Hidden symmetries of higher-dimensional black holes

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Black holes, hidden symmetries, and complete integrability

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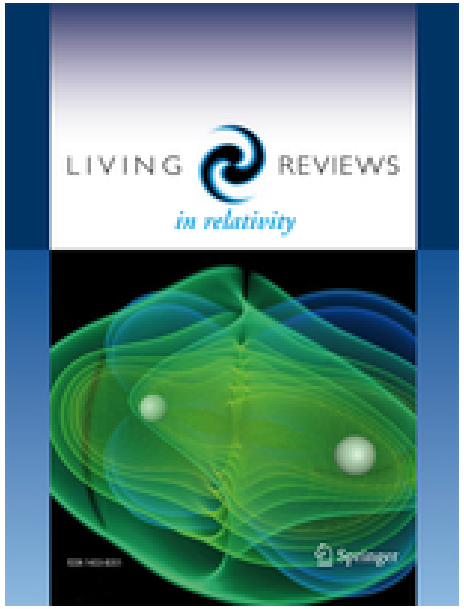

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REVIEW ARTICLE

Black holes, hidden symmetries, and complete integrability

Valeri P. Frolov[1] · Pavel Krtouš[2] · David Kubizňák[3]

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Abstract The study of higher-dimensional black holes is a subject which has recently attracted vast interest. Perhaps one of the most surprising discoveries is a realization that the properties of higher-dimensional black holes with the spherical horizon topology and described by the Kerr–NUT–(A)dS metrics are very similar to the properties of the well known four-dimensional Kerr metric. This remarkable result stems from the existence of a single object called the principal tensor. In our review we discuss explicit and hidden symmetries of higher-dimensional Kerr–NUT–(A)dS black hole spacetimes. We start with discussion of the Killing and Killing–Yano objects representing explicit and hidden symmetries. We demonstrate that the principal tensor can be used as a "seed object" which generates all these symmetries. It determines the form of the geometry, as well as guarantees its remarkable properties, such as special algebraic type of the spacetime, complete integrability of geodesic motion, and separability of the Hamilton–Jacobi, Klein–Gordon, and Dirac equations. The review also contains a discussion of different applications of the developed formalism and its possible generalizations.

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Contents

Symmetries

Symmetries on configuration space

configuration space M = space of positions

- 1-parameter family of diffeomorphisms on M generated by vector field $\boldsymbol{l}$ on M
- relevant quantities unchanged by the symmetry $\pounds_{\boldsymbol{l}}\boldsymbol{g} = 0 \qquad \pounds_{\boldsymbol{l}}\boldsymbol{A} = 0 \qquad \pounds_{\boldsymbol{l}}V = 0$

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Symmetries on phase space

phase space Γ = space of positions and momenta

- 1-parameter family of diffeomorphisms on Γ generated by vector field $\boldsymbol{Z}$ on Γ
- symplectomorphism: $\pounds_{\boldsymbol{Z}}\, \boldsymbol{\Omega} = 0$ generator is Hamiltonian flow $Z = \boldsymbol{X}_F$
- Hamiltonian unchanged by the symmetry: $\pounds_{\boldsymbol{Z}} H = 0 \quad \Leftrightarrow \quad \{F, H\} = 0$
- F is a *conserved quantity* $\pounds_{\boldsymbol{X}_H} F = 0$

Symmetries

Symmetries on configuration space

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Symmetry is generated by a conserved quantity

Symmetries on p

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Symmetries of geodesic motion

Symmetries on configuration space

configuration space M = space of positions

- generator of the symmetry $\boldsymbol{l}$
- metric conserved by the symmetry – isometry
  $\pounds_{\boldsymbol{l}}\, \boldsymbol{g} = 0$

Symmetries on phase space

phase space Γ = space of positions and momenta

- generator of the symmetry $\boldsymbol{X}_K$
- $\{K, H\} = 0$
- Hamiltonian given by the metric
  $H = \frac{1}{2m}\, \boldsymbol{g}^{ab}\, \boldsymbol{p}_a \boldsymbol{p}_b$

Symmetries of geodesic motion

Symmetries on configuration space $\Rightarrow$ Explicit symmetries

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- generator of the symmetry $\boldsymbol{l}$
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conserved quantity linear in momentum $\boldsymbol{p}$

$$L = \boldsymbol{l}^a \boldsymbol{p}_a$$

$\boldsymbol{l}$ – **Killing vector**

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Symmetries on phase space $\Rightarrow$ Hidden symmetries

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- $\{K, H\} = 0$
- Hamiltonian given by the metric
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conserved quantity monomial in momentum $\boldsymbol{p}$

$$K = \boldsymbol{k}^{a_1 \ldots a_p} \boldsymbol{p}_{a_1} \cdots \boldsymbol{p}_{a_p}$$

$\boldsymbol{k}$ – **Killing tensor**

Objects encoding symmetry

Killing vectors

$\boldsymbol{l}$ is a Killing vector iff

$$\pounds_{\boldsymbol{l}}\,\boldsymbol{g} = 0 \quad \Leftrightarrow \quad \boldsymbol{\nabla}^{(a}\boldsymbol{l}^{b)} = 0$$

Killing tensors

$\boldsymbol{k}$ is a Killing tensor of rank p iff

$$\boldsymbol{k}^{a_1\dots a_p} = \boldsymbol{k}^{(a_1\dots a_p)}$$

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- **Killing vector is a Killing tensor of rank 1**

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- Killing vector is a Killing tensor of rank 1
- **Existence of Killing tensors is highly non-trivial restriction on the geometry**

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- **Killing tensors can be build from more elementary building blocks**

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- Killing tensors can be build from more elementary building blocks

Conformal Killing–Yano forms

Conformal Killing–Yano forms

Splitting covariant derivative of a form

$$\boldsymbol{\nabla\omega} = \mathcal{A}\boldsymbol{\nabla\omega} + \mathcal{C}\boldsymbol{\nabla\omega} + \mathcal{T}\boldsymbol{\nabla\omega}$$

$$(\mathcal{A}\sigma)_{aa_1\dots a_p} = \sigma_{[aa_1\dots a_p]}$$

$$(\mathcal{C}\sigma)_{aa_1\dots a_p} = \frac{p}{D-p+1} g_{a[a_1}\sigma^{a}{}_{|a|a_2\dots a_p]}$$

$$(\mathcal{T}\sigma)_{aa_1\dots a_p} = \sigma_{aa_1\dots a_p} - \sigma_{[aa_1\dots a_p]} - \frac{p}{D-p+1} g_{a[a_1}\sigma^{a}{}_{|a|a_2\dots a_p]}$$

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$\boldsymbol{d\omega}$ antisymmetric part

$\boldsymbol{\delta\omega}$ divergence part

$\mathbf{T}\boldsymbol{\omega}$ twistor operator

Conformal Killing–Yano forms

Splitting covariant derivative of a form

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$$\boldsymbol{d}\omega \qquad \boldsymbol{\delta}\omega \qquad \mathbf{T}\omega$$

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General form	$\nabla\omega = \mathcal{A}\nabla\omega + \mathcal{C}\nabla\omega + \mathcal{T}\nabla\omega$	
Closed form	$\nabla\omega = \mathcal{C}\nabla\omega + \mathcal{T}\nabla\omega$	$\boldsymbol{d}\omega = 0$
Divergence-free co-closed form	$\nabla\omega = \mathcal{A}\nabla\omega + \mathcal{T}\nabla\omega$	$\boldsymbol{\delta}\omega = 0$
Conformal Killing–Yano form	$\nabla\omega = \mathcal{A}\nabla\omega + \mathcal{C}\nabla\omega$	$\mathbf{T}\omega = 0$
Killing–Yano form	$\nabla\omega = \mathcal{A}\nabla\omega$	$\mathbf{T}\omega = 0\,,\ \boldsymbol{\delta}\omega = 0$
Closed conformal Killing–Yano form	$\nabla\omega = \mathcal{C}\nabla\omega$	$\mathbf{T}\omega = 0\,,\ \boldsymbol{d}\omega = 0$
Harmonic form	$\nabla\omega = \mathcal{T}\nabla\omega$	$\boldsymbol{d}\omega = 0\,,\ \boldsymbol{\delta}\omega = 0$
Covariantly constant form	$\nabla\omega = 0$	$\boldsymbol{d}\omega = 0\,,\ \boldsymbol{\delta}\omega = 0\,,\ \mathbf{T}\omega = 0$

Conformal Killing–Yano forms

form $\boldsymbol{\omega}$ is a *conformal Killing–Yano form* iff for any vector $\boldsymbol{X}$ there exist forms $\boldsymbol{\kappa}$ and $\boldsymbol{\xi}$ such that

$$\boldsymbol{\nabla}_{X}\,\boldsymbol{\omega} = \boldsymbol{X}\cdot\boldsymbol{\kappa} + \boldsymbol{X}\wedge\boldsymbol{\xi}$$

then

$$\boldsymbol{\kappa} = \frac{1}{p+1}\,\boldsymbol{\nabla}\wedge\boldsymbol{\omega} \qquad\qquad \boldsymbol{\xi} = \frac{1}{D-p+1}\,\boldsymbol{\nabla}\cdot\boldsymbol{\omega}$$

Killing–Yano forms

$$\boldsymbol{\nabla}_{X}\,\boldsymbol{f} = \boldsymbol{X}\cdot\boldsymbol{\kappa}$$

$$\boldsymbol{\nabla}_{a}\boldsymbol{f}_{a_1\dots a_p} = \boldsymbol{\nabla}_{[a}\boldsymbol{f}_{a_1\dots a_p]}$$

Closed conformal Killing–Yano forms

$$\boldsymbol{\nabla}_{X}\,\boldsymbol{h} = \boldsymbol{X}\wedge\boldsymbol{\xi}$$

$$\boldsymbol{\nabla}_{a}\boldsymbol{h}_{a_1\dots a_p} = \frac{p}{D-p-1}\,\boldsymbol{g}_{a[a_1}\boldsymbol{\nabla}^{n}\boldsymbol{h}_{|n|a_2\dots a_p]}$$

Basic properties of conformal Killing–Yano forms

Killing–Yano forms | **Closed conformal Killing–Yano forms**

Killing–Yano forms		Closed conformal Killing–Yano forms
$\boldsymbol{f}$	$\overset{*}{\longleftrightarrow}$	$*\boldsymbol{f}$
$*\boldsymbol{h}$	Hodge duality	$\boldsymbol{h}$

$\boldsymbol{k}^{ab} = \boldsymbol{f}_1{}^{(a}{}_{c_2\dots c_p}\,\boldsymbol{f}_2{}^{b)c_2\dots c_p}$

is Killing tensor

$\boldsymbol{h} = \boldsymbol{h}_1 \wedge \boldsymbol{h}_2$

is closed conformal Killing–Yano form

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Principal tensor

Principal tensor $\boldsymbol{h}$ is non-degenerate closed conformal Killing–Yano 2-form

$$\boldsymbol{\nabla}_c \boldsymbol{h}_{ab} = \boldsymbol{g}_{ca} \boldsymbol{\xi}_b - \boldsymbol{g}_{cb} \boldsymbol{\xi}_a \qquad\qquad \boldsymbol{\xi}_a = \frac{1}{D-1} \boldsymbol{\nabla}^b \boldsymbol{h}_{ba}$$

Darboux frame $(\boldsymbol{e}^\mu, \hat{\boldsymbol{e}}^\mu)$

$$\boldsymbol{h} = \sum_{\mu=1}^{N} x_\mu \, \boldsymbol{e}^\mu \wedge \hat{\boldsymbol{e}}^\mu$$

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left(\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \hat{\boldsymbol{e}}^\mu \hat{\boldsymbol{e}}^\mu \right)$$

$$\boldsymbol{d} x_\mu \propto \boldsymbol{e}^\mu \qquad \hat{\boldsymbol{e}}_\mu \cdot \boldsymbol{d} x_\mu = 0$$

non-degeneracy:
x_μ are functionally independent functions

even dimension $D = 2N$
$\mu = 1, \ldots, N$

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Primary Killing vector

Principal tensor $\boldsymbol{h}$

$$\boldsymbol{\nabla}_c \boldsymbol{h}_{ab} = \boldsymbol{g}_{ca} \boldsymbol{\xi}_b - \boldsymbol{g}_{cb} \boldsymbol{\xi}_a$$

Primary Killing vector $\boldsymbol{\xi}$

$$\boldsymbol{\xi}_a = \frac{1}{D-1} \boldsymbol{\nabla}^b \boldsymbol{h}_{ba}$$

$$\pounds_{\boldsymbol{\xi}} \boldsymbol{g} = 0$$

$\boldsymbol{\xi}$ is a Killing vector

$$\Uparrow$$

a highly non-trivial consequence of integrability conditions for the principal tensor equation

$$\pounds_{\boldsymbol{\xi}} \boldsymbol{h} = 0$$

$\boldsymbol{\xi}$ preserves the principal tensor

$$\Uparrow$$

a direct consequence of the principal tensor equation

Killing tower

- **Closed conformal Killing–Yano forms $\boldsymbol{h}^{(j)}$ of rank $2j$:**

$$\boldsymbol{h}^{(j)} = \frac{1}{j!}\,\boldsymbol{h}^{\wedge j}$$

- **Killing–Yano forms $\boldsymbol{f}^{(j)}$ of rank $(D-2j)$:**

$$\boldsymbol{f}^{(j)} = *\boldsymbol{h}^{(j)}$$

- **Rank-2 Killing tensors $\boldsymbol{k}_{(j)}$:**

$$\boldsymbol{k}_{(j)}^{ab} = \frac{1}{(D-2j-1)!}\,\boldsymbol{f}^{(j)a}{}_{c_1\dots c_{D-2j-1}}\,\boldsymbol{f}^{(j)bc_1\dots c_{D-2j-1}}$$

- **Killing vectors $\boldsymbol{l}_{(j)}$:**

$$\boldsymbol{l}_{(j)} = \boldsymbol{k}_{(j)} \cdot \boldsymbol{\xi}$$

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- **Rank-2 Killing tensors $\boldsymbol{k}_{(j)}$:** **Hidden symmetries**

$$\boldsymbol{k}_{(j)}^{ab} = \frac{1}{(D{-}2j{-}1)!}\,\boldsymbol{f}^{(j)a}{}_{c_1\dots c_{D-2j-1}}\,\boldsymbol{f}^{(j)bc_1\dots c_{D-2j-1}}$$

- **Killing vectors $\boldsymbol{l}_{(j)}$:** **Explicit symmetries**

$$\boldsymbol{l}_{(j)} = \boldsymbol{k}_{(j)}\cdot\boldsymbol{\xi}$$

Killing tower

- Rank-2 Killing tensors $\boldsymbol{k}_{(j)}$ **Hidden symmetries**
- Killing vectors $\boldsymbol{l}_{(j)}$ **Explicit symmetries**

$$j = 0, \ldots, N-1$$

Symmetries Nijenhuis–Schouten commute

$$\left[\boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)}\right]_{\mathrm{NS}} = 0 \qquad \left[\boldsymbol{k}_{(i)}, \boldsymbol{l}_{(j)}\right]_{\mathrm{NS}} = 0 \qquad \left[\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}\right]_{\mathrm{NS}} = 0$$

$$\Uparrow$$

a non-trivial consequence of the Killing tower definition, of the principal tensor equation and its integrability conditions

Principal tensor geometry

- eigenvalues of the principal tensor $\boldsymbol{h}$ $\quad\Rightarrow\quad$ coordinates x_μ
- Killing vectors $\boldsymbol{l}_{(j)}$ commute and tangent to $x_\mu = \text{const}$ $\quad\Rightarrow\quad$ Kiling coordinates ψ_j

Uniqueness of the geometry

it is possible to reconstruct the metric up to N free metric functions X_μ

these functions are determined by the Einstein equations

Off-shell Kerr–NUT–(A)dS geometry

for simplicity
even dimension
$D = 2N$

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

explicit function polynomial in coordinates x_μ (symmetric polynomials)

$$A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1 \\ \nu_1<\dots<\nu_k}}^{N} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad A_\mu^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j=1 \\ \nu_1<\dots<\nu_j \\ \nu_i \neq \mu}}^{N} x_{\nu_1}^2 \dots x_{\nu_j}^2 \qquad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{N} (x_\nu^2 - x_\mu^2)$$

unspecified N metric functions of one variable

$$X_\mu = X_\mu(x_\mu)$$

Off-shell Kerr–NUT–(A)dS geometry

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Darboux frame

forms:

$$\boldsymbol{e}^\mu = \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}} \boldsymbol{d}x_\mu \qquad \hat{\boldsymbol{e}}^\mu = \left(\frac{X_\mu}{U_\mu}\right)^{\frac{1}{2}} \sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j$$

vectors:

$$\boldsymbol{e}_\mu = \left(\frac{X_\mu}{U_\mu}\right)^{\frac{1}{2}} \boldsymbol{\partial}_{x_\mu} \qquad \hat{\boldsymbol{e}}_\mu = \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}} \sum_{k=0}^{N-1} \frac{(-x_\mu^2)^{N-1-k}}{U_\mu} \boldsymbol{\partial}_{\psi_k}$$

Off-shell Kerr–NUT–(A)dS geometry

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$$\boldsymbol{e}_\mu = \Big(\frac{X_\mu}{U_\mu}\Big)^{\frac{1}{2}} \boldsymbol{\partial}_{x_\mu} \qquad \hat{\boldsymbol{e}}_\mu = \Big(\frac{U_\mu}{X_\mu}\Big)^{\frac{1}{2}} \sum_{k=0}^{N-1} \frac{(-x_\mu^2)^{N-1-k}}{U_\mu} \boldsymbol{\partial}_{\psi_k}$$

Curvature

$$\mathbf{Ric} = -\sum_{\mu=1}^{N} r_\mu \left(\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \hat{\boldsymbol{e}}^\mu \hat{\boldsymbol{e}}^\mu \right)$$

$$r_\mu = \frac{\partial}{\partial x_\mu^2} \left[\sum_{\nu=1}^{N} \frac{x_\nu^2 \big(x_\nu^{-1} X_\nu \big)_{,\nu}}{U_\nu} \right]$$

$$R = -\sum_{\nu=1}^{N} \frac{X_\nu''}{U_\nu}$$

On-shell Kerr–NUT–(A)dS geometry

for simplicity
even dimension
$D = 2N$

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

Einstein equations $\Rightarrow$

$$X_\mu = \lambda \prod_{\nu=1}^{N} (a_\nu^2 - x_\mu^2) - 2b_\mu \, x_\mu = \lambda \, \mathcal{J}(x_\mu^2) - 2b_\mu x_\mu$$

Parameters:

λ cosmological parameter related to the cosmological constant $\Lambda = (2N-1)(N-1)\lambda$

b_μ mass and NUT parameters

a_μ rotational parameters

freedom in scaling of coordinates $\Rightarrow$ one parameter can be fixed by a gauge condition

exact interpretation of parameters depends on coordinate ranges, signature, and gauge choices

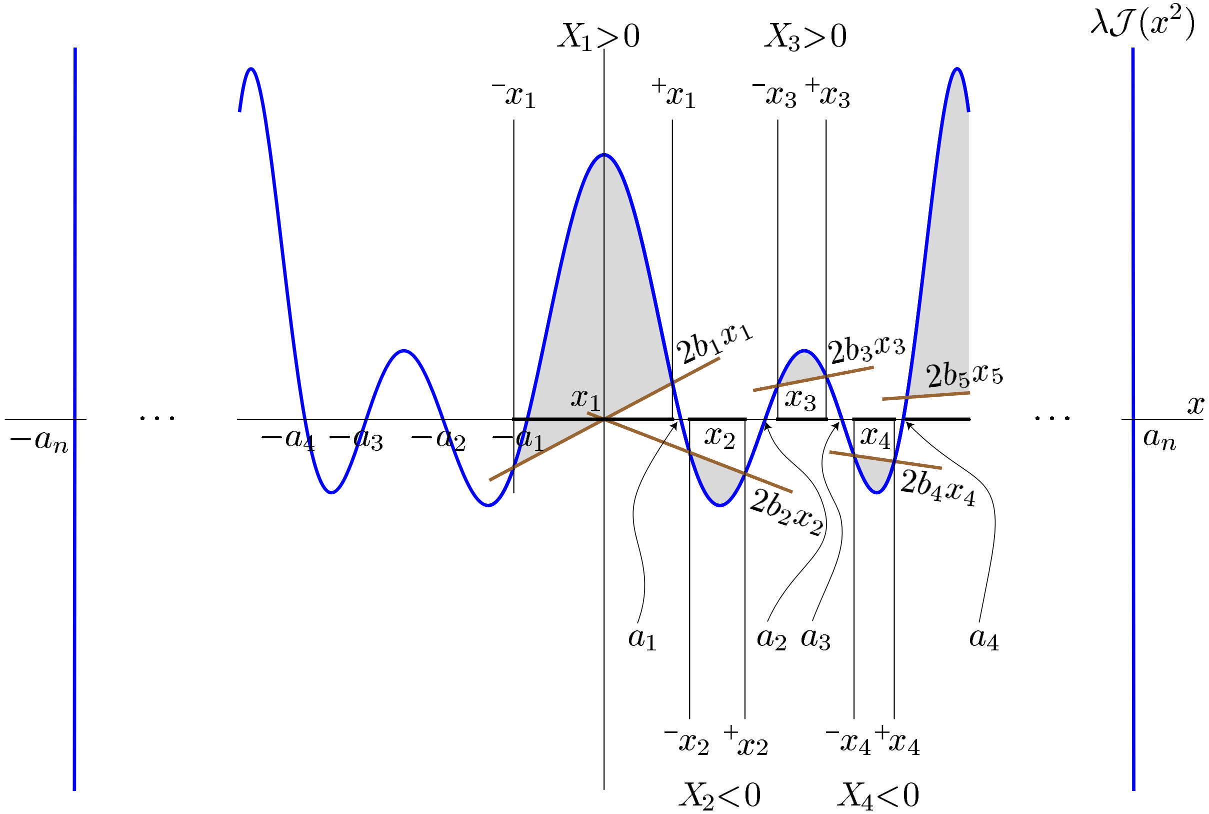
On-shell Kerr–NUT–(A)dS geometry

for simplicity
even dimension
$D = 2N$

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

Coordinate ranges:

- x_μ between roots of X_μ $\qquad X_\mu = \lambda\,\mathcal{J}(x_\mu^2) - 2b_\mu x_\mu$



On-shell Kerr–NUT–(A)dS geometry

for simplicity
even dimension
$D = 2N$

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

Coordinate ranges:

- x_μ between roots of X_μ [for $b_\mu = 0$ roots roots of X_μ are $a_\mu \quad \Rightarrow \quad x_\mu \in (a_{\mu-1}, a_\mu)$]

- ψ_j – Killing coordinates
  ϕ_μ – any liner combination of ψ_j form also Killing coordinates

  a periodic character of properly chosen coordinates ϕ_μ guarantees a regularity of rotation axes

  regularity of whole rotational axes can be achieved only for vanishing NUT parameters

Kerr–NUT–(A)dS geometry – Lorentzian signature

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

Wick rotation:

time: $\tau = \psi_0$ radial coordinate: $x_N = ir$ mass: $b_N = im$

Gauge condition:

$a_N^2 = -\frac{1}{\lambda}$ (suitable for limit $\lambda \to 0$)

New Killing coordinates:

$$t = \tau + \sum_{\bar{k}} \bar{\mathcal{A}}^{(\bar{k}+1)} \bar{\psi}_{\bar{k}} \qquad \frac{\phi_{\bar{\mu}}}{a_{\bar{\mu}}} = \lambda\tau - \sum_{\bar{k}} \big(\bar{\mathcal{A}}_{\bar{\mu}}^{(\bar{k})} - \lambda \bar{\mathcal{A}}_{\bar{\mu}}^{(\bar{k}+1)} \big) \bar{\psi}_{\bar{k}}$$

barred quantities refer to ranges of indices given by $N \to \bar{N} = N - 1$

(i.e., $\bar{\mu} = 1, \ldots, \bar{N}$ and $\bar{j} = 0, \ldots, \bar{N} - 1$, etc.)

Kerr–NUT–(A)dS geometry – Lorentzian signature

$$
\boldsymbol{g} = -\frac{\Delta_r}{\Sigma}\left(\prod_{\bar\nu}\frac{1+\lambda x_{\bar\nu}^2}{1+\lambda a_{\bar\nu}^2}\,\boldsymbol{d}t - \sum_{\bar\nu}\frac{\bar J(a_{\bar\nu}^2)}{a_{\bar\nu}(1+\lambda a_{\bar\nu}^2)\bar{\mathcal{U}}_{\bar\nu}}\boldsymbol{d}\phi_{\bar\nu}\right)^2 + \frac{\Sigma}{\Delta_r}\,\boldsymbol{d}r^2
$$

$$
+\sum_{\bar\mu}\frac{(r^2{+}x_{\bar\mu}^2)}{\Delta_{\bar\mu}/\bar U_{\bar\mu}}\,\boldsymbol{d}x_{\bar\mu}^2 + \sum_{\bar\mu}\frac{\Delta_{\bar\mu}/\bar U_{\bar\mu}}{(r^2{+}x_{\bar\mu}^2)}\left(\frac{1{-}\lambda r^2}{1{+}\lambda x_{\bar\mu}^2}\prod_{\bar\nu}\frac{1{+}\lambda x_{\bar\nu}^2}{1{+}\lambda a_{\bar\nu}^2}\,\boldsymbol{d}t + \sum_{\bar\nu}\frac{(r^2{+}a_{\bar\nu}^2)\bar J_{\bar\mu}(a_{\bar\nu}^2)}{a_{\bar\nu}(1{+}\lambda a_{\bar\nu}^2)\,\bar{\mathcal{U}}_{\bar\nu}}\boldsymbol{d}\phi_{\bar\nu}\right)^2
$$

$$
\Delta_r = -X_N = \big(1{-}\lambda r^2\big)\prod_{\bar\nu}\big(r^2{+}a_{\bar\nu}^2\big) - 2mr \qquad\qquad \Sigma = U_N = \prod_{\bar\nu}(r^2+x_{\bar\nu}^2)
$$

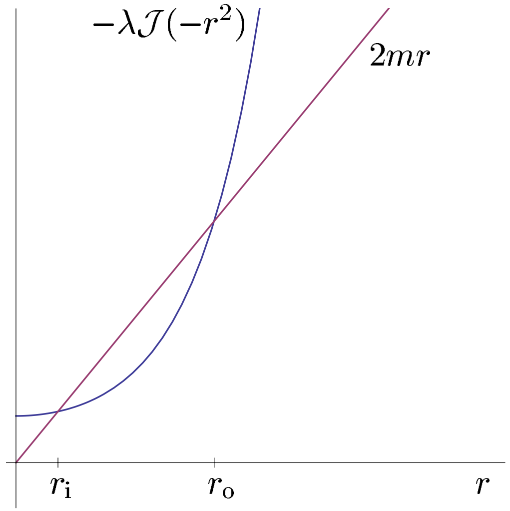
$$
\Delta_{\bar\mu} = -X_{\bar\mu} = \big(1{+}\lambda x_{\bar\mu}^2\big)\bar{\mathcal{J}}(x_{\bar\mu}^2) + 2b_{\bar\mu}x_{\bar\mu} \qquad\qquad \bar U_{\bar\mu} = \prod_{\substack{\bar\nu \\ \bar\nu\neq\bar\mu}}(x_{\bar\nu}^2 - x_{\bar\mu}^2)
$$

Kerr–NUT–(A)dS geometry – Lorentzian signature

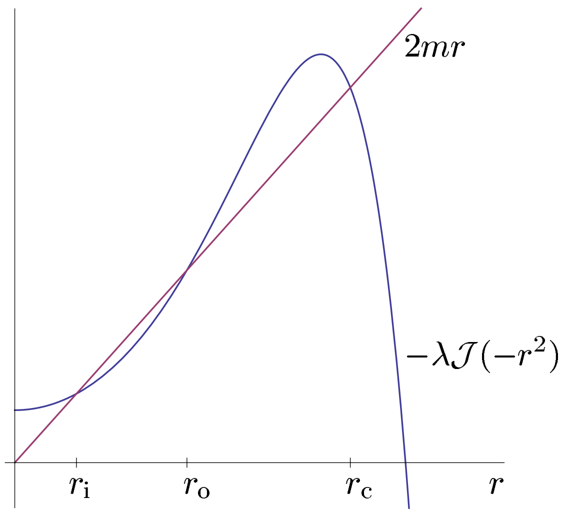
$$\boldsymbol{g} = -\frac{\Delta_r}{\Sigma}\left(\prod_{\bar\nu}\frac{1+\lambda x_{\bar\nu}^2}{1+\lambda a_{\bar\nu}^2}\,\boldsymbol{d}t - \sum_{\bar\nu}\frac{\bar J(a_{\bar\nu}^2)}{a_{\bar\nu}(1+\lambda a_{\bar\nu}^2)\bar{\mathcal{U}}_{\bar\nu}}\boldsymbol{d}\phi_{\bar\nu}\right)^2 + \frac{\Sigma}{\Delta_r}\,\boldsymbol{d}r^2$$

$$+\sum_{\bar\mu}\frac{(r^2{+}x_{\bar\mu}^2)}{\Delta_{\bar\mu}/\bar U_{\bar\mu}}\,\boldsymbol{d}x_{\bar\mu}^2 + \sum_{\bar\mu}\frac{\Delta_{\bar\mu}/\bar U_{\bar\mu}}{(r^2{+}x_{\bar\mu}^2)}\left(\frac{1{-}\lambda r^2}{1{+}\lambda x_{\bar\mu}^2}\prod_{\bar\nu}\frac{1{+}\lambda x_{\bar\nu}^2}{1{+}\lambda a_{\bar\nu}^2}\,\boldsymbol{d}t + \sum_{\bar\nu}\frac{(r^2{+}a_{\bar\nu}^2)\bar J_{\bar\mu}(a_{\bar\nu}^2)}{a_{\bar\nu}(1{+}\lambda a_{\bar\nu}^2)\,\bar{\mathcal{U}}_{\bar\nu}}\boldsymbol{d}\phi_{\bar\nu}\right)^2$$

Horizon function: $\Delta_r = \big(1{-}\lambda r^2\big)\prod_{\bar\nu}\big(r^2{+}a_{\bar\nu}^2\big) - 2mr$



$\lambda{\leq}0$



$\lambda{>}0$

Kerr–NUT geometry: $\Lambda = 0$

$$\boldsymbol{g} = -\frac{\Delta_r}{\Sigma}\Big(\boldsymbol{d}\tau + \sum_{\bar{k}} \bar{A}^{(\bar{k}+1)}\boldsymbol{d}\bar{\psi}_{\bar{k}}\Big)^2 + \frac{\Sigma}{\Delta_r}\boldsymbol{d}r^2$$

$$+\sum_{\bar{\mu}} \frac{(r^2+x_{\bar{\mu}}^2)}{\Delta_{\bar{\mu}}/\bar{U}_{\bar{\mu}}}\boldsymbol{d}x_{\bar{\mu}}^2 + \sum_{\bar{\mu}} \frac{\Delta_{\bar{\mu}}/\bar{U}_{\bar{\mu}}}{(r^2+x_{\bar{\mu}}^2)}\Big(\boldsymbol{d}\tau + \sum_{\bar{k}}\big(\bar{A}_{\bar{\mu}}^{(\bar{k}+1)} - r^2\bar{A}_{\bar{\mu}}^{(\bar{k})}\big)\boldsymbol{d}\bar{\psi}_{\bar{k}}\Big)^2$$

$$\Delta_r = -X_N = \prod_{\bar{\nu}}\big(r^2+a_{\bar{\nu}}^2\big) - 2mr \qquad \Sigma = U_N = \prod_{\bar{\nu}}(r^2 + x_{\bar{\nu}}^2)$$

$$\Delta_{\bar{\mu}} = -X_{\bar{\mu}} = \bar{\mathcal{J}}(x_{\bar{\mu}}^2) + 2b_{\bar{\mu}}x_{\bar{\mu}} \qquad \bar{U}_{\bar{\mu}} = \prod_{\substack{\bar{\nu}\\ \bar{\nu}\neq\bar{\mu}}}(x_{\bar{\nu}}^2 - x_{\bar{\mu}}^2)$$

Myers–Perry metric: $\Lambda = 0 \quad b_{\bar{\mu}} = 0$

$$\boldsymbol{g} = -\boldsymbol{d}t^2 + \frac{2mr}{\Sigma}\Big(\boldsymbol{d}t + \sum_{\bar{\nu}} a_{\bar{\nu}}\mu_{\bar{\nu}}^2 \boldsymbol{d}\phi_{\bar{\nu}}\Big)^2 + \frac{\Sigma}{\Delta_r}\boldsymbol{d}r^2 + r^2\boldsymbol{d}\mu_0^2 + \sum_{\bar{\nu}}(r^2 + a_{\bar{\nu}}^2)\Big(\boldsymbol{d}\mu_{\bar{\nu}}^2 + \mu_{\bar{\nu}}^2\boldsymbol{d}\phi_{\bar{\nu}}^2\Big)$$

$$\Delta_r = \prod_{\bar{\nu}}(r^2 + a_{\bar{\nu}}^2) - 2mr \qquad\qquad \Sigma = \Big(\mu_0^2 + \sum_{\bar{\nu}}\frac{r^2\mu_{\bar{\nu}}^2}{r^2 + a_{\bar{\nu}}^2}\Big)\prod_{\bar{\mu}}(r^2 + a_{\bar{\mu}}^2)\ ,$$

new latitudinal coordinates:

$\bar{N}$ coordinates $x_1, \dots, x_{\bar{N}}$ $\quad\rightarrow\quad$ $N+1$ constrained coordinates $\mu_0, \mu_1, \dots, \mu_{\bar{N}}$

$$\mu_{\bar{\nu}}^2 = \frac{\bar{J}(a_{\bar{\nu}}^2)}{-a_{\bar{\nu}}^2\bar{\mathcal{U}}_{\bar{\nu}}} = \frac{\prod_{\bar{\alpha}}(x_{\bar{\alpha}}^2 - a_{\bar{\nu}}^2)}{-a_{\bar{\nu}}^2\prod_{\bar{\alpha}\neq\bar{\nu}}(a_{\bar{\alpha}}^2 - a_{\bar{\nu}}^2)}\ , \quad \mu_0^2 = \frac{\bar{A}^{(\bar{N})}}{\bar{\mathcal{A}}^{(\bar{N})}} = \frac{\prod_{\bar{\alpha}} x_{\bar{\alpha}}^2}{\prod_{\bar{\alpha}} a_{\bar{\alpha}}^2}\ ,$$

constrain:

$$\sum_{\bar{\nu}=0}^{\bar{N}}\mu_{\bar{\nu}}^2 = 1$$

Schwarzschild–Tangherlini–(A)dS metric: $b_{\bar{\mu}} = 0 \quad a_\mu = 0$

$$\boldsymbol{g} = -\big(1 - \lambda r^2 - 2mr^{3-2N}\big)\,\boldsymbol{d}t^2 + \frac{1}{1 - \lambda r^2 - 2mr^{3-2N}}\,\boldsymbol{d}r^2 + r^2\,\boldsymbol{d\Omega}^2_{\bar{N}}$$

- Spherical symmetric black hole of mass m
- Horizons given by roots of the metric function
- Asymptotic character given by λ
- Radial dependence changes with the dimension

Principal tensor geometry = off-shell Kerr–NUT–(A)dS

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

explicit function polynomial in coordinates x_μ (symmetric polynomials)

$$A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1 \\ \nu_1<\dots<\nu_k}}^{N} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad A_\mu^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j=1 \\ \nu_1<\dots<\nu_j \\ \nu_i\neq\mu}}^{N} x_{\nu_1}^2 \dots x_{\nu_j}^2 \qquad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{N} (x_\nu^2 - x_\mu^2)$$

unspecified N metric functions of one variable

$$X_\mu = X_\mu(x_\mu)$$

Principal tensor geometry = off-shell Kerr–NUT–(A)dS

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu}\,\boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

explicit

$A^{(k)} =$

$\nu_1, \dots$
$\nu_1 <$

$$\boldsymbol{g} = -\frac{\Delta_r}{\Sigma}\left(\prod_{\bar\nu}\frac{1+\lambda x_{\bar\nu}^2}{1+\lambda a_{\bar\nu}^2}\,\boldsymbol{d}t - \sum_{\bar\nu}\frac{\bar J(a_{\bar\nu}^2)}{a_{\bar\nu}(1+\lambda a_{\bar\nu}^2)\bar{\mathcal{U}}_{\bar\nu}}\boldsymbol{d}\phi_{\bar\nu}\right)^2 + \frac{\Sigma}{\Delta_r}\,\boldsymbol{d}r^2$$
$$+ \sum_{\bar\mu}\frac{(r^2+x_{\bar\mu}^2)}{\Delta_{\bar\mu}/\bar U_{\bar\mu}}\,\boldsymbol{d}x_{\bar\mu}^2 + \sum_{\bar\mu}\frac{\Delta_{\bar\mu}/\bar U_{\bar\mu}}{(r^2+x_{\bar\mu}^2)}\left(\frac{1-\lambda r^2}{1+\lambda x_{\bar\mu}^2}\prod_{\bar\nu}\frac{1+\lambda x_{\bar\nu}^2}{1+\lambda a_{\bar\nu}^2}\,\boldsymbol{d}t + \sum_{\bar\nu}\frac{(r^2+a_{\bar\nu}^2)\bar J_{\bar\mu}(a_{\bar\nu}^2)}{a_{\bar\nu}(1+\lambda a_{\bar\nu}^2)\bar{\mathcal{U}}_{\bar\nu}}\boldsymbol{d}\phi_{\bar\nu}\right)^2$$

ions of one variable

$$X_\mu = X_\mu(x_\mu)$$

Principal tensor geometry = off-shell Kerr–NUT–(A)dS

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

explicit function polynomial in coordinates x_μ (symmetric polynomials)

$$A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1\\ \nu_1<\dots<\nu_k}}^{N} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad A_\mu^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j=1\\ \nu_1<\dots<\nu_j\\ \nu_i\neq\mu}}^{N} x_{\nu_1}^2 \dots x_{\nu_j}^2 \qquad U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{N} (x_\nu^2 - x_\mu^2)$$

unspecified N metric functions of one variable

$$X_\mu = X_\mu(x_\mu)$$

Non-trivial limits of Kerr–NUT–(A)dS geometry

- Limit of vanishing rotations
- Deformed black hole
- Nutty spacetimes
- Near horizon limits
- Limit of equal rotations

Limit of vanishing rotations

vanishing rotations in $\bar{N}$ directions

$$\Updownarrow$$

$a_{\bar{\mu}} \to 0$ **for these directions**

- setting $a_{\bar{\mu}} = 0$ yields degenerate ranges of coordinates
- setting $a_{\bar{\mu}} = 0$ yields degenerate metric
- a suitable scaling of coordinates necessary!

Limit of vanishing rotations

vanishing rotations in $\bar{N}$ directions

$$\Updownarrow$$

$a_{\bar{\mu}} \to 0$ for these directions

regular sector	**unspined sector**
$a_{\bar{N}+\tilde{\mu}} = \tilde{a}_{\tilde{\mu}}$	$a_{\bar{\mu}} = \varepsilon\, \bar{a}_{\bar{\mu}}$
$x_{\bar{N}+\tilde{\mu}} = \tilde{x}_{\tilde{\mu}}$	$x_{\bar{\mu}} = \varepsilon\, \bar{x}_{\bar{\mu}}$
$\phi_{\bar{N}+\tilde{\mu}} = \tilde{\phi}_{\tilde{\mu}}$	$\phi_{\bar{\mu}} = \bar{\phi}_{\bar{\mu}}$
$\tilde{\mu} = 1, \ldots, \tilde{N}$	$\bar{\mu} = 1, \ldots, \bar{N}$

$$\varepsilon \to 0$$

Limit of vanishing rotations

vanishing rotations in $\bar{N}$ directions

$$\Updownarrow$$

$a_{\bar{\mu}} \to 0$ for these directions

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$\phi_{\bar{N}+\tilde{\mu}} = \tilde{\phi}_{\tilde{\mu}}$	$\phi_{\bar{\mu}} = \bar{\phi}_{\bar{\mu}}$
$\tilde{\mu} = 1, \dots, \tilde{N}$	$\bar{\mu} = 1, \dots, \bar{N}$

$$\varepsilon \to 0$$

$$\boldsymbol{g} = \tilde{\boldsymbol{g}} + \tilde{w}^2 \bar{\boldsymbol{g}}$$

$$\tilde{\boldsymbol{g}} = \sum_{\tilde{\mu}} \left[\frac{\tilde{U}_{\tilde{\mu}}}{\tilde{X}_{\tilde{\mu}}} \boldsymbol{d}\tilde{x}_{\tilde{\mu}}^2 + \frac{\tilde{X}_{\tilde{\mu}}}{\tilde{U}_{\tilde{\mu}}} \Big(\sum_{\tilde{\nu}} \frac{\tilde{J}_{\tilde{\mu}}(\tilde{a}_{\tilde{\nu}}^2)}{\tilde{\mathcal{U}}_{\tilde{\nu}}} \boldsymbol{d}\phi_{\tilde{\nu}} \Big)^2 \right]$$

$$\tilde{X}_{\tilde{\mu}} = \lambda\, \tilde{\mathcal{J}}(\tilde{x}_{\tilde{\mu}}^2) - 2\, \tilde{b}_{\tilde{\mu}}\, \tilde{x}_{\tilde{\mu}}^{1-2\tilde{N}}$$

off-shell Kerr–NUT–(A)dS

Lorentzian part

$$\bar{\boldsymbol{g}} = \sum_{\bar{\mu}} \left[\frac{\bar{U}_{\bar{\mu}}}{\bar{X}_{\bar{\mu}}} \boldsymbol{d}\bar{x}_{\bar{\mu}}^2 + \frac{\bar{X}_{\bar{\mu}}}{\bar{U}_{\bar{\mu}}} \Big(\sum_{\bar{\nu}} \frac{\bar{J}_{\bar{\mu}}(\bar{a}_{\bar{\nu}}^2)}{\bar{\mathcal{U}}_{\bar{\nu}}} \boldsymbol{d}\phi_{\bar{\nu}} \Big)^2 \right]$$

$$\bar{X}_{\bar{\mu}} = \lambda\, \bar{\mathcal{J}}(\bar{x}_{\bar{\mu}}^2) - 2\, \bar{b}_{\bar{\mu}}\, \bar{x}_{\bar{\mu}}$$

on-shell Kerr–NUT–(A)dS

Euclidian instanton

Deformed black hole

switching-off all rotations by repeating the limiting procedure

$$\boldsymbol{g} = -f\,\boldsymbol{d}t^2 + \frac{1}{f}\boldsymbol{d}r^2 + r^2\Big[\boldsymbol{q}_{N-1} + \xi_{N-1}^2\Big(\boldsymbol{q}_{N-2} + \cdots + \xi_3^2\big(\boldsymbol{q}_2 + \xi_2^2\boldsymbol{q}_1\big)\Big)\Big]$$

$$f = 1 - \lambda r^2 - \frac{2m}{r^{2N-3}}$$

$$\boldsymbol{q}_{\bar\mu} = \frac{1}{\Delta_{\bar\mu}}\boldsymbol{d}\xi_{\bar\mu}^2 + \Delta_{\bar\mu}\,\boldsymbol{d}\phi_{\bar\mu}^2 \qquad\qquad \Delta_{\bar\mu} = 1 - \xi_{\bar\mu}^2 - 2\,\beta_{\bar\mu}\,\xi_{\bar\mu}^{-2\bar\mu+3}$$

- non-rotating
- spatially deformed

static deformed black hole

Nutty spacetimes

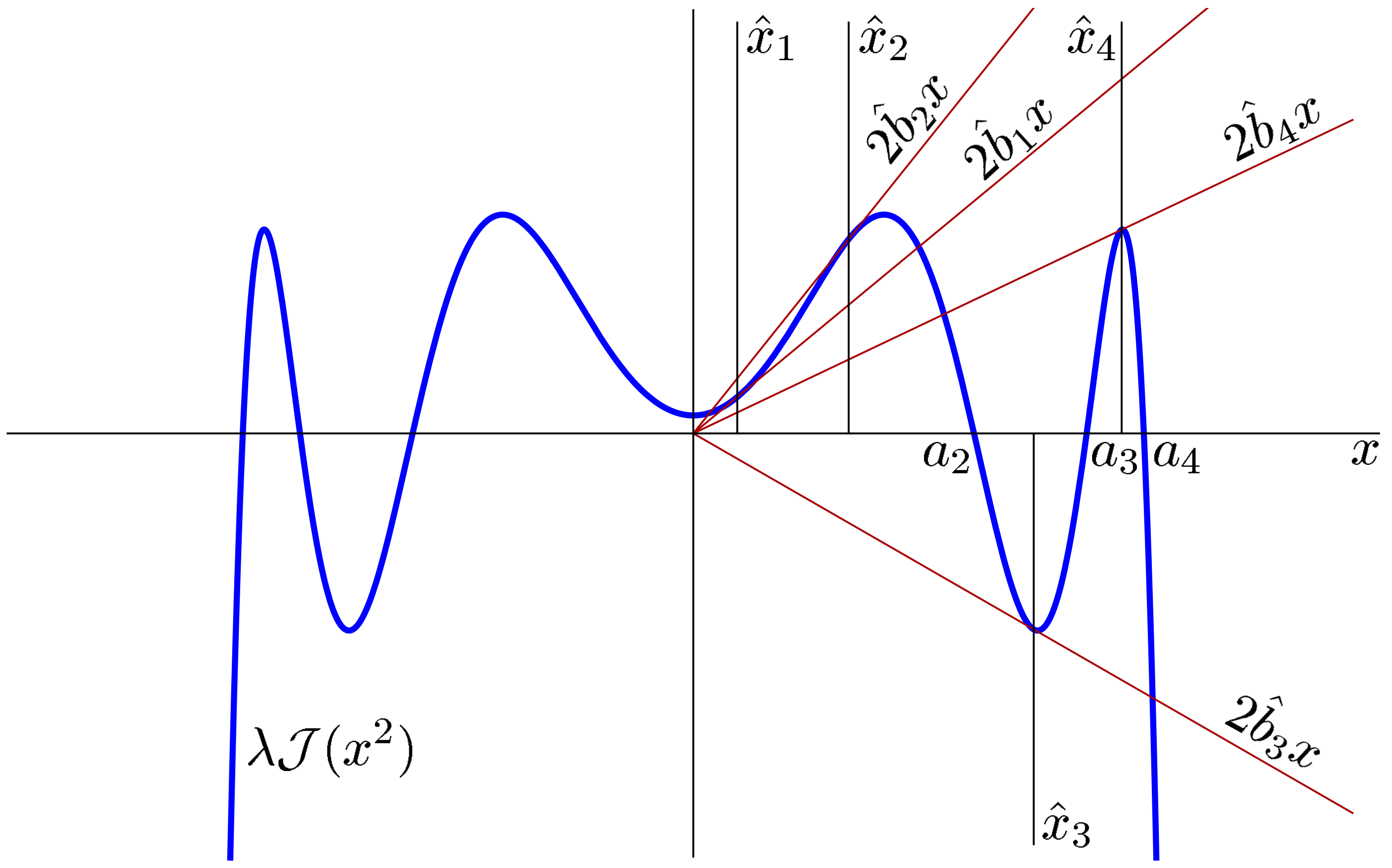
generalization of Taub–NUT–(A)dS

Nutty spacetimes

generalization of Taub–NUT–(A)dS

tangent points $\hat{x}_\mu$ are double roots of X_μ for suitable values $b_\mu = \hat{b}_\mu$

$$X_\mu = \lambda \mathcal{J}(x_{\bar{\mu}}^2) - 2\hat{b}_{\bar{\mu}} x_{\bar{\mu}}$$



Nutty spacetimes

generalization of Taub–NUT–(A)dS

spacetimes with near-critical NUTs

x-coordinate ranges near double roots $\hat{x}_{\bar{\mu}}$

$$\boldsymbol{g} = -\frac{\Delta}{\Sigma}\Big(\boldsymbol{d}t + \sum_{\bar{\mu}} \frac{2\hat{x}_{\bar{\mu}}}{\delta_{\bar{\mu}}}\,(\xi_{\bar{\mu}} - \mathring{\xi}_{\bar{\mu}})\,\boldsymbol{d}\varphi_{\bar{\mu}}\Big)^2 + \frac{\Sigma}{\Delta}\boldsymbol{d}r^2 + \sum_{\bar{\mu}} \frac{r^2 + \hat{x}_{\bar{\mu}}^2}{\delta_{\bar{\mu}}}\left(\frac{1}{1-\xi_{\bar{\mu}}^2}\,\boldsymbol{d}\xi_{\bar{\mu}}^2 + \big(1-\xi_{\bar{\mu}}^2\big)\,\boldsymbol{d}\varphi_{\bar{\mu}}^2\right)$$

$$\Delta = -\lambda \mathcal{J}(-r^2) - 2mr \qquad \Sigma = \prod_{\bar{\nu}} (r^2 + \hat{x}_{\bar{\nu}}^2) \qquad \delta_{\mu} = \lambda(2N-1)(\hat{r}^2 + \hat{x}_{\bar{\mu}}^2)$$

Near horizon limits

analogy for radial coordinate

spacetimes with near-critical mass

coordinate range of r near double root $\hat{r}$

Near horizon limits

analogy for radial coordinate

spacetimes with near-critical mass

coordinate range of r near double root $\hat{r}$

Limit of equal rotations

enhancement of symmetry in the equal-rotation sector

NS-commutation of the Killing tower

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right]$$

Principal tensor

$$\boldsymbol{h} = \sum_\mu x_\mu \, \boldsymbol{e}^\mu \wedge \hat{\boldsymbol{e}}^\mu$$

Primary Killing vector

$$\boldsymbol{\xi} = \boldsymbol{\partial}_{\psi_0}$$

Hidden symmetries – Killing tensors

$$\boldsymbol{k}_{(j)} = \sum_\mu A_\mu^{(j)} \left(\boldsymbol{e}_\mu \boldsymbol{e}_\mu + \hat{\boldsymbol{e}}_\mu \hat{\boldsymbol{e}}_\mu \right)$$

Explicit symmetries – Killing vectors

$$\boldsymbol{l}_{(j)} = \boldsymbol{\partial}_{\psi_j}$$

$$\left[\boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)}\right]_{\text{NS}} = 0 \qquad \left[\boldsymbol{k}_{(i)}, \boldsymbol{l}_{(j)}\right]_{\text{NS}} = 0 \qquad \left[\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}\right]_{\text{NS}} = 0$$

Darboux frame forms: vectors:

$$\boldsymbol{e}^\mu = \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}} \boldsymbol{d}x_\mu \qquad \hat{\boldsymbol{e}}^\mu = \left(\frac{X_\mu}{U_\mu}\right)^{\frac{1}{2}} \sum_{j=0}^{N-1} A_\mu^{(j)} \boldsymbol{d}\psi_j$$

$$\boldsymbol{e}_\mu = \left(\frac{X_\mu}{U_\mu}\right)^{\frac{1}{2}} \boldsymbol{\partial}_{x_\mu} \qquad \hat{\boldsymbol{e}}_\mu = \left(\frac{U_\mu}{X_\mu}\right)^{\frac{1}{2}} \sum_{k=0}^{N-1} \frac{(-x_\mu^2)^{N-1-k}}{U_\mu} \boldsymbol{\partial}_{\psi_k}$$

Generating functions for Killing objects

Tensors depending on an auxiliary parameter β

$$\boldsymbol{q}(\beta) = \boldsymbol{g} - \beta^2\,\boldsymbol{h}^2$$

$$A(\beta) = \left(\frac{\operatorname{Det}\boldsymbol{q}(\beta)}{\operatorname{Det}\boldsymbol{g}}\right)^{\frac{1}{2}}$$

$$\boldsymbol{k}(\beta) = A(\beta)\;\boldsymbol{q}(\beta)^{-1}$$

$$\boldsymbol{l}(\beta) = \boldsymbol{k}(\beta)\cdot\boldsymbol{\xi}$$

$$\Rightarrow$$

$$A(\beta) = \sum_j A^{(j)}\beta^{2j}$$

$$\boldsymbol{k}(\beta) = \sum_j \boldsymbol{k}_{(j)}\beta^{2j}$$

$$\boldsymbol{l}(\beta) = \sum_j \boldsymbol{l}_{(j)}\beta^{2j}$$

Vanishing NS-brackets

$$\big[\,\boldsymbol{l}(\beta_1)\,,\,\boldsymbol{l}(\beta_2)\,\big]_{\mathrm{NS}} = 0$$

$$\big[\,\boldsymbol{l}(\beta_1)\,,\boldsymbol{k}(\beta_2)\big]_{\mathrm{NS}} = 0$$

$$\big[\boldsymbol{k}(\beta_1),\boldsymbol{k}(\beta_2)\big]_{\mathrm{NS}} = 0$$

Since $\boldsymbol{k}(0) = \boldsymbol{g}$, it also implies that $\boldsymbol{l}(\beta)$ and $\boldsymbol{k}(\beta)$ are KTs.

Proof of vanishing NS-brackets

$$\nabla_a h_{bc} = g_{ab}\xi_c - g_{ac}\xi_b \qquad \Leftarrow \quad \text{CCKY form}$$

$$\nabla_c q_{ab} = 2\beta^2\big(g_{c(a}h_{b)n} + h_{c(a}g_{b)n}\big)\xi^n \qquad \Leftarrow \quad q = g - \beta^2 h^2$$

$$\nabla_a A = 2\beta^2 h_{am}k^{mn}\xi_n \qquad \Leftarrow \quad A = (\mathrm{Det}\, q / \mathrm{Det}\, g)^{\frac{1}{2}}$$

$$\nabla^c k^{ab} = \frac{2\beta^2}{A}\big(k^{ab}\, k^{cn}h_n{}^m + h^m{}_n k^{n(a}k^{b)c} - k^{m(a}k^{b)n}h_n{}^c\big)\,\xi_m \qquad \Leftarrow \quad k = A\, q^{-1}$$

$$l^a = k^{an}\xi_n$$

$\Downarrow$ substituting into NS-brackets

$$\big[k_1, k_2\big]^{abc}_{\mathrm{NS}} = 3\big(k_1^{n(a}\nabla_n k_2^{bc)} - k_2^{n(a}\nabla_n k_1^{bc)} \qquad = 0$$

$$\big[k_1,\, l_2\big]^{ab}_{\mathrm{NS}} = 2\, k_1^{n(a}\nabla_n l_2^{b)} - l_2^n \nabla_n k_1^{ab)} \qquad = \big(k_1^{am}\,(\nabla_m \xi_n)\, k_2^{nb} + k_2^{am}\,(\nabla_n \xi_m)\, k_1^{nb}\big)$$

$$\big[\, l_1\,,\, l_2\,\big]^a_{\mathrm{NS}} = l_1^n \nabla_n l_2^{a)} - l_2^n \nabla_n l_1^{a)} \qquad = \big(k_1^{am}\,(\nabla_m \xi_n)\, k_2^{nb} - k_2^{am}\,(\nabla_m \xi_n)\, k_1^{nb}\big)\;\xi_b$$

$\Uparrow$

$$\mathcal{S}(\nabla\xi) = 0 \qquad \text{and} \qquad \big[\nabla\xi, k(\beta)\big] = 0 \quad \Leftarrow \quad \big[\nabla\xi, h\big] = 0$$

where $k_1 = k(\beta_1)$, $k_2 = k(\beta_2)$, $l_1 = l(\beta_1)$, etc.

Integrability conditions for principal CCKY form

$$\boldsymbol{\nabla}_a \boldsymbol{h}_{bc} = \boldsymbol{g}_{ab}\boldsymbol{\xi}_c - \boldsymbol{g}_{ac}\boldsymbol{\xi}_b$$

$\Downarrow$ taking second derivative $2\boldsymbol{\nabla}_{[a}\boldsymbol{\nabla}_{b]}\boldsymbol{h}_{mn}$

$$\boldsymbol{R}^{ab}{}_{c[m}\,\boldsymbol{h}^c{}_{n]} = 2\delta^{[a}_{[m}\boldsymbol{\nabla}^{b]}\boldsymbol{\xi}_{n]} \qquad \text{(IC1)}$$

$\Downarrow$ Bianchi identities, contractions

$$(D-2)\boldsymbol{\nabla}_a\boldsymbol{\xi}_b = -\mathbf{Ric}_{an}\,\boldsymbol{h}^n{}_b + \frac{1}{2}\boldsymbol{h}_{mn}\boldsymbol{R}^{mn}{}_{ab} \qquad \text{(IC2)}$$

$\Downarrow$ substituting (IC2) to (IC1)

$$(D-2)\boldsymbol{R}^{ab}{}_{n[c}\,\boldsymbol{h}^n{}_{d]} - \boldsymbol{h}_{mn}\boldsymbol{R}^{mn[a}{}_{[c}\,\delta^{b]}_{d]} - 2\mathbf{Ric}^{[a}{}_n\,\delta^{b]}_{[c}\,\boldsymbol{h}^n{}_{d]} = 0 \qquad \text{(IC3)}$$

$$\mathcal{S}(\boldsymbol{\nabla}\boldsymbol{\xi}) = \big[\boldsymbol{h}, \mathbf{Ric}\big] \qquad \Leftarrow \quad \text{symmetrization of (IC2)}$$

$$(D-2)\big[\boldsymbol{\nabla}\boldsymbol{\xi}, \boldsymbol{h}\big] = \big[\boldsymbol{h}, \mathbf{Ric}\big]\cdot\boldsymbol{h} + \frac{1}{2}\big[\mathbf{Rh}, \boldsymbol{h}\big] \qquad \Leftarrow \quad \big[\text{(IC2)}, \boldsymbol{h}\big]$$

Alignment of Riemann tensor with principal CCKY form

Taking various (anti)symmetrizations and contractions of (IC3) with $\boldsymbol{h}$ one can prove:

$$\Big[\, \boldsymbol{h}\,,\, \mathbf{Rh}^{(p)} \Big] = 0 \qquad \text{where} \quad \mathbf{Rh}^{(p)a}{}_b = \boldsymbol{h}^p{}_{mn}\, \boldsymbol{R}^{mn\,a}{}_b$$

$$\mathbf{Rh} = \mathbf{Rh}^{(1)}$$

$$\Big[\, \boldsymbol{h}\,,\, \mathbf{Rich}^{(2p)} \Big] = 0 \qquad \text{where} \quad \mathbf{Rich}^{(2p)}_{ab} = \boldsymbol{h}^p{}_{mn}\, \boldsymbol{R}^m{}_a{}^n{}_b$$

$$\mathbf{Rich}^{(0)} = \mathbf{Ric}$$

Any contraction of $\boldsymbol{R}$ with any power of $\boldsymbol{h}$ commutes with $\boldsymbol{h}$

NS-commutation of Killing tower

Any contraction of $\boldsymbol{R}$ with any power of $\boldsymbol{h}$ commutes with $\boldsymbol{h}$

$$\Rightarrow \qquad \mathcal{S}(\boldsymbol{\nabla}\boldsymbol{\xi}) = \big[\boldsymbol{h}, \mathbf{Ric}\big] = 0$$

$$\big[\boldsymbol{\nabla}\boldsymbol{\xi}, \boldsymbol{h}\big] = \tfrac{1}{D-2}\Big(\big[\boldsymbol{h}, \mathbf{Ric}\big]\cdot\boldsymbol{h} + \frac{1}{2}\big[\mathbf{Rh}, \boldsymbol{h}\big]\Big) = 0$$

$\Rightarrow$ **Vanishing NS-brackets**

$$\big[\boldsymbol{k}_1, \boldsymbol{k}_2\big]_{\mathrm{NS}} = 0 \qquad \big[\boldsymbol{k}_1, \boldsymbol{l}_2\big]_{\mathrm{NS}} = 0 \qquad \big[\boldsymbol{l}_1, \boldsymbol{l}_2\big]_{\mathrm{NS}} = 0$$

$$\big[\boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)}\big]_{\mathrm{NS}} = 0 \quad \big[\boldsymbol{k}_{(i)}, \boldsymbol{l}_{(j)}\big]_{\mathrm{NS}} = 0 \quad \big[\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}\big]_{\mathrm{NS}} = 0$$

Consequences for physical systems: integrability and separability

- **Integrability of the geodesic motion**
- **Separability of the Hamilton–Jacobi equation**
- **Separability of the wave equation**
- **Separability of the Dirac equation**

Integrability of the geodesic motion

hidden and explicit symmetries define observables quadratic and linear in momenta

$$\boldsymbol{k}_{(j)} \quad \Rightarrow \quad K_j = \boldsymbol{k}^{ab}_{(j)}\, \boldsymbol{p}_a \boldsymbol{p}_b \qquad\qquad \boldsymbol{l}_{(j)} \quad \Rightarrow \quad L_j = \boldsymbol{l}^{a}_{(j)}\, \boldsymbol{p}_a$$

$$\big[\boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)}\big]_{\mathrm{NS}} = 0 \qquad \big[\boldsymbol{k}_{(i)}, \boldsymbol{l}_{(j)}\big]_{\mathrm{NS}} = 0 \qquad \big[\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}\big]_{\mathrm{NS}} = 0$$

$$\Updownarrow$$

$$\big\{K_i, K_j\big\} = 0 \qquad \big\{K_i, L_j\big\} = 0 \qquad \big\{L_i, L_j\big\} = 0$$

$$\boldsymbol{k}_{(0)} = \boldsymbol{g} \quad \Rightarrow \quad K_0 \propto H$$

K_j and L_j are conserved quantities in involution

$$\Downarrow$$

Complete integrability

Geodesics

geodesics labeled by constants K_j, L_j and initial positions

Solving for momenta

$$p_\mu = \pm \frac{\sqrt{X_\mu \tilde{K}_\mu - \tilde{L}_\mu^2}}{X_\mu}$$

$$p_j = L_j$$

$$\tilde{K}_\mu = \sum_j K_j (-x_\mu^2)^{N-1-j}$$

$$\tilde{L}_\mu = \sum_j L_j (-x_\mu^2)^{N-1-j}$$

Solving for coordinates

$$\dot{x}_\mu = \pm \frac{1}{U_\mu} \sqrt{X_\mu \tilde{K}_\mu - \tilde{L}_\mu^2}$$

$$\dot{\psi}_j = \sum_\mu \frac{(-x_\mu^2)^{N-1-j}}{U_\mu} \frac{\tilde{L}_\mu}{X_\mu}$$

can be solved numerically or

analytically using action–angle variables

Separability of the Hamilton–Jacobi equation

Static Hamilton–Jacobi equation

$$H(x, \boldsymbol{d}S) = E$$

Hamilton–Jacobi equations for conserved quantities

$$\boldsymbol{d}S \cdot \boldsymbol{k}_{(j)} \cdot \boldsymbol{d}S = K_j \qquad\qquad \boldsymbol{l}_{(j)} \cdot \boldsymbol{d}S = L_j$$

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Additive separability ansatz

$$S = \sum_\mu S_\mu + \sum_j L_j\, \psi_j \qquad\qquad S_\mu = S_\mu(\, x_\mu\, ;\ K_j,\, L_j)$$

Separated ordinary differential equations

$$(S'_\mu)^2 = \frac{\tilde{K}_\mu}{X_\mu} - \frac{\tilde{L}_\mu^2}{X_\mu^2} \qquad \Rightarrow \qquad S_\mu = \int^{x_\mu} \frac{\sqrt{X_\mu \tilde{K}_\mu - \tilde{L}_\mu^2}}{X_\mu}\, dx_\mu$$

Separability of the wave equation

Wave equation

$$\Box\,\phi = 0 \qquad\qquad \Box = -\boldsymbol{g}^{ab}\boldsymbol{\nabla}_a\boldsymbol{\nabla}_b$$

Operators corresponding to conserved quantities

$$\mathcal{K}_j = -\boldsymbol{\nabla}_a\,\boldsymbol{k}_{(j)}^{ab}\,\boldsymbol{\nabla}_b \qquad\qquad \mathcal{L}_j = -i\,\boldsymbol{l}_{(j)}^{a}\boldsymbol{\nabla}_a$$

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Commutativity of the symmetry operators

$$\big[\mathcal{K}_k, \mathcal{K}_l\big] = 0 \qquad \big[\mathcal{K}_k, \mathcal{L}_l\big] = 0 \qquad \big[\mathcal{L}_k, \mathcal{L}_l\big] = 0$$

Common eigenvalue problem

$$\mathcal{K}_j\phi = K_j\phi \qquad\qquad \mathcal{L}_j\phi = L_j\phi$$

Separability of the wave equation

Operators corresponding to conserved quantities

$$\mathcal{K}_j = -\boldsymbol{\nabla}_a \, \boldsymbol{k}_{(j)}^{ab} \, \boldsymbol{\nabla}_b \qquad \qquad \mathcal{L}_j = -i \, \boldsymbol{l}_{(j)}^{a} \boldsymbol{\nabla}_a$$

Common eigenvalue problem

$$\mathcal{K}_j \phi = K_j \phi \qquad \qquad \mathcal{L}_j \phi = L_j \phi$$

Multiplicative separability ansatz

$$\phi = \prod_{\mu} R_\mu \prod_{k=0}^{N-1+\varepsilon} \exp\big(i L_k \psi_k\big) \qquad \qquad R_\mu = R_\mu(\, x_\mu \,;\; K_j,\; L_j)$$

Separability of the wave equation

Operators corresponding to conserved quantities

$$\mathcal{K}_j = -\boldsymbol{\nabla}_a\, \boldsymbol{k}^{ab}_{(j)}\, \boldsymbol{\nabla}_b \qquad\qquad \mathcal{L}_j = -i\, \boldsymbol{l}^{a}_{(j)} \boldsymbol{\nabla}_a$$

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Separated ordinary differential equations

$$\Bigl(X_\mu R'_\mu\Bigr)' + \left(\frac{\tilde{K}_\mu}{X_\mu} - \frac{\tilde{L}^2_\mu}{X^2_\mu}\right) R'_\mu = 0$$

$$\tilde{K}_\mu = \sum_j K_j (-x^2_\mu)^{N-1-j}$$
$$\tilde{L}_\mu = \sum_j L_j (-x^2_\mu)^{N-1-j}$$

Summary

- Explicit and hidden symmetries $\Leftrightarrow$ Killing vectors and Killing tensors
- Killing tensors can be build from conformal Killing–Yano forms
- Principal tensor $\Leftrightarrow$ non-degenerate closed conformal Killing–Yano 2-form
- Killing tower $\Rightarrow$ a sequence of Killing tensors and vectors
- Uniqueness of the geometry compatible with the principal tensor
- Kerr–NUT–(A)dS metric
- Rich symmetry structure $\Rightarrow$ Integrability and Separability

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for details see

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Black holes, hidden symmetries, and complete integrability

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Thank you!