Hidden symmetries of higher-dimensional black holes

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Va	aleri P. Frolov ¹ · Pavel Krtouš ² © · David Kubizňák ³
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Al at th og	tracted vast interest. Perhaps one of the most surprising discoveries is a realization at the properties of higher-dimensional black holes with the spherical horizon topol- zy and described by the Kerr–NUT–(A)dS metrics are very similar to the properties f the well known four-dimensional Kerr metric. This remarkable result stems from
th ex sp se be fo cia se al: pc	the existence of a single object called the principal tensor. In our review we discuss cylicit and hidden symmetries of higher-dimensional Kern–NUT–(A)dS black hole bacetimes. We start with discussion of the Killing and Killing–Yano objects repre- nting explicit and hidden symmetries. We demonstrate that the principal tensor can the used as a "seed object" which generates all these symmetries. It determines the serm of the geometry, as well as guarantees its remarkable properties, such as spe- al algebraic type of the spacetime, complete integrability of geodesic motion, and parability of the Hamilton–Jacobi, Klein–Gordon, and Dirac equations. The review so contains a discussion of different applications of the developed formalism and its ssible generalizations.
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Black holes, hidden symmetries, and complete integral	oility
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Symmetries

Symmetries on configuration space

configuration space M = space of positions

- 1-parameter family of diffeomorphisms on M
- relevant quantities unchanged by the symmetry

generated by vector field \boldsymbol{l} on M $\pounds_{\boldsymbol{l}}\boldsymbol{g} = 0$ $\pounds_{\boldsymbol{l}}\boldsymbol{A} = 0$ $\pounds_{\boldsymbol{l}}V = 0$

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Symmetries on phase space

phase space Γ = space of positions and momenta

- 1-parameter family of diffeomorphisms on Γ
- symplectomorphism: $\pounds_{\mathbf{Z}} \mathbf{\Omega} = 0$
- Hamiltonian unchanged by the symmetry:
- F is a conserved quantity

generated by vector field \mathbf{Z} on Γ generator is Hamiltonian flow $Z = \mathbf{X}_F$ $\pounds_{\mathbf{Z}}H = 0 \quad \Leftrightarrow \quad \{F, H\} = 0$ $\pounds_{\mathbf{X}_H}F = 0$

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V = 0

Symmetry is generated by

a conserved quantity

Symmetries of geodesic motion

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- generator of the symmetry \boldsymbol{l}
- metric conserved by the symmetry isometry $\pounds_l g = 0$

Symmetries on phase space

phase space Γ = space of positions and momenta

- generator of the symmetry \boldsymbol{X}_K
- $\{K,H\}=0$
- Hamiltonian given by the metric

$$H = rac{1}{2m} \boldsymbol{g}^{ab} \, \boldsymbol{p}_a \boldsymbol{p}_b$$

Symmetries of geodesic motion

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\Rightarrow Explicit symmetries

conserved quantity linear in momentum \boldsymbol{p}

$$L = l^a p_a$$

l – Killing vector

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- $\{K,H\}=0$
- Hamiltonian given by the metric

 $H = \frac{1}{2m} \boldsymbol{g}^{ab} \boldsymbol{p}_a \boldsymbol{p}_b$

\Rightarrow Hidden symmetries

conserved quantity monomial in momentum \boldsymbol{p} $K = \boldsymbol{k}^{a_1 \dots a_p} \boldsymbol{p}_{a_1} \cdots \boldsymbol{p}_{a_p}$ $\boldsymbol{k} - \textbf{Killing tensor}$

Killing vectors

l is a Killing vector iff

$$\pounds_{\boldsymbol{l}} \boldsymbol{g} = 0 \quad \Leftrightarrow \quad \boldsymbol{\nabla}^{(a} \boldsymbol{l}^{b)} = 0$$

Killing tensors

 \boldsymbol{k} is a Killing tensor of rank p iff

$$\boldsymbol{k}^{a_1...a_p} = \boldsymbol{k}^{(a_1...a_p)}$$
$$\boldsymbol{\nabla}^{(a_0} \boldsymbol{k}^{a_1...a_p)} = 0$$

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• Killing vector is a Killing tensor of rank 1

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- Killing vector is a Killing tensor of rank 1
- Existence of Killing tensors is highly non-trivial restriction on the geometry

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$$\pounds_{l} g = 0 \quad \Leftrightarrow \quad \nabla^{(a} l^{b)} = 0$$

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- Killing tensors can be build from more elementary building blocks

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- Existence of Killing tensors is highly non-trivial restriction on the geometry
- Killing tensors can be build from more elementary building blocks

Conformal Killing–Yano forms

Pavel Krtouš, Charles University

Splitting covariant derivative of a form

$$oldsymbol{
abla} oldsymbol{\omega} = \mathcal{A}oldsymbol{
abla} oldsymbol{\omega} + \mathcal{C}oldsymbol{
abla} oldsymbol{\omega} + \mathcal{T}oldsymbol{
abla} oldsymbol{\omega}$$

$$(\mathcal{A}\sigma)_{aa_1\dots a_p} = \sigma_{[aa_1\dots a_p]}$$
$$(\mathcal{C}\sigma)_{aa_1\dots a_p} = \frac{p}{D-p+1}g_{a[a_1}\sigma^a{}_{|a|a_2\dots a_p]}$$
$$(\mathcal{T}\sigma)_{aa_1\dots a_p} = \sigma_{aa_1\dots a_p} - \sigma_{[aa_1\dots a_p]} - \frac{p}{D-p+1}g_{a[a_1}\sigma^a{}_{|a|a_2\dots a_p]}$$

. . .

Splitting covariant derivative of a form

 $abla \omega = \mathcal{A} \nabla \omega + \mathcal{C} \nabla \omega + \mathcal{T} \nabla \omega$

$d\omega$	antisymmetric part
$\delta \omega$	divergence part
$\mathrm{T}\omega$	twistor operator

$$(\mathcal{A}\sigma)_{aa_1\dots a_p} = \sigma_{[aa_1\dots a_p]}$$
$$(\mathcal{C}\sigma)_{aa_1\dots a_p} = \frac{p}{D-p+1}g_{a[a_1}\sigma^a{}_{|a|a_2\dots a_p]}$$
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abla} oldsymbol{\omega} = \mathcal{A} oldsymbol{
abla} oldsymbol{\omega} + \mathcal{C} oldsymbol{
abla} oldsymbol{\omega} + \mathcal{T} oldsymbol{
abla} oldsymbol{\omega}$$
 $oldsymbol{d} oldsymbol{\omega} oldsymbol{\Delta} oldsymbol{\omega} oldsymbol{ abla} oldsymbol{ abla}$

$$(\mathcal{A}\sigma)_{aa_1\dots a_p} = \sigma_{[aa_1\dots a_p]}$$
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General form	$oldsymbol{ abla} oldsymbol{ abla} oldsymbol{\omega} = \mathcal{A} oldsymbol{ abla} oldsymbol{\omega} + \mathcal{C} oldsymbol{ abla} oldsymbol{\omega} + \mathcal{T} oldsymbol{ abla} oldsymbol{\omega}$	
Closed form	$oldsymbol{ abla} oldsymbol{arphi} oldsymbol{\omega} = \mathcal{C}oldsymbol{ abla}oldsymbol{\omega} + \mathcal{T}oldsymbol{ abla}oldsymbol{\omega}$	$d\omega = 0$
Divergence-free co-closed form	$oldsymbol{ abla} oldsymbol{ abla} oldsymbol{\omega} = \mathcal{A} oldsymbol{ abla} oldsymbol{\omega} + \mathcal{T} oldsymbol{ abla} oldsymbol{\omega}$	$\boldsymbol{\delta \omega} = 0$
Conformal Killing–Yano form	$oldsymbol{ abla} oldsymbol{\omega} = \mathcal{A}oldsymbol{ abla} oldsymbol{\omega} + \mathcal{C}oldsymbol{ abla} oldsymbol{\omega}$	$\mathbf{T}\boldsymbol{\omega}=0$
Killing–Yano form	$oldsymbol{ abla} oldsymbol{\omega} = \mathcal{A}oldsymbol{ abla} oldsymbol{\omega}$	$\mathbf{T}\boldsymbol{\omega}=0,\boldsymbol{\delta}\boldsymbol{\omega}=0$
Closed conformal Killing–Yano form	$oldsymbol{ abla} oldsymbol{\omega} = \mathcal{C}oldsymbol{ abla}oldsymbol{\omega}$	$\mathbf{T}\boldsymbol{\omega}=0,\boldsymbol{d}\boldsymbol{\omega}=0$
Harmonic form	$oldsymbol{ abla} oldsymbol{\omega} = \mathcal{T}oldsymbol{ abla}oldsymbol{\omega}$	$\boldsymbol{d\omega}=0,\;\boldsymbol{\delta\omega}=0$
Covariantly constant form	$\nabla \boldsymbol{\omega} = 0$	$\boldsymbol{d\omega}=0,\;\boldsymbol{\delta\omega}=0,\;\mathbf{T}\boldsymbol{\omega}=0$

form ω is a *conformal Killing–Yano form* iff for any vector **X** there exist forms κ and ξ such that

$$abla_X\,\omega = X\cdot\kappa + X\wedge oldsymbol{\xi}$$

$$\boldsymbol{\kappa} = rac{1}{p+1} \, \boldsymbol{\nabla} \wedge \boldsymbol{\omega} \qquad \qquad \boldsymbol{\xi} = rac{1}{D-p+1} \, \boldsymbol{\nabla} \cdot \boldsymbol{\omega}$$

Killing–Yano forms

Closed conformal Killing–Yano forms

 $abla_X \, oldsymbol{f} = oldsymbol{X} \cdot oldsymbol{\kappa} \qquad
abla_X \, oldsymbol{h} = oldsymbol{X} \wedge oldsymbol{\xi}$

$$oldsymbol{
alpha}_a oldsymbol{f}_{a_1...a_p} = oldsymbol{
alpha}_{[a} oldsymbol{f}_{a_1...a_p]} = rac{p}{D-p-1} oldsymbol{g}_{a[a_1} oldsymbol{
alpha}^n oldsymbol{h}_{[n|a_2...a_p]}$$

Basic properties of conformal Killing–Yano forms

Killing–Yano forms

Closed conformal Killing–Yano forms



$$\boldsymbol{k}^{ab} = \boldsymbol{f}_1{}^{(a}{}_{c_2...c_p} \, \boldsymbol{f}_2{}^{b)c_2...c_p}$$

is Killing tensor

 $oldsymbol{h}=oldsymbol{h}_1\wedgeoldsymbol{h}_2$

is closed conformal Killing–Yano form

Basic properties of conformal Killing–Yano forms

Killing–Yano forms

Closed conformal Killing–Yano forms



$$oldsymbol{k}^{ab}=oldsymbol{f}_1{}^{(a}{}_{c_2...c_p}oldsymbol{f}_2{}^{b)c_2...c_p}$$

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is closed conformal Killing–Yano form

Principal tensor

Principal tensor h is non-degenerate closed conformal Killing–Yano 2-form

$$\boldsymbol{
abla}_{c} \, \boldsymbol{h}_{ab} = \boldsymbol{g}_{ca} \, \boldsymbol{\xi}_{b} - \boldsymbol{g}_{cb} \, \boldsymbol{\xi}_{a} \qquad \qquad \boldsymbol{\xi}_{a} = rac{1}{D-1} \boldsymbol{
abla}^{b} \boldsymbol{h}_{ba}$$

Darboux frame (e^{μ}, \hat{e}^{μ})

non-degeneracy: x_{μ} are functionally independent functions

$$oldsymbol{h} = \sum_{\mu=1}^N x_\mu \, oldsymbol{e}^\mu \wedge \hat{oldsymbol{e}}^\mu
onumber \ oldsymbol{g} = \sum_{\mu=1}^N ig(oldsymbol{e}^\mu oldsymbol{e}^\mu + \hat{oldsymbol{e}}^\mu \hat{oldsymbol{e}}^\mu ig)$$

 $dx_{\mu} \propto e^{\mu}$ $\hat{e}_{\mu} \cdot dx_{\mu} = 0$ even dimension D = 2N $\mu = 1, \dots, N$

Principal tensor

Principal tensor h is non-degenerate closed conformal Killing–Yano 2-form

$$\boldsymbol{\nabla}_{c} \boldsymbol{h}_{ab} = \boldsymbol{g}_{ca} \boldsymbol{\xi}_{b} - \boldsymbol{g}_{cb} \boldsymbol{\xi}_{a}$$
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Darboux frame (e^{μ}, \hat{e}^{μ})

non-degeneracy:

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Primary Killing vector

Principal tensor h

$$\boldsymbol{\nabla}_{c} \boldsymbol{h}_{ab} = \boldsymbol{g}_{ca} \boldsymbol{\xi}_{b} - \boldsymbol{g}_{cb} \boldsymbol{\xi}_{a}$$

Primary Killing vector $\boldsymbol{\xi}$

$$\boldsymbol{\xi}_a = rac{1}{D-1} \boldsymbol{
abla}^b \boldsymbol{h}_{ba}$$

 $\pounds_{\boldsymbol{\xi}} \, \boldsymbol{g} = 0$
 $\boldsymbol{\xi}$ is a Killing vector

↑

a highly non-trivial consequence of integrability conditions for the principal tensor equation

- $\pounds_{\pmb{\xi}} \, \pmb{h} = 0$
- $\pmb{\xi}$ preserves the principal tensor

↑

a direct consequence of the principal tensor equation

• Closed conformal Killing–Yano forms $h^{(j)}$ of rank 2j:

$$oldsymbol{h}^{(j)} = rac{1}{j!} oldsymbol{h}^{\wedge j}$$

• Killing–Yano forms
$$f^{(j)}$$
 of rank $(D-2j)$:

$$oldsymbol{f}^{(j)} = *oldsymbol{h}^{(j)}$$

• Rank-2 Killing tensors $k_{(j)}$:

$$\boldsymbol{k}_{(j)}^{ab} = \frac{1}{(D-2j-1)!} \, \boldsymbol{f}^{(j)a}{}_{c_1...c_{D-2j-1}} \, \boldsymbol{f}^{(j)bc_1...c_{D-2j-1}}$$

$$oldsymbol{l}_{(j)}=oldsymbol{k}_{(j)}oldsymbol{\cdot}oldsymbol{\xi}$$

• Closed conformal Killing–Yano forms $h^{(j)}$ of rank 2j:

$$oldsymbol{h}^{(j)}=rac{1}{j!}oldsymbol{h}^{\wedge j}$$

- Killing–Yano forms $f^{(j)}$ of rank (D 2j): $f^{(j)} = *h^{(j)}$
- Rank-2 Killing tensors $k_{(j)}$:

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• Closed conformal Killing–Yano forms $h^{(j)}$ of rank 2j:

$$\boldsymbol{h}^{(j)} = rac{1}{j!} \boldsymbol{h}^{\wedge j}$$

• Killing–Yano forms $f^{(j)}$ of rank (D-2j):

$$oldsymbol{f}^{(j)}=*oldsymbol{h}^{(j)}$$

• Rank-2 Killing tensors $k_{(j)}$:

Hidden symmetries

$$\boldsymbol{k}_{(j)}^{ab} = \frac{1}{(D-2j-1)!} \, \boldsymbol{f}^{(j)a}{}_{c_1...c_{D-2j-1}} \, \boldsymbol{f}^{(j)bc_1...c_{D-2j-1}}$$

• Killing vectors $\boldsymbol{l}_{(j)}$:

Explicit symmetries

$$oldsymbol{l}_{(j)} = oldsymbol{k}_{(j)} \cdot oldsymbol{\xi}$$

- Rank-2 Killing tensors $\boldsymbol{k}_{(j)}$
- Killing vectors $\boldsymbol{l}_{(j)}$

Hidden symmetries Explicit symmetries

$$j=0,\ldots,N-1$$

Symmetries Nijenhuis–Schouten commute

$$\left[\boldsymbol{k}_{(i)},\boldsymbol{k}_{(j)}\right]_{\rm NS} = 0 \qquad \left[\boldsymbol{k}_{(i)},\boldsymbol{l}_{(j)}\right]_{\rm NS} = 0 \qquad \left[\boldsymbol{l}_{(i)},\boldsymbol{l}_{(j)}\right]_{\rm NS} = 0$$

↑

a non-trivial consequence of the Killing tower definition, of the principal tensor equation and its integrability conditions

Principal tensor geometry

- eigenvalues of the principal tensor h \Rightarrow coordinates x_{μ}
- Killing vectors $l_{(j)}$ commute and tangent to $x_{\mu} = \text{const} \implies \text{Killing coordinates } \psi_j$

Uniqueness of the geometry

it is possible to reconstruct the metric up to N free metric functions X_{μ}

these functions are determined by the Einstein equations

Off-shell Kerr–NUT–(A)dS geometry

for simplicity even dimension D = 2N

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_{\mu}}{X_{\mu}} \boldsymbol{d} x_{\mu}^2 + \frac{X_{\mu}}{U_{\mu}} \left(\sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d} \psi_j \right)^2 \right]$$

explicit function polynomial in coordinates x_{μ} (symmetric polynomials)

$$A^{(k)} = \sum_{\substack{\nu_1, \dots, \nu_k = 1 \\ \nu_1 < \dots < \nu_k}}^N x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad A^{(j)}_{\mu} = \sum_{\substack{\nu_1, \dots, \nu_j = 1 \\ \nu_1 < \dots < \nu_j \\ \nu_i \neq \mu}}^N x_{\nu_1}^2 \dots x_{\nu_j}^2 \qquad U_{\mu} = \prod_{\substack{\nu = 1 \\ \nu \neq \mu}}^N (x_{\nu}^2 - x_{\mu}^2)$$

unspecified N metric functions of one variable

$$X_{\mu} = X_{\mu}(x_{\mu})$$

Off-shell Kerr–NUT–(A)dS geometry

for simplicity even dimension D = 2N

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_{\mu}}{X_{\mu}} \boldsymbol{d} x_{\mu}^{2} + \frac{X_{\mu}}{U_{\mu}} \left(\sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d} \psi_{j} \right)^{2} \right]$$

Darboux frame

forms:

vectors:

$$\boldsymbol{e}^{\mu} = \left(\frac{U_{\mu}}{X_{\mu}}\right)^{\frac{1}{2}} \boldsymbol{d}x_{\mu} \qquad \hat{\boldsymbol{e}}^{\mu} = \left(\frac{X_{\mu}}{U_{\mu}}\right)^{\frac{1}{2}} \sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d}\psi_{j} \qquad \qquad \boldsymbol{e}_{\mu} = \left(\frac{X_{\mu}}{U_{\mu}}\right)^{\frac{1}{2}} \boldsymbol{\partial}_{x_{\mu}} \qquad \hat{\boldsymbol{e}}_{\mu} = \left(\frac{U_{\mu}}{X_{\mu}}\right)^{\frac{1}{2}} \sum_{k=0}^{N-1} \frac{(-x_{\mu}^{2})^{N-1-k}}{U_{\mu}} \boldsymbol{\partial}_{\psi_{k}}$$

Off-shell Kerr–NUT–(A)dS geometry

for simplicity even dimension D = 2N

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_{\mu}}{X_{\mu}} \boldsymbol{d} x_{\mu}^2 + \frac{X_{\mu}}{U_{\mu}} \left(\sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d} \psi_j \right)^2 \right]$$

Darboux frame

forms:

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$$\boldsymbol{e}^{\mu} = \left(\frac{U_{\mu}}{X_{\mu}}\right)^{\frac{1}{2}} \boldsymbol{d}x_{\mu} \qquad \hat{\boldsymbol{e}}^{\mu} = \left(\frac{X_{\mu}}{U_{\mu}}\right)^{\frac{1}{2}} \sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d}\psi_{j} \qquad \qquad \boldsymbol{e}_{\mu} = \left(\frac{X_{\mu}}{U_{\mu}}\right)^{\frac{1}{2}} \boldsymbol{\partial}_{x_{\mu}} \qquad \hat{\boldsymbol{e}}_{\mu} = \left(\frac{U_{\mu}}{X_{\mu}}\right)^{\frac{1}{2}} \sum_{k=0}^{N-1} \frac{(-x_{\mu}^{2})^{N-1-k}}{U_{\mu}} \boldsymbol{\partial}_{\psi_{k}}$$

Curvature

$$\begin{split} \mathbf{Ric} &= -\sum_{\mu=1}^{N} r_{\mu} \left(\boldsymbol{e}^{\mu} \boldsymbol{e}^{\mu} + \hat{\boldsymbol{e}}^{\mu} \hat{\boldsymbol{e}}^{\mu} \right) \\ R &= -\sum_{\nu=1}^{N} \frac{X_{\nu}''}{U_{\nu}} \end{split}$$
On-shell Kerr–NUT–(A)dS geometry

for simplicity even dimension D = 2N

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_{\mu}}{X_{\mu}} \boldsymbol{d} x_{\mu}^{2} + \frac{X_{\mu}}{U_{\mu}} \left(\sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d} \psi_{j} \right)^{2} \right]$$

Einstein equations \Rightarrow

$$X_{\mu} = \lambda \prod_{\nu=1}^{N} (a_{\nu}^{2} - x_{\mu}^{2}) - 2b_{\mu} x_{\mu} = \lambda \mathcal{J}(x_{\mu}^{2}) - 2b_{\mu} x_{\mu}$$

Parameters:

 λ

 b_{μ}

cosmological parameter related to the cosmological constant $\Lambda = (2N - 1)(N - 1)\lambda$ mass and NUT parameters

 a_{μ} rotational parameters

freedom in scaling of coordinates \Rightarrow one parameter can be fixed by a gauge condition exact interpretation of parameters depends on coordinate ranges, signature, and gauge choices

On-shell Kerr–NUT–(A)dS geometry

for simplicity even dimension D = 2N

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_{\mu}}{X_{\mu}} \boldsymbol{d} x_{\mu}^2 + \frac{X_{\mu}}{U_{\mu}} \left(\sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d} \psi_j \right)^2 \right]$$

Coordinate ranges:

• x_{μ} between roots of X_{μ}

 $X_{\mu} = \lambda \, \mathcal{J}(x_{\mu}^2) - 2b_{\mu}x_{\mu}$



On-shell Kerr–NUT–(A)dS geometry

for simplicity even dimension D = 2N

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_{\mu}}{X_{\mu}} \boldsymbol{d} x_{\mu}^{2} + \frac{X_{\mu}}{U_{\mu}} \left(\sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d} \psi_{j} \right)^{2} \right]$$

Coordinate ranges:

• x_{μ} between roots of X_{μ} [for $b_{\mu} = 0$ roots roots of X_{μ} are $a_{\mu} \Rightarrow x_{\mu} \in (a_{\mu-1}, a_{\mu})$]

• ψ_j – Killing coordinates ϕ_{μ} – any liner combination of ψ_j form also Killing coordinates

a periodic character of properly chosen coordinates ϕ_{μ} guarantees a regularity of rotation axes

regularity of whole rotational axes can be achieved only for vanishing NUT parameters

Kerr–NUT–(A)dS geometry – Lorentzian signature

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_{\mu}}{X_{\mu}} \boldsymbol{d} x_{\mu}^2 + \frac{X_{\mu}}{U_{\mu}} \left(\sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d} \psi_j \right)^2 \right]$$

Wick rotation:

time: $\tau = \psi_0$ radial coordinate: $x_N = ir$ mass: $b_N = im$

Gauge condition:

$$a_N^2 = -\frac{1}{\lambda}$$
 (suitable for limit $\lambda \to 0$)

New Killing coordinates:

$$t = \tau + \sum_{\bar{k}} \bar{\mathcal{A}}^{(\bar{k}+1)} \bar{\psi}_{\bar{k}} \qquad \qquad \frac{\phi_{\bar{\mu}}}{a_{\bar{\mu}}} = \lambda \tau - \sum_{\bar{k}} \left(\bar{\mathcal{A}}^{(\bar{k})}_{\bar{\mu}} - \lambda \bar{\mathcal{A}}^{(\bar{k}+1)}_{\bar{\mu}} \right) \bar{\psi}_{\bar{k}}$$

barred quantities refer to ranges of indices given by $N \to \overline{N} = N - 1$

(i.e.,
$$\bar{\mu} = 1, ..., \bar{N}$$
 and $\bar{j} = 0, ..., \bar{N} - 1$, etc.)

Kerr–NUT–(A)dS geometry – Lorentzian signature

$$\begin{split} \boldsymbol{g} &= -\frac{\Delta_r}{\Sigma} \left(\prod_{\bar{\nu}} \frac{1 + \lambda x_{\bar{\nu}}^2}{1 + \lambda a_{\bar{\nu}}^2} \, \boldsymbol{d}t - \sum_{\bar{\nu}} \frac{\bar{J}(a_{\bar{\nu}}^2)}{a_{\bar{\nu}}(1 + \lambda a_{\bar{\nu}}^2) \bar{\mathcal{U}}_{\bar{\nu}}} \boldsymbol{d}\phi_{\bar{\nu}} \right)^2 + \frac{\Sigma}{\Delta_r} \, \boldsymbol{d}r^2 \\ &+ \sum_{\bar{\mu}} \frac{(r^2 + x_{\bar{\mu}}^2)}{\Delta_{\bar{\mu}}/\bar{\mathcal{U}}_{\bar{\mu}}} \, \boldsymbol{d}x_{\bar{\mu}}^2 + \sum_{\bar{\mu}} \frac{\Delta_{\bar{\mu}}/\bar{\mathcal{U}}_{\bar{\mu}}}{(r^2 + x_{\bar{\mu}}^2)} \left(\frac{1 - \lambda r^2}{1 + \lambda x_{\bar{\mu}}^2} \prod_{\bar{\nu}} \frac{1 + \lambda x_{\bar{\nu}}^2}{1 + \lambda a_{\bar{\nu}}^2} \, \boldsymbol{d}t + \sum_{\bar{\nu}} \frac{(r^2 + a_{\bar{\nu}}^2) \bar{\mathcal{J}}_{\bar{\mu}}(a_{\bar{\nu}}^2)}{a_{\bar{\nu}}(1 + \lambda a_{\bar{\nu}}^2) \bar{\mathcal{U}}_{\bar{\nu}}} \boldsymbol{d}\phi_{\bar{\nu}} \right)^2 \end{split}$$

$$\Delta_{r} = -X_{N} = (1 - \lambda r^{2}) \prod_{\bar{\nu}} (r^{2} + a_{\bar{\nu}}^{2}) - 2mr \qquad \Sigma = U_{N} = \prod_{\bar{\nu}} (r^{2} + x_{\bar{\nu}}^{2})$$
$$\Delta_{\bar{\mu}} = -X_{\bar{\mu}} = (1 + \lambda x_{\bar{\mu}}^{2}) \bar{\mathcal{J}}(x_{\bar{\mu}}^{2}) + 2b_{\bar{\mu}} x_{\bar{\mu}} \qquad \bar{U}_{\bar{\mu}} = \prod_{\bar{\nu} \neq \bar{\mu}} (x_{\bar{\nu}}^{2} - x_{\bar{\mu}}^{2})$$

Pavel Krtouš, Charles University

Kerr–NUT–(A)dS geometry – Lorentzian signature

$$\begin{split} \boldsymbol{g} &= -\frac{\Delta_r}{\Sigma} \left(\prod_{\bar{\nu}} \frac{1 + \lambda x_{\bar{\nu}}^2}{1 + \lambda a_{\bar{\nu}}^2} \, \boldsymbol{d}t - \sum_{\bar{\nu}} \frac{\bar{J}(a_{\bar{\nu}}^2)}{a_{\bar{\nu}}(1 + \lambda a_{\bar{\nu}}^2) \bar{\mathcal{U}}_{\bar{\nu}}} \boldsymbol{d}\phi_{\bar{\nu}} \right)^2 + \frac{\Sigma}{\Delta_r} \, \boldsymbol{d}r^2 \\ &+ \sum_{\bar{\mu}} \frac{(r^2 + x_{\bar{\mu}}^2)}{\Delta_{\bar{\mu}}/\bar{U}_{\bar{\mu}}} \, \boldsymbol{d}x_{\bar{\mu}}^2 + \sum_{\bar{\mu}} \frac{\Delta_{\bar{\mu}}/\bar{U}_{\bar{\mu}}}{(r^2 + x_{\bar{\mu}}^2)} \left(\frac{1 - \lambda r^2}{1 + \lambda x_{\bar{\mu}}^2} \prod_{\bar{\nu}} \frac{1 + \lambda x_{\bar{\nu}}^2}{1 + \lambda a_{\bar{\nu}}^2} \, \boldsymbol{d}t + \sum_{\bar{\nu}} \frac{(r^2 + a_{\bar{\nu}}^2) \bar{J}_{\bar{\mu}}(a_{\bar{\nu}}^2)}{a_{\bar{\nu}}(1 + \lambda a_{\bar{\nu}}^2) \, \bar{\mathcal{U}}_{\bar{\nu}}} \boldsymbol{d}\phi_{\bar{\nu}} \right)^2 \end{split}$$

Horizon function:

 $\Delta_r = \left(1 - \lambda r^2\right) \prod_{\bar{\nu}} \left(r^2 + a_{\bar{\nu}}^2\right) - 2mr$



Kerr–NUT geometry:

$$\Lambda = 0$$

$$\begin{split} \boldsymbol{g} &= -\frac{\Delta_r}{\Sigma} \Big(\boldsymbol{d}\tau + \sum_{\bar{k}} \bar{A}^{(\bar{k}+1)} \boldsymbol{d}\bar{\psi}_{\bar{k}} \Big)^2 + \frac{\Sigma}{\Delta_r} \boldsymbol{d}r^2 \\ &+ \sum_{\bar{\mu}} \frac{(r^2 + x_{\bar{\mu}}^2)}{\Delta_{\bar{\mu}}/\bar{U}_{\bar{\mu}}} \boldsymbol{d}x_{\bar{\mu}}^2 + \sum_{\bar{\mu}} \frac{\Delta_{\bar{\mu}}/\bar{U}_{\bar{\mu}}}{(r^2 + x_{\bar{\mu}}^2)} \Big(\boldsymbol{d}\tau + \sum_{\bar{k}} \big(\bar{A}^{(\bar{k}+1)}_{\bar{\mu}} - r^2 \bar{A}^{(\bar{k})}_{\bar{\mu}} \big) \boldsymbol{d}\bar{\psi}_{\bar{k}} \Big)^2 \end{split}$$

$$\Delta_{r} = -X_{N} = \prod_{\bar{\nu}} \left(r^{2} + a_{\bar{\nu}}^{2} \right) - 2mr \qquad \Sigma = U_{N} = \prod_{\bar{\nu}} \left(r^{2} + x_{\bar{\nu}}^{2} \right)$$
$$\Delta_{\bar{\mu}} = -X_{\bar{\mu}} = \bar{\mathcal{J}}(x_{\bar{\mu}}^{2}) + 2b_{\bar{\mu}}x_{\bar{\mu}} \qquad \bar{U}_{\bar{\mu}} = \prod_{\bar{\nu}\neq\bar{\mu}} \left(x_{\bar{\nu}}^{2} - x_{\bar{\mu}}^{2} \right)$$

Myers–Perry metric:

$$\Lambda = 0 \quad b_{\bar{\mu}} = 0$$

$$\boldsymbol{g} = -\boldsymbol{d}t^2 + \frac{2mr}{\Sigma} \Big(\boldsymbol{d}t + \sum_{\bar{\nu}} a_{\bar{\nu}} \mu_{\bar{\nu}}^2 \boldsymbol{d}\phi_{\bar{\nu}} \Big)^2 + \frac{\Sigma}{\Delta_r} \boldsymbol{d}r^2 + r^2 \boldsymbol{d}\mu_0^2 + \sum_{\bar{\nu}} (r^2 + a_{\bar{\nu}}^2) \Big(\boldsymbol{d}\mu_{\bar{\nu}}^2 + \mu_{\bar{\nu}}^2 \boldsymbol{d}\phi_{\bar{\nu}}^2 \Big)$$

$$\Delta_r = \prod_{\bar{\nu}} (r^2 + a_{\bar{\nu}}^2) - 2mr \qquad \Sigma = \left(\mu_0^2 + \sum_{\bar{\nu}} \frac{r^2 \mu_{\bar{\nu}}^2}{r^2 + a_{\bar{\nu}}^2}\right) \prod_{\bar{\mu}} (r^2 + a_{\bar{\mu}}^2) ,$$

new latitudinal coordinates:

 $\bar{N} \text{ coordinates } x_1, \dots, x_{\bar{N}} \longrightarrow N+1 \text{ constrained coordinates } \mu_0, \mu_1, \dots, \mu_{\bar{N}}$ $\mu_{\bar{\nu}}^2 = \frac{\bar{J}(a_{\bar{\nu}}^2)}{-a_{\bar{\nu}}^2 \bar{\mathcal{U}}_{\bar{\nu}}} = \frac{\prod_{\bar{\alpha}} (x_{\bar{\alpha}}^2 - a_{\bar{\nu}}^2)}{-a_{\bar{\nu}}^2 \prod_{\bar{\alpha} \neq \bar{\nu}} (a_{\bar{\alpha}}^2 - a_{\bar{\nu}}^2)} , \quad \mu_0^2 = \frac{\bar{\mathcal{A}}^{(\bar{N})}}{\bar{\mathcal{A}}^{(\bar{N})}} = \frac{\prod_{\bar{\alpha}} x_{\bar{\alpha}}^2}{\prod_{\bar{\alpha}} a_{\bar{\alpha}}^2} ,$

constrain:

$$\sum_{\bar{\nu}=0}^{\bar{N}}\mu_{\bar{\nu}}^2=1$$

Schwarzschild–Tangherlini–(A)dS metric: $b_{\bar{\mu}} = 0$ $a_{\mu} = 0$

$$g = -(1 - \lambda r^2 - 2mr^{3-2N}) dt^2 + \frac{1}{1 - \lambda r^2 - 2mr^{3-2N}} dr^2 + r^2 d\Omega_{\bar{N}}^2$$

- Spherical symmetric black hole of mass m
- Horizons given by roots of the metric function
- Asymptotic character given by λ
- Radial dependence changes with the dimension

Principal tensor geometry = off-shell Kerr-NUT-(A)dS

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_{\mu}}{X_{\mu}} \boldsymbol{d} x_{\mu}^2 + \frac{X_{\mu}}{U_{\mu}} \left(\sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d} \psi_j \right)^2 \right]$$

explicit function polynomial in coordinates x_{μ} (symmetric polynomials)

$$A^{(k)} = \sum_{\substack{\nu_1, \dots, \nu_k = 1\\\nu_1 < \dots < \nu_k}}^N x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad A^{(j)}_{\mu} = \sum_{\substack{\nu_1, \dots, \nu_j = 1\\\nu_1 < \dots < \nu_j\\\nu_i < \dots < \nu_j}}^N x_{\nu_1}^2 \dots x_{\nu_j}^2 \qquad U_{\mu} = \prod_{\substack{\nu = 1\\\nu \neq \mu}}^N (x_{\nu}^2 - x_{\mu}^2)$$

unspecified N metric functions of one variable

$$X_{\mu} = X_{\mu}(x_{\mu})$$

Principal tensor geometry = off-shell Kerr-NUT-(A)dS

$$\begin{split} \boldsymbol{g} &= \sum_{\mu=1}^{N} \left[\frac{U_{\mu}}{X_{\mu}} \, \boldsymbol{d}x_{\mu}^{2} + \frac{X_{\mu}}{U_{\mu}} \left(\sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d}\psi_{j} \right)^{2} \right] \\ & \text{explicit} \\ A^{(k)} &= \left[\sum_{\substack{\nu_{1}, \cdots \\ \nu_{1} < \\ \nu_{2} < \\ \nu_{1} < \\ \nu_{1} < \\ \nu_{2} < \\ \nu_{1} < \\ \nu_{1} < \\ \nu_{2} < \\ \nu_{2} < \\ \nu_{1} < \\ \nu_{2} < \\ \nu_{2} < \\ \nu_{1} < \\ \nu_{2} < \\ \nu_$$

Principal tensor geometry = off-shell Kerr-NUT-(A)dS

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_{\mu}}{X_{\mu}} \boldsymbol{d} x_{\mu}^2 + \frac{X_{\mu}}{U_{\mu}} \left(\sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d} \psi_j \right)^2 \right]$$

explicit function polynomial in coordinates x_{μ} (symmetric polynomials)

$$A^{(k)} = \sum_{\substack{\nu_1, \dots, \nu_k = 1\\\nu_1 < \dots < \nu_k}}^N x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad A^{(j)}_{\mu} = \sum_{\substack{\nu_1, \dots, \nu_j = 1\\\nu_1 < \dots < \nu_j\\\nu_i < \dots < \nu_j}}^N x_{\nu_1}^2 \dots x_{\nu_j}^2 \qquad U_{\mu} = \prod_{\substack{\nu = 1\\\nu \neq \mu}}^N (x_{\nu}^2 - x_{\mu}^2)$$

unspecified N metric functions of one variable

$$X_{\mu} = X_{\mu}(x_{\mu})$$

Non-trivial limits of Kerr–NUT–(A)dS geometry

- Limit of vanishing rotations
- Deformed black hole
- Nutty spacetimes
- Near horizon limits
- Limit of equal rotations

Limit of vanishing rotations

- setting $a_{\bar{\mu}} = 0$ yields degenerate ranges of coordinates
- setting $a_{\bar{\mu}} = 0$ yields degenerate metric
- a suitable scaling of coordinates necessary!

Limit of vanishing rotations

regular sector	unspined sector
$a_{\bar{N}+\tilde{\mu}} = \tilde{a}_{\tilde{\mu}}$	$a_{ar{\mu}} = arepsilon ar{a}_{ar{\mu}}$
$x_{\bar{N}+\tilde{\mu}} = \tilde{x}_{\tilde{\mu}}$	$x_{\bar{\mu}} = \varepsilon \bar{x}_{\bar{\mu}}$
$\phi_{ar{N}+ ilde{\mu}}= ilde{\phi}_{ ilde{\mu}}$	$\phi_{ar{\mu}} = ar{\phi}_{ar{\mu}}$
$ ilde{\mu}=1,\ldots, ilde{N}$	$ar{\mu}=1,\ldots,ar{N}$

 $\varepsilon \to 0$

Limit of vanishing rotations

vanishing rotations in \overline{N} directions

 $a_{\bar{\mu}} \rightarrow 0$ for these directions

regular sectorunspined sector $a_{\bar{N}+\tilde{\mu}} = \tilde{a}_{\tilde{\mu}}$ $a_{\bar{\mu}} = \varepsilon \bar{a}_{\bar{\mu}}$ $x_{\bar{N}+\tilde{\mu}} = \tilde{x}_{\tilde{\mu}}$ $x_{\bar{\mu}} = \varepsilon \bar{x}_{\bar{\mu}}$ $\phi_{\bar{N}+\tilde{\mu}} = \tilde{\phi}_{\tilde{\mu}}$ $\phi_{\bar{\mu}} = \phi_{\bar{\mu}}$ $\tilde{\mu} = 1, \dots, \tilde{N}$ $\bar{\mu} = 1, \dots, \bar{N}$

 $\varepsilon \to 0$

$$\boldsymbol{g} = \tilde{\boldsymbol{g}} + \tilde{w}^2 \bar{\boldsymbol{g}}$$

$$\begin{split} \tilde{\boldsymbol{g}} &= \sum_{\tilde{\mu}} \left[\frac{\tilde{U}_{\tilde{\mu}}}{\tilde{X}_{\tilde{\mu}}} \boldsymbol{d} \tilde{x}_{\tilde{\mu}}^2 + \frac{\tilde{X}_{\tilde{\mu}}}{\tilde{U}_{\tilde{\mu}}} \left(\sum_{\tilde{\nu}} \frac{\tilde{J}_{\tilde{\mu}}(\tilde{a}_{\tilde{\nu}}^2)}{\tilde{\mathcal{U}}_{\tilde{\nu}}} \boldsymbol{d} \phi_{\tilde{\nu}} \right)^2 \right] \qquad \qquad \bar{\boldsymbol{g}} = \sum_{\bar{\mu}} \left[\frac{\bar{U}_{\bar{\mu}}}{\bar{X}_{\bar{\mu}}} \boldsymbol{d} \bar{x}_{\bar{\mu}}^2 + \frac{\bar{X}_{\bar{\mu}}}{\bar{U}_{\bar{\mu}}} \left(\sum_{\bar{\nu}} \frac{\bar{J}_{\bar{\mu}}(\bar{a}_{\tilde{\nu}}^2)}{\bar{\mathcal{U}}_{\bar{\nu}}} \boldsymbol{d} \phi_{\bar{\nu}} \right)^2 \right] \\ \tilde{X}_{\tilde{\mu}} = \lambda \, \tilde{\mathcal{J}}(\tilde{x}_{\tilde{\mu}}^2) - 2 \, \tilde{b}_{\tilde{\mu}} \, \tilde{x}_{\tilde{\mu}}^{1-2\tilde{N}} \qquad \qquad \bar{X}_{\bar{\mu}} = \lambda \, \bar{\mathcal{J}}(\bar{x}_{\bar{\mu}}^2) - 2 \, \bar{b}_{\bar{\mu}} \, \bar{x}_{\bar{\mu}} \end{split}$$

off-shell Kerr–NUT–(A)dS Lorentzian part on-shell Kerr–NUT–(A)dS Euclidian instanton

Deformed black hole

switching-off all rotations by repeating the limiting procedure

$$oldsymbol{g} = -foldsymbol{d}t^2 + rac{1}{f}oldsymbol{d}r^2 + r^2 \Big[oldsymbol{q}_{N-1} + \xi_{N-1}^2 \Big(oldsymbol{q}_{N-2} + \dots + \xi_3^2 \Big(oldsymbol{q}_2 + \xi_2^2 oldsymbol{q}_1\Big)\Big)\Big] \ f = 1 - \lambda r^2 - rac{2m}{r^{2N-3}} \ oldsymbol{q}_{ar{\mu}} = rac{1}{\Delta_{ar{\mu}}}oldsymbol{d}\xi_{ar{\mu}}^2 + \Delta_{ar{\mu}}oldsymbol{d}\phi_{ar{\mu}}^2 \qquad \Delta_{ar{\mu}} = 1 - \xi_{ar{\mu}}^2 - 2\,eta_{ar{\mu}}\,\xi_{ar{\mu}}^{-2ar{\mu}+3}$$

- \bullet non-rotating
- spatially deformed

static deformed black hole

Nutty spacetimes



generalization of Taub–NUT–(A)dS

Nutty spacetimes

generalization of Taub–NUT–(A)dS

tangent points \hat{x}_{μ} are double roots of X_{μ} for suitable values $b_{\mu} = \hat{b}_{\mu}$

 $X_{\mu} = \lambda \mathcal{J}(x_{\bar{\mu}}^2) - 2\hat{b}_{\bar{\mu}}x_{\bar{\mu}}$





Nutty spacetimes

generalization of Taub–NUT–(A)dS

spacetimes with near-critical NUTs x-coordinate ranges near double roots $\hat{x}_{\bar{\mu}}$

$$\begin{split} \boldsymbol{g} &= -\frac{\Delta}{\Sigma} \Big(\boldsymbol{d}t + \sum_{\bar{\mu}} \frac{2\hat{x}_{\bar{\mu}}}{\delta_{\bar{\mu}}} \left(\xi_{\bar{\mu}} - \mathring{\xi}_{\bar{\mu}} \right) \boldsymbol{d}\varphi_{\bar{\mu}} \Big)^2 + \frac{\Sigma}{\Delta} \boldsymbol{d}r^2 + \sum_{\bar{\mu}} \frac{r^2 + \hat{x}_{\bar{\mu}}^2}{\delta_{\bar{\mu}}} \left(\frac{1}{1 - \xi_{\bar{\mu}}^2} \, \boldsymbol{d}\xi_{\bar{\mu}}^2 + \left(1 - \xi_{\bar{\mu}}^2 \right) \boldsymbol{d}\varphi_{\bar{\mu}}^2 \right) \\ \Delta &= -\lambda \mathcal{J}(-r^2) - 2mr \qquad \Sigma = \prod_{\bar{\nu}} (r^2 + \hat{x}_{\bar{\nu}}^2) \qquad \delta_{\mu} = \lambda (2N - 1)(\hat{r}^2 + \hat{x}_{\bar{\mu}}^2) \end{split}$$

Near horizon limits

analogy for radial coordinate

spacetimes with near-critical mass

coordinate range of r near double root \hat{r}

Near horizon limits

analogy for radial coordinate

spacetimes with near-critical mass

coordinate range of r near double root \hat{r}

Limit of equal rotations

enhancement of symmetry in the equal-rotation sector

NS-commutation of the Killing tower

$$\boldsymbol{g} = \sum_{\mu=1}^{N} \left[\frac{U_{\mu}}{X_{\mu}} \boldsymbol{d} x_{\mu}^2 + \frac{X_{\mu}}{U_{\mu}} \left(\sum_{j=0}^{N-1} A_{\mu}^{(j)} \boldsymbol{d} \psi_j \right)^2 \right]$$

Primary Killing vector

 $oldsymbol{\xi} = oldsymbol{\partial}_{\psi_0}$

Hidden symmetries – Killing tensors

 $oldsymbol{h} = \sum_{\mu} x_{\mu} \, oldsymbol{e}^{\mu} \wedge \hat{oldsymbol{e}}^{\mu}$

 $oldsymbol{k}_{(j)} = \sum_{\mu} A^{(j)}_{\mu} \left(oldsymbol{e}_{\mu} oldsymbol{e}_{\mu} + \hat{oldsymbol{e}}_{\mu} \hat{oldsymbol{e}}_{\mu}
ight)$

Explicit symmetries – Killing vectors

 $oldsymbol{l}_{(j)} = oldsymbol{\partial}_{\psi_j}$

$$\left[\boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)}\right]_{\rm NS} = 0 \qquad \left[\boldsymbol{k}_{(i)}, \boldsymbol{l}_{(j)}\right]_{\rm NS} = 0 \qquad \left[\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}\right]_{\rm NS} = 0$$

Darboux frame

forms:

vectors:

$$\boldsymbol{e}^{\mu} = \left(\frac{U_{\mu}}{X_{\mu}}\right)^{\frac{1}{2}} \boldsymbol{d}x_{\mu} \qquad \hat{\boldsymbol{e}}^{\mu} = \left(\frac{X_{\mu}}{U_{\mu}}\right)^{\frac{1}{2}} \sum_{j=0}^{N-1} A^{(j)}_{\mu} \boldsymbol{d}\psi_{j} \qquad \qquad \boldsymbol{e}_{\mu} = \left(\frac{X_{\mu}}{U_{\mu}}\right)^{\frac{1}{2}} \boldsymbol{\partial}_{x_{\mu}} \qquad \hat{\boldsymbol{e}}_{\mu} = \left(\frac{U_{\mu}}{X_{\mu}}\right)^{\frac{1}{2}} \sum_{k=0}^{N-1} \frac{(-x_{\mu}^{2})^{N-1-k}}{U_{\mu}} \boldsymbol{\partial}_{\psi_{k}}$$

Pavel Krtouš, Charles University

Hidden symmetries of higher-dimensional black holes, Prague, December 5, 2017

NS-commutation of the Killing tower 37

Generating functions for Killing objects

Tensors depending on an auxiliary parameter β

$$\begin{aligned} \boldsymbol{q}(\beta) &= \boldsymbol{g} - \beta^2 \, \boldsymbol{h}^2 \\ A(\beta) &= \left(\frac{\operatorname{Det} \boldsymbol{q}(\beta)}{\operatorname{Det} \boldsymbol{g}}\right)^{\frac{1}{2}} & A(\beta) &= \sum_j A^{(j)} \beta^{2j} \\ \boldsymbol{k}(\beta) &= A(\beta) \, \boldsymbol{q}(\beta)^{-1} & \boldsymbol{k}(\beta) &= \sum_j \boldsymbol{k}_{(j)} \beta^{2j} \\ \boldsymbol{l}(\beta) &= \boldsymbol{k}(\beta) \cdot \boldsymbol{\xi} & \boldsymbol{l}(\beta) &= \sum_j \boldsymbol{l}_{(j)} \beta^{2j} \end{aligned}$$

Vanishing NS-brackets

$$\begin{bmatrix} \boldsymbol{l}(\beta_1), \, \boldsymbol{l}(\beta_2) \end{bmatrix}_{\rm NS} = 0$$
$$\begin{bmatrix} \boldsymbol{l}(\beta_1), \, \boldsymbol{k}(\beta_2) \end{bmatrix}_{\rm NS} = 0$$
$$\begin{bmatrix} \boldsymbol{k}(\beta_1), \, \boldsymbol{k}(\beta_2) \end{bmatrix}_{\rm NS} = 0$$
Since $\boldsymbol{k}(0) = \boldsymbol{g}$, it also implies that $\boldsymbol{l}(\beta)$ and $\boldsymbol{k}(\beta)$ are KTs

Proof of vanishing NS-brackets

where
$$k_1 = k(\beta_1), k_2 = k(\beta_2), l_1 = l(\beta_1),$$
 etc.

Integrability conditions for principal CCKY form

$$\nabla_{a}\boldsymbol{h}_{bc} = \boldsymbol{g}_{ab}\boldsymbol{\xi}_{c} - \boldsymbol{g}_{ac}\boldsymbol{\xi}_{b}$$

$$\downarrow \qquad \text{taking second derivative } 2\nabla_{[a}\nabla_{b]}\boldsymbol{h}_{mn}$$

$$\boldsymbol{R}^{ab}_{\ c[m} \boldsymbol{h}^{c}_{\ n]} = 2\delta^{[a}_{[m}\nabla^{b]}\boldsymbol{\xi}_{n]} \qquad (\text{IC1})$$

$$\downarrow \qquad \text{Bianchi identities, contractions}$$

$$(D-2)\nabla_{a}\boldsymbol{\xi}_{b} = -\mathbf{Ric}_{an}\boldsymbol{h}^{n}_{\ b} + \frac{1}{2}\boldsymbol{h}_{mn}\boldsymbol{R}^{mn}_{\ ab} \qquad (\text{IC2})$$

$$\downarrow \qquad \text{substituting (IC2) to (IC1)}$$

$$(D-2)\boldsymbol{R}^{ab}_{\ n[c}\boldsymbol{h}^{n}_{\ d]} - \boldsymbol{h}_{mn}\boldsymbol{R}^{mn[a}_{\ [c}\delta^{b]}_{\ d]} - 2\mathbf{Ric}^{[a}_{\ n}\delta^{b]}_{\ [c}\boldsymbol{h}^{n}_{\ d]} = 0 \qquad (\text{IC3})$$

$$\mathcal{S}(\nabla \boldsymbol{\xi}) = \begin{bmatrix} \boldsymbol{h}, \operatorname{Ric} \end{bmatrix} \qquad \Leftarrow \qquad \text{symmetrization of (IC2)}$$
$$(D-2)[\nabla \boldsymbol{\xi}, \boldsymbol{h}] = \begin{bmatrix} \boldsymbol{h}, \operatorname{Ric} \end{bmatrix} \cdot \boldsymbol{h} + \frac{1}{2} [\operatorname{Rh}, \boldsymbol{h}] \qquad \Leftarrow \qquad [(\operatorname{IC2}), \boldsymbol{h}]$$

Alignment of Riemann tensor with principal CCKY form

Taking various (anti)symmetrizations and contractions of (IC3) with h one can prove:

$$\begin{bmatrix} \boldsymbol{h} , \mathbf{R} \mathbf{h}^{(p)} \end{bmatrix} = 0 \qquad \text{where} \quad \mathbf{R} \mathbf{h}^{(p)a}{}_{b} = \boldsymbol{h}^{p}{}_{mn} \, \boldsymbol{R}^{mna}{}_{b} \\ \mathbf{R} \mathbf{h} = \mathbf{R} \mathbf{h}^{(1)} \\ \end{bmatrix}$$
$$\begin{bmatrix} \boldsymbol{h} , \mathbf{Rich}^{(2p)} \end{bmatrix} = 0 \qquad \text{where} \quad \mathbf{Rich}^{(2p)}_{ab} = \boldsymbol{h}^{p}{}_{mn} \, \boldsymbol{R}^{m}{}_{a}{}^{n}{}_{b} \end{cases}$$

Any contraction of R with any power of h commutes with h

 $\operatorname{Rich}^{(0)} = \operatorname{Ric}$

NS-commutation of Killing tower

Any contraction of ${m R}$ with any power of ${m h}$ commutes with ${m h}$

$$\Rightarrow \qquad \mathcal{S}(\nabla \boldsymbol{\xi}) = \left[\boldsymbol{h}, \mathbf{Ric}\right] = 0$$
$$\left[\nabla \boldsymbol{\xi}, \boldsymbol{h}\right] = \frac{1}{D-2} \left(\left[\boldsymbol{h}, \mathbf{Ric}\right] \cdot \boldsymbol{h} + \frac{1}{2} \left[\mathbf{Rh}, \boldsymbol{h}\right] \right) = 0$$

\Rightarrow Vanishing NS-brackets

$$\begin{bmatrix} \boldsymbol{k}_1, \boldsymbol{k}_2 \end{bmatrix}_{\text{NS}} = 0 \qquad \begin{bmatrix} \boldsymbol{k}_1, \boldsymbol{l}_2 \end{bmatrix}_{\text{NS}} = 0 \qquad \begin{bmatrix} \boldsymbol{l}_1, \boldsymbol{l}_2 \end{bmatrix}_{\text{NS}} = 0$$
$$\begin{bmatrix} \boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)} \end{bmatrix}_{\text{NS}} = 0 \qquad \begin{bmatrix} \boldsymbol{k}_{(i)}, \boldsymbol{l}_{(j)} \end{bmatrix}_{\text{NS}} = 0 \qquad \begin{bmatrix} \boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)} \end{bmatrix}_{\text{NS}} = 0$$

Consequences for physical systems: integrability and separability

- Integrability of the geodesic motion
- Separability of the Hamilton–Jacobi equation
- Separability of the wave equation
- Separability of the Dirac equation

Integrability of the geodesic motion

hidden and explicit symmetries define observables quadratic and linear in momenta

$$\boldsymbol{k}_{(j)} \quad \Rightarrow \quad K_j = \boldsymbol{k}_{(j)}^{ab} \, \boldsymbol{p}_a \boldsymbol{p}_b \qquad \boldsymbol{l}_{(j)} \quad \Rightarrow \quad L_j = \boldsymbol{l}_{(j)}^a \, \boldsymbol{p}_a$$

$$\begin{bmatrix} \boldsymbol{k}_{(i)}, \boldsymbol{k}_{(j)} \end{bmatrix}_{\rm NS} = 0 \qquad \begin{bmatrix} \boldsymbol{k}_{(i)}, \boldsymbol{l}_{(j)} \end{bmatrix}_{\rm NS} = 0 \qquad \begin{bmatrix} \boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)} \end{bmatrix}_{\rm NS} = 0$$

$$\begin{cases} K_i, K_j \end{bmatrix} = 0 \qquad \begin{cases} K_i, L_j \end{bmatrix} = 0 \qquad \begin{cases} L_i, L_j \end{bmatrix} = 0$$

$$oldsymbol{k}_{(0)} = oldsymbol{g} \qquad \Rightarrow \qquad K_0 \propto H$$

 K_j and L_j are conserved quantities in involution

Geodesics

geodesics labeled by constants K_j , L_j and initial positions

Solving for momenta

$$p_{\mu} = \pm \frac{\sqrt{X_{\mu}\tilde{K}_{\mu} - \tilde{L}_{\mu}^2}}{X_{\mu}}$$
$$p_j = L_j$$



Solving for coordinates

$$\dot{x}_{\mu} = \pm \frac{1}{U_{\mu}} \sqrt{X_{\mu} \tilde{K}_{\mu} - \tilde{L}_{\mu}^2}$$
$$\dot{\psi}_j = \sum_{\mu} \frac{(-x_{\mu}^2)^{N-1-j}}{U_{\mu}} \frac{\tilde{L}_{\mu}}{X_{\mu}}$$

can be solved numerically or analytically using action–angle variables

Separability of the Hamilton–Jacobi equation

Static Hamilton–Jacobi equation

$$H(x, dS) = E$$

Hamilton–Jacobi equations for conserved quantities

$$dS \cdot k_{(j)} \cdot dS = K_j$$
 $l_{(j)} \cdot dS = L_j$

Separability of the Hamilton–Jacobi equation

Static Hamilton–Jacobi equation

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Hamilton–Jacobi equations for conserved quantities

$$dS \cdot k_{(j)} \cdot dS = K_j$$
 $l_{(j)} \cdot dS = L_j$

Additive separability ansatz

$$S = \sum_{\mu} S_{\mu} + \sum_{j} L_{j} \psi_{j} \qquad S_{\mu} = S_{\mu}(x_{\mu}; K_{j}, L_{j})$$

Separated ordinary differential equations

$$(S'_{\mu})^{2} = \frac{\tilde{K}_{\mu}}{X_{\mu}} - \frac{\tilde{L}_{\mu}^{2}}{X_{\mu}^{2}} \qquad \Rightarrow \qquad S_{\mu} = \int^{x_{\mu}} \frac{\sqrt{X_{\mu}\tilde{K}_{\mu} - \tilde{L}_{\mu}^{2}}}{X_{\mu}} dx_{\mu}$$

Separability of the wave equation

Wave equation

$$\Box \phi = 0 \qquad \qquad \Box = -\boldsymbol{g}^{ab} \boldsymbol{\nabla}_a \boldsymbol{\nabla}_b$$

Operators corresponding to conserved quantities

$$\mathcal{K}_j = -oldsymbol{
abla}_a \, oldsymbol{k}_{(j)}^{ab} \, oldsymbol{
abla}_b \qquad \qquad \mathcal{L}_j = -i \, oldsymbol{l}_{(j)}^a oldsymbol{
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Commutativity of the symmetry operators

$$\left[\mathcal{K}_k, \mathcal{K}_l\right] = 0$$
 $\left[\mathcal{K}_k, \mathcal{L}_l\right] = 0$ $\left[\mathcal{L}_k, \mathcal{L}_l\right] = 0$

Common eigenvalue problem

$$\mathcal{K}_j \phi = K_j \phi$$
 $\mathcal{L}_j \phi = L_j \phi$

Separability of the wave equation

Operators corresponding to conserved quantities

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Common eigenvalue problem

$$\mathcal{K}_j \phi = K_j \phi$$
 $\mathcal{L}_j \phi = L_j \phi$

Multiplicative separability ansatz

$$\phi = \prod_{\mu} R_{\mu} \prod_{k=0}^{N-1+\varepsilon} \exp(iL_k \psi_k) \qquad \qquad R_{\mu} = R_{\mu}(x_{\mu}; K_j, L_j)$$
Separability of the wave equation

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Separated ordinary differential equations

$$\left(X_{\mu}R'_{\mu}\right)' + \left(\frac{\tilde{K}_{\mu}}{X_{\mu}} - \frac{\tilde{L}_{\mu}^2}{X_{\mu}^2}\right)R'_{\mu} = 0$$

$$\tilde{K}_{\mu} = \sum_{j} K_{j} (-x_{\mu}^{2})^{N-1-j}$$
$$\tilde{L}_{\mu} = \sum_{j} L_{j} (-x_{\mu}^{2})^{N-1-j}$$

- Explicit and hidden symmetries \Leftrightarrow Killing vectors and Killing tensors
- Killing tensors can be build from conformal Killing–Yano forms
- Principal tensor \Leftrightarrow non-degenerate closed conformal Killing–Yano 2-form
- Killing tower \Rightarrow a sequence of Killing tensors and vectors
- Uniqueness of the geometry compatible with the principal tensor
- Kerr–NUT–(A)dS metric
- Rich symmetry structure \Rightarrow Integrability and Separability

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for details see

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Black holes, hidden symmetries, and complete integrability

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Thank you!