Gyratonic solutions in Kundt class

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contents:

- *pp*-waves and Kundt waves
- gyratons and gyratonic matter
- gyratons on Minkowski and AdS spacetimes
- gyratons on ('Melvin-ized') direct-product spacetimes
- gyratons in higher dimensions

Waves propagating with speed of light

- Spacetime with a null coordinate u
- Field with a phase dependence $\phi(u)$
- A possible dependence on transversal directions $\phi(u, x^i)$

pp-waves

$$g = -2H\mathrm{d}u^2 - \mathrm{d}u \vee \mathrm{d}r + \mathrm{d}\zeta \vee \mathrm{d}\bar{\zeta}$$

notation:
$\alpha \vee \beta = \alpha\beta + \beta\alpha$

Kundt waves

$$g = -2H\mathrm{d}u^2 - \mathrm{d}u \vee \mathrm{d}r + (W\mathrm{d}\zeta + \bar{W}\mathrm{d}\bar{\zeta}) \vee \mathrm{d}u + \mathrm{d}\zeta \vee \mathrm{d}\bar{\zeta}$$

H and W have no or just a trivial dependence on r
ζ, $\bar{\zeta}$ – transversal directions

Null fluid

- Null hypersurface $u =$ constant generated by a null congruence k
- k is algebraically special direction of the geometry
- Standard matter fields aligned with k

phenomenological description of a null dust or of a coherent light beam:

$$T = j_u \, \mathrm{d}u^2$$

Gyratonic matter

- generalization of null fluid
- allowing intrinsic rotation (spin) of the fluid
  (e.g., circularly polarized light)
- allowing transversal transfer and accumulation of energy
  (sort of heating or cooling processes)

$$T = j_u \, \mathrm{d}u^2 + j \vee \mathrm{d}u$$

j_u	– energy density of the fluid (transverse scalar)
$j = j_i \, \mathrm{d}x^i$	– intrinsic energy transfer (transverse 1-form)

local energy conservation:

$$\nabla \cdot T = 0$$

- in general, problems with causality (energy conditions violated)
- no heating/cooling = no accumulation of energy
  = j_u constant along the null congruence k

Gyratons

Gravitational waves which include a response to gyratonic matter

- we restrict to the class of Kundt waves
- we allow additional aligned electromagnetic field

Gyratons on Minkowski spacetime

V. P. Frolov, W. Israel, A. Zelnikov: *Gravitational field of relativistic gyratons*, Phys.Rev. D **72** (2005) 084031

- flat Minkowski background
- general dimension
- Kundt form of the full metric

metric:

$$g = -2H\mathrm{d}u^2 - \mathrm{d}u \vee \mathrm{d}r + a \vee \mathrm{d}u + q$$

gyratonic matter:

$$T = j_u \mathrm{d}u^2 + j \vee \mathrm{d}u$$

u	– null coordinate
$k = \partial_r$	– null congruence
r	– complementary time coordinate = affine parameter of the congruence k
x^i	– transverse coordinates
$q = \sum_i \mathrm{d}x^i \mathrm{d}x^i$	– flat transverse metric
$H(u, x^i)$	– characteristic *pp*-wave term (scalar function)
$a(u, x^i)$	– characteristic gyratonic term (transverse 1-form)
$H = 0 \quad a = 0$	– Minkowski background

Gyratons on Minkowski spacetimes

Field equations:

$$\operatorname{div} f = j$$

$$-\triangle H = j_u + \frac{1}{2} f^2 + \partial_u \operatorname{div} a$$

$f = \mathrm{d}a$	– ‘strength’ of the gyratonic 1-form a (transverse 2-form)
$f^2 = \frac{1}{2} f^{ij} f_{ij}$	– transverse square of F
div	– transverse divergence (with respect to the transverse metric q)
$\triangle$	– transverse laplace (with respect to the transverse metric q)

Properties:

- field equations are formulated in terms of transverse space
- separation of equation for a and H
- linear structure of equations (solvable in terms of the Green functions)
- in 4 dimensions, in vacuum region, the gyratonic term a is locally gauge trivial
- it is not globaly gauge trivial (a nontrivial integral characteristic related to the sources)
- belongs to the class of VSI spacetimes

Gyratons on direct-product spacetime

H. Kadlecova, A. Zelnikov, P. Krtouš, J. Podolský: *Gyratons on direct-product spacetimes*, Phys.Rev. D **80** (2009) 024004

- special subclass of 4-dimensional Kundt spacetimes
- additional electromagnetic field and non-trivial cosmological constant
- direct-product background

metric:

$$g = -2H\,\mathrm{d}u^2 - \mathrm{d}u \vee \mathrm{d}r + a \vee \mathrm{d}u + q$$

electromagnetic field:

$$F = E\,\mathrm{d}r \wedge \mathrm{d}u + B\,\epsilon + \mathrm{d}u \wedge \sigma$$

gyratonic matter:

$$T = j_u\,\mathrm{d}u^2 + j \vee \mathrm{d}u$$

$q(u, x^i)$	– general transverse metric
$H(r, u, x^i)$	– characteristic pp-wave term (scalar function)
$a(u, x^i)$	– characteristic gyratonic term (transverse 1-form)
$\epsilon(u, x^i)$	– transverse Levi-Civita tensor (transverse 2-form)
$E,\ B$	– constants characterizing the background EM field
$\sigma(r, u, x^i)$	– transverse 1-form describing a deformation of EM field due to the gyraton
$V = \frac{1}{2}\big(E^2 + B^2\big)$	– energy density of EM field

Transverse space

$u = \text{constant}$	– null hypersurfaces of a ‘constant phase’
$k = -^{\sharp}\mathrm{d}u$	– null congruence = direction of the ray propagation
r	– affine parametrization of the congruence k
$u, r = \text{constant}$	– transverse space = the ‘wave front’ of the gyraton

Characteristic features of the gyratons:

- transverse geometry is independent of the gyraton
- all equation can be formulated on the transverse space

r-dependence and gauge

geometrical assumptions and the field equations $\Rightarrow$

$$\partial_r q = 0 \qquad \partial_r a = 0 \qquad \partial_r \sigma = 0$$

coordinate gauge freedom $\Rightarrow$

$$\partial_u q = 0 \qquad \operatorname{div} a = 0$$

the energy conservation for gyratonic matter $\Rightarrow$

$$j_u = r \operatorname{div} j + \iota \qquad \partial_r j = 0$$

$\operatorname{div} j$ – intrinsic energy accumulation

$\iota(u, x^i)$ – initial energy density

transverse trace of the Einstein equations $\Rightarrow$

$$H = -\tfrac{1}{2}\,\Lambda_-\, r^2 + g\, r + h$$

$g(u, x^i)$ and $h(u, x^i)$ characterize gravitational wave of the gyraton

notation: $\Lambda_\pm = \Lambda \pm V$

Field equations

- equations formulated on the transverse space
- equations separate – they can be solved subsequently
- linear structure of the equations
- solvable in terms of the Green functions on the transverse space

order of solving of the field equation:

1. transverse metric q
2. gyratonic term a (gravitational response to intrinsic spin)
3. metric function g (gravitational response intrinsic energy transfer)
4. electromagnetic field σ (EM response to graviton)
5. metric function h (gravitational wave contribution)

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$$\text{scalar curvature of } q = 2\Lambda_-$$

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$$\frac{1}{2}\,\triangle\, b + \Lambda_+\, b = -\mathrm{rot}\, j \qquad \mathrm{rot}\, a = b \qquad \mathrm{div}\, a = 0$$

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$$\triangle\, g = \operatorname{div} j$$

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$$\operatorname{rot}\sigma = 0 \qquad \operatorname{div}\sigma = -B \operatorname{rot} a$$

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$$\triangle\, h = \iota - \tfrac{1}{2}\, b^2 + \Lambda_-\, a^2 - 2\, a \cdot \mathrm{d}g + (\sigma - E\, a)^2$$

Properties of the gyraton

- geometry of the transverse space is independent of the gyraton
- spacetime polynomial curvature invariants are independent of the gyraton (CSI spacetimes)
- algebraic type II solutions
- null congruence k is geodesic, expansion-free, sheer-free, twist-free, recurrent and algebraically special

Backgrounds

background spacetime given by

$$h = 0 \qquad g = 0 \qquad a = 0$$

$$\Downarrow$$

direct-product spacetimes

product of two 2-dimensional spaces of constant curvatures:

- Lorentzian space spanned by coordinates r, u – curvature $2\Lambda_-$
- Riemaniann space spanned by coordinates x^i – curvature $2\Lambda_+$

special subcases:

Λ	V	Λ_-	Λ_+	geometry	spacetime
$= 0$	$= 0$	0	0	$M_2 \times E^2$	Minkowski
> 0	$= 0$	Λ	Λ	$dS_2 \times S^2$	Nariai
< 0	$= 0$	Λ	Λ	$AdS_2 \times H^2$	anti-Nariai
$= 0$	> 0	$-V$	V	$AdS_2 \times S^2$	Bertotti–Robinson
> 0	$= \Lambda$	0	2Λ	$M_2 \times S^2$	Plebański–Hacyan
< 0	$= \|\Lambda\|$	2Λ	0	$AdS_2 \times E^2$	Plebański–Hacyan

‘Melvinization’ of gyratons on direct-product spacetimes

P. Krtouš, H. Kadlecová: in preparation

- switching on a stronger electromagnetic field
- solutions from the Kundt class
- transverse metric with rotational symmetry

spacetime metric:

$$g = \Sigma^2\Big(-2H\,\mathrm{d}u^2 - \mathrm{d}u \vee \mathrm{d}r + a \vee \mathrm{d}u + \mathrm{d}\rho^2\Big) + \frac{S^2}{\Sigma^2}\,\mathrm{d}\varphi^2$$

electromagnetic field:

$$F = E\,\mathrm{d}r \wedge \mathrm{d}u + \Sigma^{-2}\,B\,\epsilon + \mathrm{d}u \wedge \sigma$$

gyratonic matter:

$$T = j_u\,\mathrm{d}u^2 + j \vee \mathrm{d}u$$

ρ, φ	– radial and angular transverse coordinates
$\Sigma(\rho)$	– a new metric function amplifying influence of the EM field
$S(\rho)$	– a new transverse metric function

Transverse geometry

transverse metric:

$$q = \Sigma^2 \, \mathrm{d}\rho^2 + \frac{S^2}{\Sigma^2} \, \mathrm{d}\varphi^2$$

field equations $\Rightarrow$

$$\Sigma_{,\rho} = \gamma \, S \qquad\qquad \Sigma_{,\rho} = \left[-\frac{1}{3} \Lambda \, \Sigma^4 + \alpha \, \Sigma^2 + \beta \, \Sigma - V \right]^{1/2}$$

$\alpha, \beta, \gamma =$ constant   solution in terms of inverse of Euler integrals

a new radial coordinate x (Plebański form of the metric)

$$\Sigma = 1 + \gamma \, x \qquad \Rightarrow \qquad q = \frac{1}{G} \, \mathrm{d}\rho^2 + G \, \mathrm{d}\varphi^2$$

G is an explicit rational function of x

regularity at the axis $\Rightarrow$

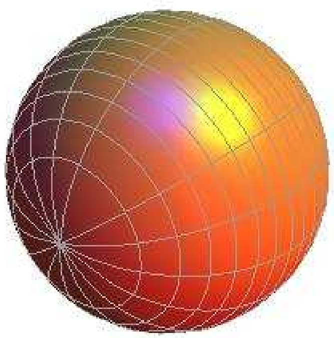
independent constants Λ, V, γ

$$\alpha = \Lambda - V + 2\gamma \qquad\qquad \beta = -\frac{2}{3}\Lambda + 2V - 2\gamma$$

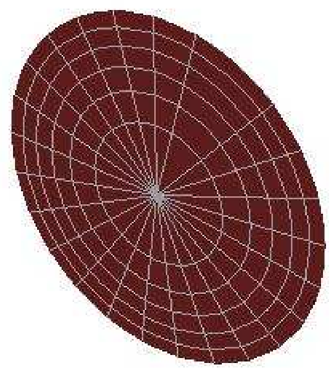
Transverse geometry

$$\gamma = 0 \quad \Rightarrow \quad \Sigma = 1 \quad \Rightarrow \quad \text{direct-product spacetimes}$$

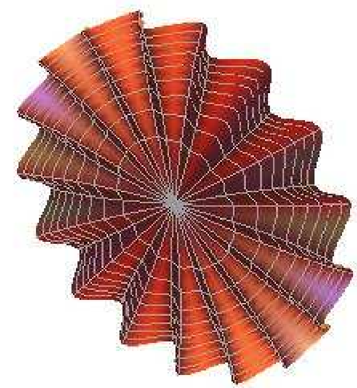
embedding diagrams of the transverse space:



$\Lambda_+ > 0$



$\Lambda_+ = 0$

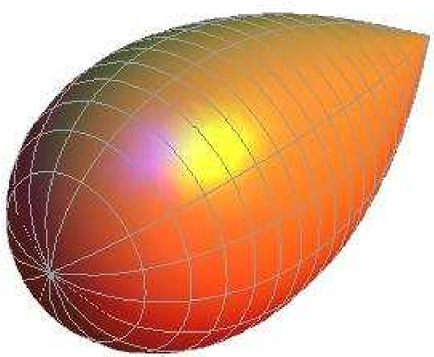


$\Lambda_+ < 0$

Transverse geometry

$$\gamma \neq 0 \qquad \Lambda > 0 \qquad \Rightarrow \qquad \text{closed space}$$

embedding diagrams of the transverse space:

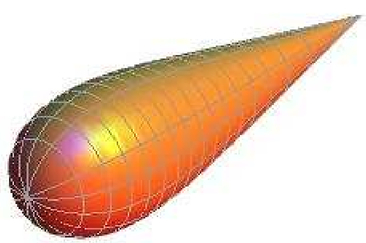


Transverse geometry

$$\gamma \neq 0 \qquad \Lambda = 0$$

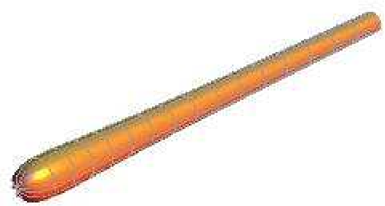
embedding diagrams of the transverse space:

closed space



$V > 2\gamma$

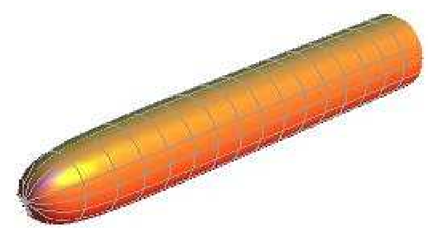
asymptotically closed space



$V = 2\gamma$

Melvin universe

asymptotically cylindrical space



$V < 2\gamma$

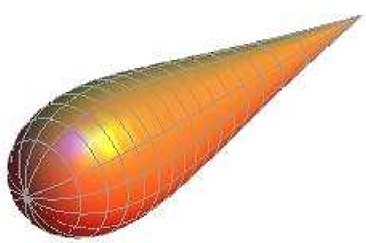
H. Kadlecova, P. Krtouš: *Gyratons on Melvin spacetime*, arXiv:1006.1794

Transverse geometry

$\gamma \neq 0 \qquad \Lambda < 0$

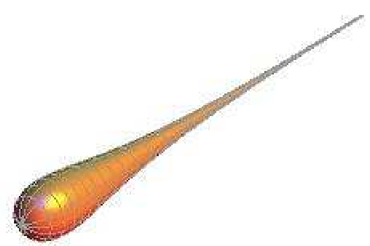
embedding diagrams of the transverse space:

closed space



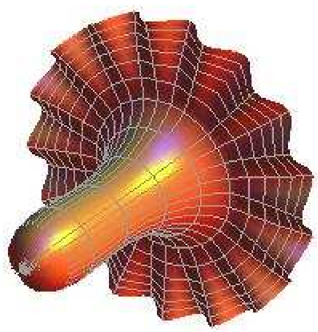
$\gamma < \gamma_*$

asymptotically closed space



$\gamma = \gamma_*$

open space



$\gamma > \gamma_*$

Spacetime curvature invariants

- spacetime polynomial curvature invariants are independent of the gyraton
- in general, these invariants are not constant !!!!!

A. Coley, S. Hervik, N. Pelavas: *Lorentzian manifolds and scalar curvature invariants*, arXiv:1003.2373

Kundt family of spacetimes in higher dimensions

P. Krtouš, A. Zelnikov, H. Kadlecova, J. Podolský: *Higher-dimensional Kundt waves and gyratons*, in preparation

J. Podolský, M. Žofka: *General Kundt spacetimes in higher dimensions*, Class. Quantum Grav. 26, 105008 (2009)

Generalization to higher dimensions $D = d + 2$:

general formalism of $2{+}d$-splitting of spacetimes admitting a null hypersurface

metric:

$$g = -2H\,\mathrm{d}u^2 - \mathrm{d}u \vee \mathrm{d}r + a \vee \mathrm{d}u + q$$

assumptions:

$$\dot{q} = 0 \qquad \dot{a} = 0$$

Ricci tensor:

notation:
$\dot{f} = \partial_r f$
$\mathring{f} = \partial_u f$
$\theta_u = \frac{1}{2} q^{ij} \mathring{q}_{ij}$

$$^D\mathrm{Ric}_{rr} = 0$$
$$^D\mathrm{Ric}_{r\mathsf{T}} = 0$$
$$^D\mathrm{Ric}_{ru} = \ddot{H}$$
$$^D\mathrm{Ric}_{u\mathsf{T}} = -\frac{1}{2}\mathrm{div}\,\mathrm{d}a + \mathrm{d}\dot{H} + \frac{1}{2}\mathrm{div}\,\mathring{q} - \mathrm{d}\theta_u$$
$$^D\mathrm{Ric}_{uu} = \triangle H + (\mathrm{d}a)\bullet(\mathrm{d}a) + \frac{1}{2}\ddot{H}\left(H + \frac{1}{2}a^2\right) + \mathrm{div}\,\dot{a} - \frac{1}{2}\mathring{q}\bullet\mathring{q} - 2\dot{H}\theta_u - \mathring{\theta}_u$$
$$^D\mathrm{Ric}_{\mathsf{TT}} = \mathrm{Ric}$$

Field equations

properties:

- the field equations formulated on the transverse space
- decoupling of the field equations
- semi-linearity of the field equations

new features:

- nontrivial transverse geometry q
  (e.g., gauge freedom does not eliminate u-dependence of q)
- nontrivial magnetic field B
- compatibility of the transverse geometry and of the magnetic field
  (e.g., $B = 0$ or direct-product structure of two dimensional spaces)

References:

- V. P. Frolov, W. Israel, A. Zelnikov: *Gravitational field of relativistic gyratons*, Phys.Rev. D **72** (2005) 084031
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- H. Kadlecova, A. Zelnikov, P. Krtouš, J. Podolský: *Gyratons on direct-product spacetimes*, Phys.Rev. D **80** (2009) 024004
- H. Kadlecova, P. Krtouš: *Gyratons on Melvin spacetime*, arXiv:1006.1794
- P. Krtouš, H. Kadlecová: *Gravitational waves on Melvinized direct-product spacetimes*, in preparation
- P. Krtouš, A. Zelnikov, H. Kadlecova, J. Podolský: *Higher-dimensional Kundt waves and gyratons*, in preparation

cf. also talks by *Hedvika Kadlecová* and *Martin Žofka*