Charged particle in higher dimensional weakly charged rotating black hole spacetime

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We study charged particle motion in weakly charged higher dimensional black holes. To describe the electromagnetic field we use a test field approximation and the higher dimensional Kerr-NUT-(A)dS metric as a background geometry. It is shown that for a special configuration of the electromagnetic field, the equations of motion of charged particles are completely integrable. The vector potential of such a field is proportional to one of the Killing vectors (called a primary Killing vector) from the ''Killing tower'' of symmetry generating objects which exists in the background geometry. A free constant in the definition of the adopted electromagnetic potential is proportional to the electric charge of the higher dimensional black hole. The full set of independent conserved quantities in involution is found. We demonstrate that Hamilton-Jacobi equations are separable, as is the corresponding Klein-Gordon equation and its symmetry operators.

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I. INTRODUCTION

In this paper we describe an interesting class of spacetimes where the equations of motion of charged particles allow a complete separation of variables. Namely, we study weakly charged rotating higher dimensional black holes. We assume that a background geometry is a solution of the (vacuum) Einstein equations and include the electromagnetism as a test field which does not affect the geometry. It is well known that in the four-dimensional case this approach can be useful. The reason is that for known charged elementary particles the ratio of the charge to mass is very large. As a result, a test electromagnetic field, which does not change the black hole geometry, can dramatically change the motion of charged particles (see e.g. [1,2] and references therein). Another application of the test electromagnetic field approximation is the study of the gyromagnetic ratio of higher dimensional rotating black holes [3].

We study charged particle motion in a spacetime with a test electromagnetic field. The conserved quantities for particle motion under the influence of extremal fields have been recently studied in [4]. We focus on the case when the background geometry describes a rotating higher dimensional black hole [5], and its generalization with ''NUT'' parameters and/or with a nontrivial cosmological constant [6,7]. The Kerr-NUT-(anti-)de Sitter metrics in higher dimensions have been extensively studied recently. In particular, it was demonstrated that they have a number of ''miraculous'' properties which make them similar to their four-dimensional ''cousin.'' It was found that the most general solution of the Einstein equations with the cosmological constant, describing higher dimensional rotating black holes with NUT parameters—Kerr-NUT-(A)dS metric—possesses a nondegenerate closed two-form of the conformal Killing-Yano tensor (principal Killing-Yano tensor) [8,9]. It was shown that this object generates a ''tower'' of conserved quantities [10–13] which makes the geodesic equations completely integrable in these spacetimes [11,12,14]. Later it was shown that the Hamilton-Jacobi and Klein-Gordon equations are separable [15]. Analogous results on separability for other field equations in this background have been obtained in [16–21]. For a general review, see [9]. The presence of a principal Killing-Yano tensor imposes restrictions on the form of the metric. Namely, the metric of the spacetime can be written in the form where the only freedom is a set of functions of one variables. This result was proved in [14,22,23].

In this paper we demonstrate that weak charges of the higher dimensional black hole solution do not change their remarkable property: The equations of a charged particle motion remain completely integrable. In the four-dimensional case this result is not surprising: The motion of charged particles in the Kerr-Newman spacetime has the same property [24], and our result can thus be obtained by linearization. In five dimensions our results might be related to the complete integrability of the particle motion equations in black hole solutions of the Chern-Simons version of Maxwell-Einstein equations [25,26]. In higher dimensions the obtained results are much less trivial.

The paper is organized as follows. Section II contains some preliminary material. Required information concerning spacetimes with a nondegenerate principal Killing-Yano tensor is collected in Sec. III. The adopted ansatz for a test electromagnetic field in this geometry is described in Sec. IV. In Sec. V, we prove that the motion of a

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charged particle in the electromagnetic field, generated by the primary Killing vector, is completely integrable. In Secs. VI and VII, we prove the separability of the Hamilton-Jacobi equations and of all symmetry operators of the Klein-Gordon operator. For simplicity, we give the proofs in the even-dimensional case, but they are valid in any number of dimensions. The changes required in the odd-dimensional case are discussed in Sec. VIII. The results of the paper are briefly summarized in the last section.

II. PRELIMINARIES

Consider a particle with a mass μ and a charge q, which is moving in the electromagnetic field $\boldsymbol{F} = d\boldsymbol{A}$. Its equation of motion is

$$\mu \frac{D^2 x^a}{d\tau^2} = qF^a{}_b \frac{Dx^b}{d\tau}. \tag{1}$$

Here, $D/d\tau$ is the covariant derivative with respect to the proper time τ. It is useful to introduce the affine parameter $\lambda = \tau/\mu$. Denoting by the dot a covariant derivative with respect to the parameter λ, the equation of motion can be rewritten as

$$\ddot{x}^a = qF^a{}_b \dot{x}^b. \tag{2}$$

It is well known that the symmetries of the background geometry guarantee the existence of conserved quantities even for motion under the influence of the electromagnetic field, provided that the electromagnetic field satisfies some consistency conditions. Let us recall that a Killing vector $\boldsymbol{\xi}$ and a rank two Killing tensor $\boldsymbol{k}$ satisfy the equations

$$\nabla_{(a}\xi_{b)} = 0, \qquad k_{ab} = k_{(ab)}, \qquad \nabla_{(a}k_{bc)} = 0. \tag{3}$$

If the spacetime possesses a Killing vector $\boldsymbol{\xi}$, the component of the canonical momentum along the Killing vector, i.e., the quantity

$$p_\xi = \xi^a(g_{ab}\dot{x}^b + qA_a), \tag{4}$$

is conserved if the vector potential $\boldsymbol{A}$ is Lie-conserved along $\boldsymbol{\xi}$,

$$\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{A} = [\boldsymbol{\xi}, \boldsymbol{A}] = 0. \tag{5}$$

The component of the velocity along $\boldsymbol{\xi}$,

$$u_\xi = \xi_a \dot{x}^a, \tag{6}$$

is conserved if

$$\xi^n F_{an} = 0. \tag{7}$$

This second condition can be generalized to quantities quadratic in velocities if the background geometry has a rank two Killing tensor $\boldsymbol{k}$.[1] Namely, the quantity

$$\dot{x}^a k_{ab} \dot{x}^b \tag{8}$$

is conserved if[2]

$$k_{(a}{}^n F_{b)n} = 0. \tag{9}$$

Since the metric $\boldsymbol{g}$ is the Killing tensor trivially satisfying this condition, we get obvious conservation of the norm of the velocity $\dot{x}^a g_{ab} \dot{x}^b$.

It will be useful to work also with Hamiltonian formalism. The equation of motion (2) follows from the Lagrangian

$$L = \tfrac{1}{2} g_{ab}\dot{x}^a\dot{x}^b + qA_a\dot{x}^a. \tag{10}$$

To write a Hamiltonian, one defines the momentum

$$p_a = \frac{\partial L}{\partial \dot{x}^a} = g_{ab}\dot{x}^b + qA_a, \tag{11}$$

and the corresponding Hamiltonian reads

$$H = \tfrac{1}{2} g^{ab}(p_a - qA_a)(p_b - qA_b). \tag{12}$$

Since it does not depend on λ, the Hamiltonian is the integral of motion. For our choice of the affine parameter λ, one finds that its value is given by

$$H = -\tfrac{1}{2}\mu^2. \tag{13}$$

The conservation law (13) with the Hamiltonian (12) implies the following Hamilton-Jacobi equation for the classical action $\mathcal{S} = -\frac{1}{2}\lambda\mu^2 + S(x^a)$:

$$-\mu^2 = g^{ab}(\partial_a S - qA_a)(\partial_b S - qA_b). \tag{14}$$

From the same Hamiltonian, one obtains the equation for a charged massive field φ by substituting $p_a \to -i\nabla_a$. The corresponding Klein-Gordon equation is

$$[[\nabla_a - iqA_a]g^{ab}[\nabla_b - iqA_b] - \mu^2]\varphi = 0. \tag{15}$$

Consider now a Ricci-flat spacetime, $\mathbf{Ric} = 0$. In the Lorentz gauge $\nabla_n A^n = 0$, the Maxwell equation $\nabla_n F^{an} = 0$ reads $\nabla_n \nabla^n A_a = 0$. The Killing vector $\boldsymbol{\xi}$ obeys the same equation $\nabla_n \nabla^n \xi_a = 0$. This means that the Killing vector field can be used as a potential of a special test electromagnetic field $\boldsymbol{A}$,

$$\boldsymbol{A} = Q\boldsymbol{\xi}. \tag{16}$$

Here, Q is a normalization constant parametrizing the strength of the field.

Let us assume that the background spacetime is even more special, namely, that it allows the separability of uncharged Hamilton-Jacobi and Klein-Gordon equations. It is natural to ask what happens with these equations when one considers the system with the test Killing electromagnetic field (16).

[1] The analogy of condition (5) for a quadratic quantity generalizing (4) is not so straightforward. It involves Schouten-Nijenhuis brackets $[\boldsymbol{k}, \boldsymbol{A}]_{\mathrm{SN}}$, as could be expected, but also some additional nontrivial conditions on $\boldsymbol{k}$, $\boldsymbol{A}$, and their first derivatives, cf. [4].

[2] A related condition in terms of a Killing-Yano tensor generating $\boldsymbol{k}$ can be found in [27].

If the separation takes place with respect to the Killing coordinate corresponding to the Killing vector ξ,

$$\xi^a \partial_a S = \Psi, \qquad \xi^a \partial_a \varphi = \Psi\varphi, \tag{17}$$

with Ψ being the separation constant, the charged Hamilton-Jacobi (14) and Klein-Gordon equations (15) take the form

$$g^{ab}\partial_a S \partial_b S + M^2 = 0, \tag{18}$$

$$[g^{ab}\nabla_a\nabla_b - M^2]\varphi = 0. \tag{19}$$

Here, the function M^2 is given by

$$M^2 = \mu^2 - 2e\Psi + e^2\xi^2, \tag{20}$$

with $e = qQ$.

These equations clearly resemble the uncharged case. Thus in the presence of the test Killing electromagnetic field the Hamilton-Jacobi and Klein-Gordon equations preserve their form with only the constant μ^2 replaced by the function M^2. Evidently, the constant shift $-2e\Psi$ does not affect the complete separability property of the initial equations. A nontrivial obstacle to the separability can create the term ξ^2. We shall describe now a physically interesting case when the complete separability is not broken by the external Killing electromagnetic field.

III. HIGHER DIMENSIONAL BLACK HOLE GEOMETRY

Rotating black hole solutions in higher dimensions belong to a broader class of spacetimes studied in [6,7,22,23]. In even dimensions $D = 2n$, the geometry of such spacetimes is described by the metric

$$\boldsymbol{g} = \sum_{\mu=1}^{n}\left[\frac{U_\mu}{X_\mu}\boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu}\left(\sum_{j=0}^{n-1} A_\mu^{(j)}\boldsymbol{d}\psi_j\right)^2\right]. \tag{21}$$

Here x_μ, $\mu = 1, \ldots, n$, correspond to radial and "azimuthal" directions and ψ_k, $k = 0, \ldots, n-1$ to temporal and longitudinal directions, namely, $\psi_0 = t$. The radial coordinate and some other quantities are rescaled by the imaginary unit i in order to bring the metric into a more symmetric form, cf. e.g. [6]; however, the metric is real. The signature of the metric depends on the signs of the metric functions. We use Latin indices from the beginning of the alphabet to label the whole coordinate set: $\{x^a\} = \{x_\mu, \psi_k\}$.

The functions U_μ and $A_\mu^{(k)}$ are defined as follows:

$$A_\mu^{(k)} = \sum_{\substack{\nu_1,\ldots,\nu_k=1\\ \nu_1<\ldots<\nu_k,\nu_i\neq\mu}}^{n} x_{\nu_1}^2 \cdots x_{\nu_k}^2, \qquad U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(x_\nu^2 - x_\mu^2). \tag{22}$$

These functions satisfy the important relations [15]

$$\sum_{\mu=1}^{n} A_\mu^{(i)} \frac{(-x_\mu^2)^{n-1-j}}{U_\mu} = \delta_j^i, \qquad \sum_{j=0}^{n-1} A_\mu^{(j)} \frac{(-x_\nu^2)^{n-1-j}}{U_\nu} = \delta_\mu^\nu, \tag{23}$$

for $i, j = 0, \ldots, n-1$ and $\mu, \nu = 1, \ldots, n$.

The quantities X_μ, $\mu = 1, \ldots, n$, are functions of a single variable; that is, each X_μ depends only on the variable x_μ, $X_\mu = X_\mu(x_\mu)$. However, if these functions are not specified, the metric (21) does not satisfy the vacuum Einstein equations and we speak about the so-called "off-shell" geometry. The vacuum (with a cosmological constant) black hole geometry is recovered [6,7] by setting

$$X_\mu = b_\mu x_\mu + \sum_{k=0}^{n} c_k x_\mu^{2k}. \tag{24}$$

The constants c_k and b_μ are then related to angular momenta, mass, NUT parameters, and the cosmological constant (which is proportional to c_n).

We can write the metric (21) in the diagonal form

$$\boldsymbol{g} = \sum_{\mu=1}^{n}\left(\frac{U_\mu}{X_\mu}\boldsymbol{\epsilon}^\mu\boldsymbol{\epsilon}^\mu + \frac{X_\mu}{U_\mu}\boldsymbol{\epsilon}^{\hat\mu}\boldsymbol{\epsilon}^{\hat\mu}\right), \tag{25}$$

introducing the non-normalized one-forms $\{\boldsymbol{\epsilon}^\mu, \boldsymbol{\epsilon}^{\hat\mu}\}$,

$$\boldsymbol{\epsilon}^\mu = \boldsymbol{d}x_\mu, \qquad \boldsymbol{\epsilon}^{\hat\mu} = \sum_{j=0}^{n-1} A_\mu^{(j)}\boldsymbol{d}\psi_j. \tag{26}$$

In this frame, the Ricci tensor for the off-shell geometry is diagonal,

$$\mathbf{Ric} = \sum_{\mu=1}^{n} r_\mu\left(\frac{U_\mu}{X_\mu}\boldsymbol{\epsilon}^\mu\boldsymbol{\epsilon}^\mu + \frac{X_\mu}{U_\mu}\boldsymbol{\epsilon}^{\hat\mu}\boldsymbol{\epsilon}^{\hat\mu}\right), \tag{27}$$

where

$$r_\mu = -\frac{1}{2x_\mu}\left[\sum_{\nu=1}^{n}\frac{x_\nu^2(x_\nu^{-1}X_\nu)_{,\nu}}{U_\nu}\right]_{,\mu}. \tag{28}$$

For the Einstein spacetime, polynomials (24) lead to a constant value $r_\mu = -(2n-1)c_n = \Lambda/(n-1)$.

The off-shell geometry (21) is endowed with a lot of symmetries. The symmetry set, forming a "Killing tower," is generated by a single object called a principal conformal Killing-Yano tensor [9,10]. This is a nondegenerate closed conformal Killing-Yano two-form $\boldsymbol{h}$,

$$\boldsymbol{h} = \sum_{k=1}^{n} x^\mu \boldsymbol{\epsilon}^\mu \wedge \boldsymbol{\epsilon}^{\hat\mu}. \tag{29}$$

The explicit symmetries are encoded by the Killing vectors $\boldsymbol{l}_{(k)}$, $k = 0, \ldots, n-1$,

$$\boldsymbol{l}_{(k)} = \boldsymbol{\partial}_{\psi_k}. \tag{30}$$

The geometry also possesses hidden symmetries encoded by the second rank Killing tensors $\boldsymbol{k}_{(j)}$, $j = 0, \ldots, n-1$, which in their covariant form read

$$\boldsymbol{k}_{(j)} = \sum_{\mu=1}^{n} A_{\mu}^{(j)} \left(\frac{U_\mu}{X_\mu} \boldsymbol{\epsilon}^\mu \boldsymbol{\epsilon}^\mu + \frac{X_\mu}{U_\mu} \boldsymbol{\epsilon}^{\hat{\mu}} \boldsymbol{\epsilon}^{\hat{\mu}} \right). \tag{31}$$

In particular, for $j = 0$, the Killing tensor reduces to the metric $k_{(0)ab} = g_{ab}$.

The Killing vector $\boldsymbol{\xi}$,

$$\xi^a = \frac{1}{D-1} \nabla_n h^{na}, \tag{32}$$

is called the primary Killing vector, since it turns out that it is the first in the tower of the Killing vectors defined above, $\boldsymbol{\xi} = \boldsymbol{l}_{(0)} = \boldsymbol{\partial}_{\psi_0}$. All Killing vectors can actually be obtained from the primary Killing vector using, for our purposes, the important relations

$$l_{(k)}^a = k_{(k)}^{an} \xi_n. \tag{33}$$

IV. TEST ELECTROMAGNETIC FIELD

In higher dimensions, a generalization of the above-described geometry to the case of arbitrary rotating charged black holes is not known. However, we can at least investigate the weakly charged black hole, i.e., the neutral black hole spacetime with a test electromagnetic field satisfying the Maxwell equations in such a background. Such an approximation is plausible since even the electromagnetic field that is small enough not to influence the background geometry can cause significant changes in the particle motion thanks to a large charge-to-mass ratio q/μ for typical particles.

Test electromagnetic fields on the background (21) have been studied, e.g., in [28]. However, in this paper we concentrate on the special Killing electromagnetic field; i.e., we assume the electromagnetic field with the vector potential (16) given by a Killing vector $\boldsymbol{\xi}$. In general, the electric current generating such a field is $J^a = 2Q\,\mathrm{Ric}^a{}_b \xi^b$. For Ricci-flat spacetimes the Killing electromagnetic field is thus source-free, and for the Einstein spaces the electric current is aligned along the generating Killing vector.

In the following we will investigate the electromagnetic field generated by the primary Killing vector $\boldsymbol{\xi}$ on the black hole background (21). Since we are interested in the motion of the particle with charge q, we use the following parametrization:

$$q\boldsymbol{A} = e\boldsymbol{\xi}. \tag{34}$$

The constant e/q parametrizes the field strength and it is proportional to the test charge of the black hole.

The results derived in the following sections do not depend on the nature of the source $\boldsymbol{J}$ of the primary Killing electromagnetic field. They hold for a general off-shell geometry (21). In such a general case, the corresponding electric current is

$$q\boldsymbol{J} = 2e \sum_{\mu=1}^{n} r_\mu \boldsymbol{\epsilon}_{\hat{\mu}}. \tag{35}$$

It represents the source distributed, in general, in the whole spacetime, which is not very reasonable. Therefore, physically the most interesting case is when the vacuum Einstein equations $\mathbf{Ric} = 0$ are satisfied, so that $\boldsymbol{J} = 0$.

In this case, the electromagnetic field (34) belongs to the class of electromagnetic fields studied in [28]. Namely, the potential (34) is gauge equivalent to the choice $e_\mu = e/q\, b_\mu$ of the constants e_μ parametrizing the field in [28], with b_μ from (24). In four dimensions, it can also be obtained from the electromagnetic field of the Kerr-Newman solution by linearization.

V. PHASE SPACE DESCRIPTION OF THE PARTICLE MOTION

The motion of the particle in the spacetime $\mathcal{M}$ with the metric (21) can be described in phase space represented as the cotangent bundle $\mathbf{T}^*\mathcal{M}$. The basic variable is the one-form of canonical momenta $\boldsymbol{p}$, components p_a of which are canonically conjugate to x^a, $a = 1, \ldots, D$.

The motion in the absence of the electromagnetic field was studied in [11,12], and it was shown that the Killing vectors (30) and tensors (31) generate functionally independent and mutually Poisson-commuting observables which are linear and quadratic in momentum $\boldsymbol{p}$, namely,

$$^0L_{(k)} = l_{(k)}^a p_a, \qquad ^0K_{(j)} = k_{(j)}^{ab} p_a p_b. \tag{36}$$

The commutation relations of the observables (36) are equivalent to nontrivial geometrical relations among the Killing vectors and tensors[3]:

$$l_{(k)}^n \nabla_n l_{(l)}^a - l_{(l)}^n \nabla_n l_{(k)}^a = 0, \tag{37}$$

$$l_{(k)}^n \nabla_n k_{(j)}^{ab} = k_{(j)}^{an} \nabla_n l_{(k)}^b + k_{(j)}^{bn} \nabla_n l_{(k)}^a, \tag{38}$$

$$k_{(i)}^{n(a} \nabla_n k_{(j)}^{bc)} - k_{(j)}^{n(a} \nabla_n k_{(i)}^{bc)} = 0. \tag{39}$$

The geodesic motion of an uncharged particle is generated by the Hamiltonian ^0H which is essentially one of these observables,

$$^0H = \tfrac{1}{2}\, {}^0K_{(0)} = \tfrac{1}{2} p_a g^{ab} p_b. \tag{40}$$

All the observables $^0L_{(k)}$ and $^0K_{(j)}$ are thus conserved quantities (i.e., integrals of motion) for the geodesic motion. They are independent and in involution. As a result,

[3]These relations correspond to vanishing Schouten-Nijenhuis brackets among all tensors $\boldsymbol{l}_{(k)}$ and $\boldsymbol{k}_{(j)}$.

according to the Liouville theorem the geodesic motion is completely integrable.

However, in this paper we want to investigate the motion of a charged particle modified by the electromagnetic field (34). Such a motion is described by the Hamiltonian (12), i.e.,

$$ {}^{e}H = \tfrac{1}{2}(p_a - e\xi_a)g^{ab}(p_b - e\xi_b). \tag{41}$$

We demonstrate now that the corresponding equations of motion are also completely integrable. However, in this case the integrals of motion must be modified. Namely, we define new observables,

$$ {}^{e}L_{(k)} = l^a_{(k)}p_a, \quad {}^{e}K_{(j)} = k^{ab}_{(j)}(p_a - e\xi_a)(p_b - e\xi_b). \tag{42}$$

We show that these observables are in involution,

$$\{{}^{e}L_{(l)}, {}^{e}L_{(k)}\} = 0, \qquad \{{}^{e}L_{(l)}, {}^{e}K_{(j)}\} = 0,$$
$$\{{}^{e}K_{(i)}, {}^{e}K_{(j)}\} = 0. \tag{43}$$

And since ${}^{e}H = \frac{1}{2}{}^{e}K_{(0)}$, they form a complete set of conserved quantities for motion of the particle under the influence of the electromagnetic field.

The modified observables (42) can be related to the observables (36) as

$$\begin{aligned} {}^{e}L_{(k)} &= {}^{0}L_{(k)} \equiv L_{(k)}, \\ {}^{e}K_{(j)} &= {}^{0}K_{(j)} - 2eL_{(j)} + e^2 k^{ab}_{(j)}\xi_a\xi_b. \end{aligned} \tag{44}$$

After plugging these expressions into Eqs. (43) and using the fact that the quantities (36) commute with each other, it remains to prove that $\{L_{(k)}, k^{ab}_{(j)}\xi_a\xi_b\} = 0$ and $\{{}^{0}K_{(i)}, k^{ab}_{(j)}\xi_a\xi_b\} + \{k^{ab}_{(i)}\xi_a\xi_b, {}^{0}K_{(j)}\} = 0$. Evaluating the Poisson brackets[4] we can translate these equalities to the language of tensors on the spacetime,

$$\begin{aligned} &l^n_{(k)}\nabla_n(k^{ab}_{(j)}\xi_a\xi_b) = 0, \\ &k^{cn}_{(i)}\nabla_n(k^{ab}_{(j)}\xi_a\xi_b) - k^{cn}_{(j)}\nabla_n(k^{ab}_{(i)}\xi_a\xi_b) = 0. \end{aligned} \tag{45}$$

The first equality can be easily proved realizing that, thanks to (33), $k^{ab}_{(j)}\xi_a\xi_b = g_{ab}l^a_{(j)}\xi^b$ and all the quantities $\boldsymbol{l}_{(j)}$, $\boldsymbol{\xi}$, and $\boldsymbol{g}$ are Lie-conserved along $\boldsymbol{l}_{(k)}$.

In the second equality one has to perform the covariant derivatives on the tensor products in their argument and use the identity

$$\begin{aligned} k^{cn}_{(i)}\nabla_n k^{ab}_{(j)} - k^{cn}_{(j)}\nabla_n k^{ab}_{(i)} = &-(k^{an}_{(i)}\nabla_n k^{bc}_{(j)} + k^{bn}_{(i)}\nabla_n k^{ac}_{(j)}) \\ &+ (k^{an}_{(j)}\nabla_n k^{bc}_{(i)} + k^{bn}_{(j)}\nabla_n k^{ac}_{(i)}), \end{aligned} \tag{46}$$

which follows from Eq. (39). Substituting $(\nabla_n k^{ac}_{(j)})\xi_a = \nabla_n l^c_{(j)} - (\nabla_n\xi_a)k^{ac}_{(j)}$ [cf. (33)], we finally get

$$\begin{aligned} &2k^{ca}_{(i)}(\nabla_a\xi_b)k^{bn}_{(j)}\xi_n + 2k^{ca}_{(j)}(\nabla_a\xi_b)k^{bn}_{(i)}\xi_n \\ &\quad - (\text{terms with } i \text{ and } j \text{ exchanged}) = 0. \end{aligned} \tag{47}$$

At the end, we used the definition (3) of the Killing vector.

We thus conclude that the observables (42) are in involution, and therefore they also commute with the Hamiltonian. They are functionally independent, which follows from the independence of variables (36), which was proven in [10,11]. Therefore, the Hamiltonian describes completely integrable motion with linear and quadratic integrals of motion.

VI. HAMILTON-JACOBI EQUATIONS

Alternatively, instead of the phase space descriptions we can use Hamilton-Jacobi theory to describe the particle motion. Namely, for each conserved quantity we can write down the Hamilton-Jacobi equation for the Hamilton-Jacobi function (classical action) S. It is obtained by substituting $\boldsymbol{dS}$ for the momentum $\boldsymbol{p}$ in the definitions of the conserved quantities:

$$ l^a_{(k)}\partial_a S = \Psi_k, \tag{48}$$

$$ k^{ab}_{(j)}(\partial_a S - e\xi_a)(\partial_b S - e\xi_b) = \Xi_j. \tag{49}$$

Here, Ψ_k and Ξ_j are constants of motion.

Now we show that all these equations can be simultaneously solved by the separability ansatz for S,

$$ S = \sum_{\mu=1}^{n} S_\mu(x_\mu) + \sum_{k=0}^{n-1}\Psi_k\psi_k, \tag{50}$$

where the functions $S_\mu(x_\mu)$ are functions of a single variable x_μ.

For $e = 0$ such a separability of the Hamilton-Jacobi equations was proved in [15,17]. Adding the electromagnetic field, generated by the primary Killing vector, does not change the first set of equations (48). It is trivially solved by the ansatz (50). The equations quadratic in $\boldsymbol{dS}$ can be written as [cf. Eq. (44)]

$$ k^{ab}_{(j)}\partial_a S\partial_b S - 2e\xi^a\partial_a S + e^2 k^{ab}_{(j)}\xi_a\xi_b = \Xi_j. \tag{51}$$

Plugging in the separability ansatz (50) and using the relation

$$ k^{ab}_{(j)}\xi_a\xi_b = \sum_{\mu=1}^{n}\frac{A^{(j)}_\mu}{U_\mu}X_\mu, \tag{52}$$

one obtains

[4] In covariant formalism we have $\{A, B\} = \nabla_n A\partial^n B - \partial^n A\nabla_n B$, where ∇_n is the covariant derivative "ignoring" the momentum dependence of the phase space observables A, B (taking a parallelly transported $\boldsymbol{p}$ as a constant) and ∂^n is a derivative with respect to the momentum $\boldsymbol{p}$, cf., for example, the Appendix of [12].

$$\sum_{\mu=1}^{n}\frac{A_{\mu}^{(j)}}{U_{\mu}}X_{\mu}\Big(S_{\mu}'^{2}+X_{\mu}^{-2}\Big(\sum_{k=0}^{n-1}\Psi_{k}(-x_{\mu}^{2})^{n-1-k}\Big)^{2}+e^{2}\Big)=\Xi_{j}+2e\Psi_{j}, \tag{53}$$

where the prime denotes the derivative of S_{μ} with respect to its single argument. We multiply both sides of Eq. (53) by $(-x_{\mu}^{2})^{n-1-j}$ and sum over j. Since Eq. (23) tells us that the matrix $(-x_{\mu}^{2})^{n-1-j}$ is inverse to $A_{\mu}^{(j)}/U_{\mu}$, we obtain

$$X_{\mu}(S_{\mu}'^{2}+X_{\mu}^{-2}\tilde{\Psi}_{\mu}^{2}+e^{2})=\tilde{\Xi}_{\mu}+2e\tilde{\Psi}_{\mu}. \tag{54}$$

Here we introduced the polynomial functions $\tilde{\Psi}_{\mu}$ and $\tilde{\Xi}_{\mu}$ of one variable x_{μ} with coefficients given by Ψ_{k} and Ξ_{j}, respectively:

$$\tilde{\Psi}_{\mu}=\sum_{k=0}^{n-1}\Psi_{k}(-x_{\mu}^{2})^{n-1-k},\qquad \tilde{\Xi}_{\mu}=\sum_{k=0}^{n-1}\Xi_{k}(-x_{\mu}^{2})^{n-1-k}. \tag{55}$$

Equation (54) gives an ordinary differential equation for each S_{μ},

$$(S_{\mu}')^{2}=\frac{\tilde{\Xi}_{\mu}}{X_{\mu}}-\Big(\frac{\tilde{\Psi}_{\mu}}{X_{\mu}}-e\Big)^{2}. \tag{56}$$

Hence, the functions S_{μ} satisfying these equations generate, through (50), the solution of all Hamilton-Jacobi equations.

VII. SEPARABILITY AND SYMMETRY OPERATORS OF THE KLEIN-GORDON EQUATION

A field analogue of the spinless particle motion is a scalar field governed by the Klein-Gordon equation. In the presence of the electromagnetic field it must be modified into (15). For the electromagnetic field (34) we thus get

$$[[\nabla_{a}-ie\xi_{a}]g^{ab}[\nabla_{b}-ie\xi_{b}]-\mu^{2}]\varphi=0. \tag{57}$$

The Klein-Gordon operator has been obtained from the Hamiltonian by the substitution $\boldsymbol{p}\rightarrow -i\boldsymbol{\nabla}$. In a similar manner we introduce a set of operators generated by the observables (42),

$$\begin{aligned}{}^{e}\mathcal{L}_{(k)}&=-il_{(k)}^{a}\nabla_{a},\\ {}^{e}\mathcal{K}_{(j)}&=-[\nabla_{a}-ie\xi_{a}]k_{(j)}^{ab}[\nabla_{b}-ie\xi_{b}],\end{aligned} \tag{58}$$

which turn out to be commuting with each other.

$$[{}^{e}\mathcal{L}_{(k)},{}^{e}\mathcal{L}_{(k)}]=0,\qquad [{}^{e}\mathcal{L}_{(k)},{}^{e}\mathcal{K}_{(j)}]=0,\qquad [{}^{e}\mathcal{K}_{(i)},{}^{e}\mathcal{K}_{(j)}]=0. \tag{59}$$

Since the charged Klein-Gordon operator in (57) is simply related to ${}^{e}\mathcal{K}_{(0)}$, it also means that these operators are symmetry operators of this Klein-Gordon operator.

In the absence of the electromagnetic field the commutation relations (59) were proved in [17]. The electromagnetic field modifies only the operators ${}^{e}\mathcal{K}_{(j)}$, and we can write

$$\begin{aligned}{}^{e}\mathcal{L}_{(k)}&={}^{0}\mathcal{L}_{(k)}\equiv\mathcal{L}_{(k)},\\ {}^{e}\mathcal{K}_{(j)}&={}^{0}\mathcal{K}_{(j)}-2e\mathcal{L}_{(j)}+e^{2}k_{(j)}^{ab}\xi_{a}\xi_{b}.\end{aligned} \tag{60}$$

Here we used (33) and the fact that the Killing vectors have vanishing divergence, $\nabla_{a}l_{(j)}^{a}=0$.

We can write operators (58) as a linear combination of operators $\tilde{\mathcal{L}}_{(k)}$ and ${}^{e}\tilde{\mathcal{K}}_{(j)}$:

$$\mathcal{L}_{(k)}=\sum_{\mu=1}^{n}\frac{A_{\mu}^{(k)}}{U_{\mu}}\tilde{\mathcal{L}}_{(\mu)},\qquad {}^{e}\mathcal{K}_{(j)}=\sum_{\mu=1}^{n}\frac{A_{\mu}^{(j)}}{U_{\mu}}{}^{e}\tilde{\mathcal{K}}_{(\mu)}, \tag{61}$$

with

$$\begin{aligned}\tilde{\mathcal{L}}_{(\mu)}&=\sum_{j=0}^{n-1}(-x_{\mu}^{2})^{n-1-j}\mathcal{L}_{(j)},\\ {}^{e}\tilde{\mathcal{K}}_{(\mu)}&=\sum_{j=0}^{n-1}(-x_{\mu}^{2})^{n-1-j}{}^{e}\mathcal{K}_{(j)}.\end{aligned} \tag{62}$$

It was shown in [17] that the operators ${}^{0}\tilde{\mathcal{K}}_{(j)}$ have the form

$${}^{0}\tilde{\mathcal{K}}_{(\mu)}=\Big[\tilde{\mathcal{X}}_{(j)}+\frac{1}{X_{\mu}}\tilde{\mathcal{L}}_{(j)}^{2}\Big], \tag{63}$$

with

$$\tilde{\mathcal{X}}_{(\mu)}=-\frac{\partial}{\partial x_{\mu}}\Big[X_{\mu}\frac{\partial}{\partial x_{\mu}}\Big]. \tag{64}$$

The relations (52) and (60) allow us to rewrite the modified operators ${}^{e}\tilde{\mathcal{K}}_{(j)}$ in a similar way,

$${}^{e}\tilde{\mathcal{K}}_{(\mu)}=\Big[\tilde{\mathcal{X}}_{(\mu)}+\frac{1}{X_{\mu}}[\tilde{\mathcal{L}}_{(\mu)}-eX_{\mu}]^{2}\Big]. \tag{65}$$

None of the above operators depend on the Killing coordinates ψ_{k}. Operators with a label μ, ${}^{e}\tilde{\mathcal{K}}_{(\mu)}$, $\tilde{\mathcal{L}}_{(\mu)}$, and $\tilde{\mathcal{X}}_{(\mu)}$, besides $\partial/\partial\psi_{k}$ depend only on the corresponding coordinate x_{μ} and the derivative $\partial/\partial x_{\mu}$. They do not contain x_{ν} or $\partial/\partial x_{\nu}$ for $\nu\neq\mu$. As a result, the operators commute among themselves,

$$[\tilde{\mathcal{L}}_{(\mu)},\tilde{\mathcal{L}}_{(\nu)}]=0,\qquad [\tilde{\mathcal{L}}_{(\mu)},{}^{e}\tilde{\mathcal{K}}_{(\nu)}]=0,\qquad [{}^{e}\tilde{\mathcal{K}}_{(\mu)},{}^{e}\tilde{\mathcal{K}}_{(\nu)}]=0, \tag{66}$$

for $\mu\neq\nu$. Using (62) and the fact that the coefficients in front of the operators depend just on x_{μ}, a simple argument shows that (66) implies the commutation (59), cf. [17].

Having the set of commuting operators, we can look for common eigenfunctions.

$$\mathcal{L}_{(k)}\varphi = \Psi_k\varphi, \qquad {}^e\mathcal{K}_{(j)}\varphi = \Xi_j\varphi \tag{67}$$

with eigenvalues Ψ_k and Ξ_j. These eigenfunctions can be found using the separability ansatz [15]

$$\varphi = \prod_{\mu=1}^{n} R_\mu(x_\mu)\prod_{k=0}^{n-1}\exp(i\Psi_k\psi_k), \tag{68}$$

where the functions $R_\mu(x_\mu)$ depend on a single variable x_μ. Indeed, substituting (68) into (67), the equations for $\mathcal{L}_{(k)}$ are trivially satisfied and the equations for ${}^e\mathcal{K}_{(j)}$ give

$$\sum_{\nu=1}^{n}\frac{A_\nu^{(j)}}{U_\nu}\Big(\frac{1}{R_\nu}\tilde{\mathcal{X}}_{(\nu)}R_\nu + \frac{1}{X_\nu}(\tilde{\Psi}_\nu - eX_\nu)^2\Big) = \Xi_j. \tag{69}$$

Here $\tilde{\Psi}_\mu$ (and $\tilde{\Xi}_\mu$ below) are again given by (55). Summing these equations with the coefficients $(-x_\mu^2)^{n-1-j}$ leads to an equivalent set of conditions:

$$(X_\mu R_\mu')' + \Big(\tilde{\Xi}_\mu - \frac{1}{X_\mu}(\tilde{\Psi}_\nu - eX_\nu)^2\Big)R_\mu = 0. \tag{70}$$

These are ordinary differential equations in the variable x_μ for the functions R_μ which guarantee that (68) solves the eigenvalue problem (67).

In particular, for ${}^e\mathcal{K}_{(0)}$ we obtain the separability of the Klein-Gordon equation, which was, in the absence of the electromagnetic field, shown in [15].

It is straightforward to check that the semiclassical (geometrical-optic) approximation of the eigenvalue conditions (67) leads to the Hamilton-Jacobi equations (48) and (49), where we have to identify

$$\varphi = \exp(iS), \quad \text{i.e.,} \quad R_\mu = \exp(iS_\mu). \tag{71}$$

VIII. ODD SPACETIME DIMENSIONS

Until now, we considered even-dimensional spacetimes. However, the obtained results are valid in any number of dimensions. This can be easily shown by performing similar calculations. Only some of the equations have to be slightly changed. Here we present a short overview of the required changes.

In the spacetime dimension $D = 2n + 1$, there exists an additional angular coordinate ψ_n, and the metric (21) contains an additional term [6,7,22,23],

$$\begin{aligned}\boldsymbol{g} = \sum_{\mu=1}^{n}\Big[\frac{U_\mu}{X_\mu}\boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu}\Big(\sum_{j=0}^{n-1}A_\mu^{(j)}\boldsymbol{d}\psi_j\Big)^2\Big]\\ + \frac{c}{A^{(n)}}\Big(\sum_{k=0}^{n}A^{(k)}\boldsymbol{d}\psi_k\Big)^2.\end{aligned} \tag{72}$$

Here, $A^{(k)}$ is defined as

$$A^{(k)} = \sum_{\substack{\nu_1,\ldots,\nu_k=1\\ \nu_1<\ldots<\nu_k}}^{n} x_{\nu_1}^2\cdots x_{\nu_k}^2, \tag{73}$$

and c is an auxiliary constant which can be modified by a coordinate transformation.

The geometry has an additional Killing vector $\boldsymbol{l}_{(n)}$ [given again by (30)] but the same second rank Killing tensors. The additional symmetry generates an additional conserved quantity $L_{(n)}$ linear in momentum. It clearly Poisson-commutes with all the other conserved observables. Similarly, we have an additional symmetry operator $\mathcal{L}_{(n)}$ commuting with other symmetry operators.

The separability ansatz for the Hamilton-Jacobi equations (50) and for the eigenvalue problem of the symmetry operators (68) changes just by including the term $\Psi_n\psi_n$. The ordinary differential equations for S_μ and R_μ, however, acquire additional nontrivial terms which can be partially hidden in the redefinition of the polynomials $\tilde{\Psi}_\mu$ and $\tilde{\Xi}_\mu$. Namely, S_μ and R_μ have to satisfy

$$\begin{aligned}&(S_\mu')^2 = \frac{\tilde{\Xi}_\mu}{X_\mu} - \Big(\frac{\tilde{\Psi}_\mu}{X_\mu} - e\Big)^2,\\ &(X_\mu R_\mu')' + \frac{X_\mu}{x_\mu}R_\mu' + \Big(\tilde{\Xi}_\mu - \frac{1}{X_\mu}(\tilde{\Psi}_\nu - eX_\nu)^2\Big)R_\mu = 0,\end{aligned} \tag{74}$$

with

$$\begin{aligned}\tilde{\Psi}_\mu &= \sum_{k=0}^{n}\Psi_k(-x_\mu^2)^{n-1-k},\\ \tilde{\Xi}_\mu &= \sum_{k=0}^{n}\Xi_k(-x_\mu^2)^{n-1-k},\end{aligned} \tag{75}$$

where we set $\Xi_n = c^{-1}\Psi_n^2$ (cf. [15,17]).

IX. SUMMARY

To summarize, we proved that the dynamical equations for a charged particle in a weakly charged Kerr-NUT-(A) dS spacetime are completely integrable. We also demonstrated that the Hamilton-Jacobi and Klein-Gordon equations are completely separable in such a space. The proof essentially used the remarkable properties of the geometry, namely, the existence of the principal conformal Killing-Yano tensor, which generates the "Killing tower" of symmetries. It should be emphasized that the developed formalism works only for the test electromagnetic field generated by the primary Killing vector. The test fields connected with other Killing vectors do not possess these nice properties.

Let us make some general remarks that are connected with our results. Complete integrability of dynamical equations is quite rare. Liouville integrability implies that a solution can be written by applying a finite number of steps which include algebraic operations and integration. In such a case the phase space is regularly foliated by trajectories. The geodesic motion in the Kerr-NUT-(A)dS spacetimes is a new, physically interesting example of completely integrable systems. In this paper we demonstrated that these nice properties remain valid if one includes a special type of test electromagnetic field generated by a primary Killing vector. This generalization allows one to study the motion of charged particles in weakly charged higher dimensional black holes. The results might also be interesting for ‘‘physical’’ applications, for example, for the study of the Hawking radiation of charged rotating black holes in higher dimensions. They might also give some hints for the search of more general, possibly self-consistent solutions of electrovacuum Einstein equations and their supersymmetric generalizations.

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