde{N}$ to be functions of t, r only and $\tilde{\nu}^A$ and γ_{AB} to be functions of all coordinates. Moreover, $\tilde{\nu}_A$ is naturally a 1-form while previously ν^A was naturally a vector. The inverse now reads

$$\tilde{g}^{-1} = -\frac{1}{\tilde{N}\tilde{f}}\partial_t^2 + \tilde{f}\partial_r^2 + \tilde{\gamma}^{AB}(\partial_A - \tilde{\nu}_A\partial_t)(\partial_B + \tilde{\nu}_B\partial_t) . \tag{B.2}$$

Notice that the structures are more or less the same, except by comparing the ∂_t^2, $\partial_t\partial_A$, $\partial_A\partial_B$ terms we have

$$(\tilde{f}\tilde{N})^{-1} - \tilde{\nu}^A\tilde{\gamma}_{AB}\tilde{\nu}^B = (fN)^{-1} , \quad \tilde{\nu}^A = -(fN)^{-1}\nu^A , \quad \tilde{\gamma}^{AB} - \tilde{f}\tilde{N}\tilde{\nu}^A\tilde{\nu}^B = \gamma^{AB} , \tag{B.3}$$

where $\gamma^{AB} = (\gamma^{-1})^{AB}$ and $\tilde{\gamma}^{AB} = (\tilde{\gamma}^{-1})^{AB}$. Thus one can translate from one description to another. However, if the transverse metric is fixed (e.g. in both case setting it to be the metric on the sphere) then symmetries in one description will not be symmetries in the other unless $\tilde{\nu}^2 \equiv \tilde{\nu}^A\tilde{\gamma}_{AB}\tilde{\nu}^B$ is x^A independent.

In this case the decomposition of the SN bracket (2.10) becomes

$$\begin{aligned}
[\hat{X}, g^{-1}]_{\text{SN}}^{\mu_1\ldots\mu_{p+1}}\partial_{(\mu_1\ldots\mu_{p+1})} - [\![X, (\tilde{f}\tilde{N})^{-1} - \tilde{\nu}^2]\!]_{\text{SN}}^{A_1\ldots A_{p-1}}\ &\partial_{(t}\partial_t\partial_{A_1}\ldots\partial_{A_{p-1})} \\
+ [\![X, \tilde{f}]\!]_{\text{SN}}^{A_1\ldots A_{p-1}}\ &\partial_{(r}\partial_r\partial_{A_1}\ldots\partial_{A_{p-1})} \\
+ 2\left((\tilde{f}\tilde{N})^{-1}\partial_t X^{A_1\ldots A_p} - [\![X, \tilde{\nu}]\!]_{\text{SN}}^{A_1\ldots A_p}\right)&\partial_{(t}\partial_{A_1}\ldots\partial_{A_p)} \\
- 2\tilde{f}\left(\partial_r X^{A_1\ldots A_p}\right)\ &\partial_{(r}\partial_{A_1}\ldots\partial_{A_p)} \\
+ 2\tilde{\nu}^{(A_1}\partial_t X^{A_2\ldots A_{p+1})}&\partial_{(A_1}\ldots\partial_{A_{p+1})} \\
+ [\![X, \tilde{\gamma}^{-1}]\!]_{\text{SN}}^{A_1\ldots A_{p+1}}&\partial_{(A_1}\ldots\partial_{A_{p+1})} .
\end{aligned} \tag{B.4}$$

Hence with this dual choice of metric ansatz, the conditions for a symmetry of $\mathcal{S}$ to lift to the full spacetime $\mathcal{M}$ are *almost* exactly the same as previously. Namely, $[\![X, \tilde{f}\tilde{N}]\!]_{\text{SN}} = 0$ (again automatically satisfied as $\tilde{f}, \tilde{N}$ depend on t, r only, i.e. $\tilde{f} = \tilde{f}(t,r)$ and $\tilde{N} = \tilde{N}(t,r)$), and

$$\partial_t X = 0 , \quad \partial_r X = 0 , \quad [\![X, \tilde{\nu}^2]\!]_{\text{SN}} = 0 , \quad [\![X, \tilde{\nu}]\!]_{\text{SN}} = 0 . \tag{B.5}$$

It is worth now discussing the third condition in more detail. If $\tilde{\nu}^2$ is x^A independent then it is trivial. But, in the case where $X = \sum_i \xi_i \otimes \xi_i$ for Killing vectors ξ_i, it reduces to the requirement

$$\sum_i \left[\!\left[\xi_i \otimes \xi_i, \tilde{\nu}^2\right]\!\right]^A = 2\tilde{\nu}_B \sum_i \xi_i^A [\xi_i, \tilde{\nu}]^B \,. \tag{B.6}$$

This is different to last condition of (B.5) (which as we have seen previously is given by (2.27)) because there is no symmetrization over the indices $\{AB\}$. Thus for the dual metric ansatz we actually have a *stronger* requirement for a symmetry to lift to the full spacetime.

As we have remarked in the main text, the Myers-Perry metrics with equal rotation parameters [55] are in the form of (B.1). We note moreover that, the Killing tensors of the Kerr-NUT-AdS black are almost of this form. See [33] wherein, the Killing tensors are shown to be Killing tensors of the induced metric on the t =const. hypersurfaces, i.e. a co-dimension *one* foliation.

B.2 Comparison with Taub-NUT metrics

An example of a spacetime that takes also the form (B.1) is the Taub-NUT metric [59, 60]

$$ds^2 = -f(r)(dt - 2n\cos\theta d\phi)^2 + \frac{dr^2}{f(r)} + (r^2+n^2)(d\theta^2 + \sin^2\theta d\phi^2) \,, \tag{B.7}$$

where $f(r) = (r^2+n^2-2mr)/(r^2+n^2)$. However, one can check that the corresponding vector

$$\tilde{\nu} = \frac{-2n\cos\theta}{(r^2+n^2)\sin^2\theta}\partial_\phi \,, \tag{B.8}$$

is not a symmetry of the metric $\tilde{\gamma} = \tilde{\gamma}_{AB}dx^A dx^B = (r^2+n^2)(d\theta^2 + \sin^2\theta d\phi^2)$. These are simply the Killing vectors of the 2-sphere

$$L_x = -\sin\phi\,\partial_\theta - \cos\phi\cot\theta\partial_\phi \,, \quad L_y = \cos\phi\,\partial_\theta - \sin\phi\cot\theta\,\partial_\phi \,, \quad L_z = \partial_\phi \,. \tag{B.9}$$

Note that, as mentioned in subsection 4.3, the Taub-NUT metric actually admits 4 full Killing vectors — see, e.g. appendix A of [63] for details. Moreover, the total angular momentum (our natural candidate for lifting)

$$C = L_x^2 + L_y^2 + L_z^2 = \partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\phi^2 = (r^2+n^2)\gamma^{-1} \,, \tag{B.10}$$

does not commute with ν^A, i.e.,

$$[\![\nu, C]\!]_{\mathrm{SN}} \neq 0 \,. \tag{B.11}$$

Finally, since $C^{AB} = (r^2+n^2)\gamma^{AB}$ the third condition of (B.5) becomes simply that

$$\partial_A(\nu^B\nu_B) = \frac{4n^2}{(r^2+n^2)}\partial_A(\cot^2\theta) \neq 0 \,, \tag{B.12}$$

which is also explicitly violated.

Therefore, $\hat{C}$ is not one of the Killing tensors of the Taub-NUT spacetime. See, e.g. [61, 62] for a complete discussion of the symmetries of the Taub-NUT spacetime. We also mention that exactly the same calculations apply to the Page instanton [74] (or any other such Euclideanization of the Taub-NUT), that is, the symmetries do *not* lift from a base space according to the procedure discussed in this appendix.

C Schouten-Nijenhuis brackets

Here we collect various properties of the SN bracket defined over any manifold $\mathcal{N}$ which we employ in the main text. We use the early greek alphabet indices α, β, γ on $\mathcal{N}$ and note that the properties apply equally to quantities on $\mathcal{M}$ or $\mathcal{S}$. These properties of the SN bracket are due to the one-to-one correspondence with the Poisson bracket which satisfies the same kind of properties.

Namely, consider the Poisson bracket on the space of functions on the cotangent bundle $\mathcal{T}^*\mathcal{N}$, defined in Darboux coordinates

$$\{f, g\} \equiv \frac{\partial f}{\partial x^\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial x^\alpha}\,. \tag{C.1}$$

Such a Poisson bracket is clearly antisymmetric and bilinear and satisfies the following properties:

$$\text{i) Jacobi:} \qquad \{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0\,, \tag{C.2}$$

$$\text{ii) Leibniz:} \qquad \{fg, h\} = \{f, h\}g + f\{g, h\}\,. \tag{C.3}$$

Thus, it defines a Leibniz Lie algebra.

We can then use the isomorsphism between the space of monomials on the contangent bundle and the space of symmetric tensors to define the SN bracket.[16] That is, for symmetric rank-p/rank-q tensors $A^{\alpha_1\dots\alpha_p}$ and $B^{\alpha_1\dots\alpha_q}$ define their corresponding monomials of momentum on the cotangent bundle as $a = A^{\alpha_1\dots\alpha_p}p_{\alpha_1}\dots p_{\alpha_p}$ and $b = B^{\alpha_1\dots\alpha_q}p_{\alpha_1}\dots p_{\alpha_q}$. Then from their Poisson bracket

$$c \equiv \{a, b\} = -[A, B]_{\mathrm{SN}}^{\alpha_1\dots\alpha_{p+q-1}}p_{\alpha_1}\dots p_{\alpha_{p+q-1}}\,, \tag{C.4}$$

we have the SN bracket naturally defined

$$[A, B]_{\mathrm{SN}}^{\alpha_1\dots\alpha_{p+q-1}} \equiv pA^{\gamma(\alpha_1\dots\alpha_{p-1}}\partial_\gamma B^{\alpha_p\dots\alpha_{p+q-1})} - qB^{\gamma(\alpha_1\dots\alpha_{q-1}}\partial_\gamma A^{\alpha_q\dots\alpha_{q+p-1})}\,. \tag{C.5}$$

For a scalar $A = \Phi$ it is clear that

$$[A, \Phi]_{\mathrm{SN}} = pA^{\gamma\alpha_1\dots\alpha_{p-1}}\partial_\gamma\Phi\,. \tag{C.6}$$

Thus the SN bracket reduces to the Lie derivative when $A = \xi^\mu$ and in particular the vanishing of the SN bracket is equivalent to the Killing vector equation when also $B = g^{\mu\nu}$. For our interests a p-symmetry of the manifold $\mathcal{N}$ is a p-symmetric tensor $K^{\mu_1,\dots,\mu_p}$ for which the SN bracket with the inverse metric, G^{-1} on $\mathcal{N}$ vanishes:

$$[K, G^{-1}] = 0 \iff \nabla^{\mathcal{N}}_{(\alpha_1}K_{\alpha_2\dots\alpha_{p+1})} = 0\,. \tag{C.7}$$

That is K is a rank-p Killing tensor. We also denote the vector space of rank-p Killing tensors over $\mathcal{M}$ as $\mathcal{K}^p(\mathcal{M})$.

[16] Unfortunately, in the mathematical community the SN bracket often refers to a similar construction but for anti-symmetric tensors. In the physics community this is almost never the case.

JHEP07(2025)098

Notice, that due to the isomorphism with the Poisson bracket a Killing tensor is equivalent to the quantity $k \equiv K^{\alpha_1 \dots \alpha_p} p_{\alpha_1} \dots p_{\alpha_p}$ being a constant of motion for the geodesic Hamiltonian $g \equiv \frac{1}{2} G^{\alpha_1 \alpha_2} p_{\alpha_1, \alpha_2}$

$$\{k, g\} = 0 \,. \tag{C.8}$$

Now, the SN bracket defines a Lie algebra since it satisfies the Jacobi identity

$$[[A, B]_{\mathrm{SN}}, C]_{\mathrm{SN}} + [[C, A]_{\mathrm{SN}}, B]_{\mathrm{SN}} + [[B, C]_{\mathrm{SN}}, A]_{\mathrm{SN}} = 0 \,. \tag{C.9}$$

This implies that, a non-trivial SN bracket of any two symmetries K_1 and K_2 generates a new non-trivial symmetry, given by $K_3 \equiv [K_1, K_2]_{\mathrm{SN}} \neq 0$

$$[K_3, G^{-1}]_{\mathrm{SN}} = [[K_1, K_2]_{\mathrm{SN}}, G^{-1}]_{\mathrm{SN}} = 0 \,. \tag{C.10}$$

Thus, we can keep generating new symmetries of the manifold by iterating SN bracket as in the main text.

Next, noting that the multiplication of two monomials of momenta corresponds to the symmetrized tensor product $\odot$ (e.g. for two vectors $X \odot Y = \frac{1}{2}(X \otimes Y + Y \otimes X)$ of their corresponding tensors we have generically a Leibniz rule

$$\{ab, c\} = \{a, c\}b + a\{b, c\} = -[A \odot B, C]_{\mathrm{SN}}^{\alpha_1 \dots A_{p+q+r-1}} p_1 \dots p_{\alpha_{p+q+r-1}} \tag{C.11}$$

$$= -([A, C]_{\mathrm{SN}}^{\alpha_1 \dots \alpha_{p+q-1}} B^{\alpha_{p+q} \dots \alpha_{p+q+r-1}} + A^{\alpha_1 \dots \alpha_p} [B, C]_{\mathrm{SN}}^{\alpha_{p+1} \dots \alpha_{p+q+r-1}}) p_1 \dots p_{\alpha_{p+q+r-1}} \,, \tag{C.12}$$

so that

$$[A \odot B, C]_{\mathrm{SN}} = [A, C]_{\mathrm{SN}} \odot B + A \odot [B, C]_{\mathrm{SN}} \,. \tag{C.13}$$

Together, these properties mean that the SN bracket forms a graded Lie algebra (see e.g. [19, 20]) on the space of Killing tensors of $\mathcal{M}$,

$$\mathcal{K}(\mathcal{M}) := \bigoplus_{i=0} \mathcal{K}^i(\mathcal{M}) \,, \tag{C.14}$$

where $\mathcal{K}^0 = \mathbb{R}$ and $\mathcal{K}^1$ is the space Killing vectors/isometries over $\mathcal{M}$. Finally we remark, in manifolds of constant Riemann curvature, i.e. symmetric spaces, the representation of the algebra of Killing tensors is well studied, see e.g. [15–20]. In particular, there is an isomorphism between a certain quotient of the universal covering algebra of the symmetry group, and the space of Killing tensors with the Lie bracket realized by the SN bracket (see Theorem 4.13 and corollary 4.14 of [21]).

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