[D. HILDITCH OCTOBER 2024]

PRAGUE MINICOURSE IN NR

- 4 LECTURES FROM ME [THEN TOMAS TAKES OVER]

I :
II :
III :
IV

OVERVIEW / CAUCHY PROBLEM, CONSTRAINTS.
WP OF THE IVP, FORMULATIONS OF GR.
BASICS OF BBH ID, OVERVIEW OF BINARY EVOLUTIONS.

BOOKS : ALCUBIERRE, BAUMGARTE - SHAPIRO, GOURGOULHON, SHIBATA. [HILDITCH NOTES]

FOUNDATIONS OF NR [MINI-COURSE, LECTURE I].

SEEK TO UNDERSTAND SOLUTION SPACE OF GR :

$$G_{ab} = 8\pi T_{ab}$$

- CURVATURE RELATED TO MATTER DISTRIBUTION.
- COUPLED NONLINEAR PDE SYSTEM.

STRATEGIES : "SOLUTIONS IN HAND" VS. PURE MATH / ABSTRACT

THINKING ON PAPER
- EXACT SOLUTIONS (SYMMETRY)
- PERTURBATION THEORY [LINEARIZE / PN]

- EXISTENCE, UNIQUENESS, ESTIMATES. BUT NOT "IN HAND".

BOTH HAVE LIMITATIONS, BUT NR CAN HELP :

NUMERICAL RELATIVITY : "THE USE OF NUMERICAL METHODS TO FIND APPROXIMATE SOLUTIONS THAT CONVERGE TO THE CONTINUUM SOLUTION AS YOU USE MORE COMPUTATIONAL RESOURCES"

- COMPLEMENTARY : ONLY APPROACH FOR PART OF PHASE SPACE.

HISTORY OF NR

1950s. ADM
1960s. FIRST NUMERICS. HAHN & LINDQ. '64
1970s. AXISYMMETRY. YORK 3+1 DECOMPOSITION
1980s. FIRST FACULTY.
1990s. CRITICAL PHENOMENA / GRAND CHALLENGE.

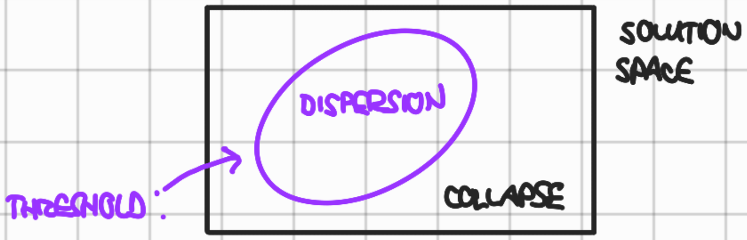
2000s. COMPACT BINARY BREAKTHROUGH
2010s. MODELING. DETECTION OF GWs.
2020s. ASTROPHYSICS. SEMI-GLOBAL. BH THRESHOLD.

OVERVIEW OF PIPELINE

1. PHYSICAL PROBLEM } FOR MY TALKS: THE BH THRESHOLD
2. FORMULATION
3. PDE ANALYSIS } DISCUSS IN A LITTLE MORE DETAIL
4. SELECT NUMERICAL METHOD
5. IMPLEMENTATION. [VALIDATION].
6. ERROR EVALUATION
7. PHYSICAL INTERPRETATION. } COLLAPSE EXAMPLES: CRITICAL PHENOMENA. FORMATION MECHANISM

EXPAND ON PIPELINE

1. PROBLEM OF INTEREST: THE BH THRESHOLD



2. FORMULATION: CAUCHY PROBLEM

- VARIOUS INTERESTING PDE PROBLEMS. BUT THE CAUCHY PROBLEM IS BY FAR THE MOST IMPORTANT IN NR.

FREE-EVOLUTION FORMULATIONS OF MAXWELL

$$\begin{cases} \partial_t A_i = -\Pi_i + \partial_i \phi \\ \partial_t \Pi_i = -\Delta A_i + \partial_i \partial_j A^j \\ M = \partial^i \Pi_i = 0 \end{cases}$$

ϕ ARBITRARY. GAUGE.

$\Rightarrow \partial_t M = 0$

[CONSTRAINTS CLOSED]

$$\begin{cases} \partial_t A_i = -\Pi_i + \partial_i \phi \\ \partial_t \Pi_i = -\Delta A_i + \partial_i \partial_j A^j + \partial_i Z \\ \partial_t \phi = \mu [\partial_i A^i + Z] \\ \partial_t Z = M - \kappa Z \\ Z = M = \partial^i \Pi_i = 0 \end{cases}$$

$\Rightarrow \partial_t M = \Delta Z$

[STILL CLOSED]

RECIPE:
- GAUGE CHOICE
- CONSTRAINTS WITH EoMs
- COUPLE CONSTRAINTS.

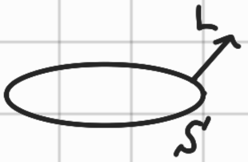
CAN ARRIVE @ "WAVE EQUATIONS".

7. PHYSICAL INTERPRETATION : APPARENT & EVENT HORIZONS.

- EVENT HORIZONS "TELEOLOGICAL" $\Rightarrow$ NO DIRECT ACCESS @ RUN-TIME.
- APPARENT HORIZON LOCAL IN TIME [FOLIATION DEPENDENT].

L^a OUTGOING NULL NORMAL TO S [HYPERSURFACE ORTH], GEODESIC.

EXPANSION $\chi \equiv \nabla_a L^a$.



$\chi < 0$, S "OUTER TRAPPED"

$\chi = 0$, S MARGINALLY OUTER TRAPPED. $\rightsquigarrow$ OUTERMOST MOTS IS AH

RELEVANCE: WCC $\Rightarrow$ AH INSIDE EH.

BH FORMATION $\quad \mathcal{L}_L \chi = -\frac{1}{2}\chi^2 - \hat{\chi}^2 - 8\pi T_{LL}$ RAYCHAUDHURI

THE IVP IN GR [LECTURE II]

"STANDARD" NUMERICAL METHODS ARE DESIGNED FOR SYSTEMS THAT LOOK LIKE

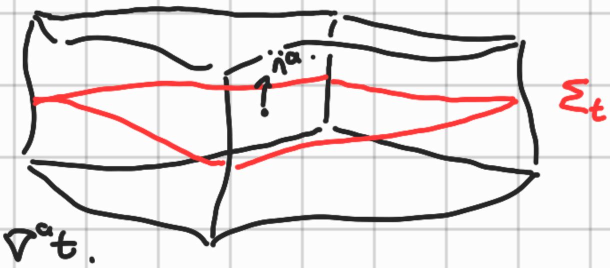
$$\partial_T u = A^p \partial_p u + S .$$

TIME DER. SPATIAL DERIVATIVES LOWER ORDER TERMS.

BUT GR: $\quad G_{ab} = 8\pi T_{ab}$ $\rightsquigarrow$ INTRODUCE LOCAL TIME COORDINATE t. ASSUMING WE HAVE A SPACETIME (M, g) GIVEN. THEN:

EVERYTHING MIXED UP. GEOMETRICAL

- GEOMETRIC 3+1 DECOMPOSITION:



COORDINATES $x^\mu = (t, x^i)$

METRIC

LAPSE $-\alpha^{-2} = \nabla^a t \nabla_a t$.

FUTURE DIRECTED UNIT NORMAL $n^a = -\alpha \nabla^a t$.

PROJECTION / SPATIAL METRIC $\gamma_{ab} = g_{ab} + n_a n_b$.

CHECK $\gamma_a{}^b \gamma_b{}^c = \gamma_a{}^c$.

[EVERY TENSOR CAN BE DECOMPOSED. EX. $V^a = -(V\cdot n) n^a + (\gamma \cdot V)^a$]

SPATIAL TENSOR

TIME VECTOR $t^a \equiv \partial_t^a = \alpha n^a + \beta^a$.

SHIFT MEASURES $t^a \neq \alpha n^a$.

CONNECTION

EXTRINSIC CURVATURE $K_{ab} = -\gamma_a{}^c \nabla_c n_b$

$\gamma_a{}^b \nabla_b \{\text{SPATIAL TENSOR}\} = D_a + K_{ab}$ TERMS [SPATIAL LEVI-CIV.]

$n^a \nabla_a \{\text{SPATIAL}\} \sim \mathcal{L}_n \{\text{SPATIAL}\} + K_{ab}$ TERMS.

CLEANING AND COUNTING:

- IN ADAPTED COORDINATES $x^\mu = (t, x^i)$ NEED ONLY KEEP SPATIAL COMPONENTS OF SPATIAL TENSORS.
- 4D CHRISTOFFELS ${}^{(4)}\Gamma^\alpha{}_{\beta\gamma}$ - 40 COMPONENTS

3+1 DECOMPOSED: $D_a \to \Gamma^k{}_{ij}$ 18

K_{ij} + 6

$\partial_\mu \alpha, \partial_\mu \beta^i$ + 16 = 40 COMPONENTS.

CURVATURE

CURVATURE DECOMPOSITION

$\gamma_a^e \gamma_b^f \gamma_c^g \gamma_d^h \, {}^{(4)}R_{efgh} = R_{abcd} + 2 K_{a[c} K_{d]b}$ GAUSS EQN.

$\gamma_a^e \gamma_b^f \gamma_c^g n^h \, {}^{(4)}R_{efgh} = -2 D_{[a} K_{b]c}$ CODAZZI-MAINARDI EQN.

$\gamma_a^e n^f \gamma_c^g n^h \, {}^{(4)}R_{efgh} = \mathcal{L}_n K_{ac} + \alpha^{-1} D_a D_c \alpha + K_a{}^b K_{bc}$ RICCI EQN.

$[\nabla_n n_a = D_a \ln \alpha .]$

DERIVATION: $\gamma_a^e \gamma_b^f \gamma_c^g n^h \, {}^{(4)}R_{efgh} = 2 \gamma_a^e \gamma_b^f \gamma_c^g \nabla_{[e} \nabla_{f]} n_g$

$= -2 \gamma_a^e \gamma_b^f \gamma_c^g \nabla_{[e} \left(K_{f]g} + n_f a_g \right)$

$= -2 D_{[a} K_{b]c} + 2 a_c K_{[ab]}$ OTHERS SIMILAR.

$K_{ab} = -\gamma_a{}^c \nabla_c n_b = -\gamma_a{}^c \gamma_b{}^d \nabla_c n_d = \alpha \gamma_a{}^c \gamma_b{}^d \nabla_c \nabla_d t$

$\Rightarrow \quad K_{[ab]} = \alpha \gamma_a{}^c \gamma_b{}^d \nabla_{[c} \nabla_{d]} t = 0$

NOW APPLY THESE TO GR:

$G_{ab} = 8\pi T_{ab}$

$\| $

${}^{(4)}R_{ab} - \frac{1}{2} g_{ab} {}^{(4)}R$

$\|$

${}^{(4)}R_{acb}{}^c$ NEED CONTRACTIONS

DECOMPOSE STRESS-ENERGY

$T_{nn} = \rho .$

$T_{n \perp a} = -j_a$

$T_{\perp a \perp b} = S_{ab}$

TRACE GAUSS EQN:

$$\gamma^{bd}\gamma^{ac}\left[\perp {}^{(4)}R_{abcd}\right] = R + K^2 - K_{ij}K^{ij}$$

$$\parallel$$

$$^{(4)}R + 2n^a n^b \, {}^{(4)}R_{ab}$$ THEN WE HAVE:

EES: $G_{ab} = 8\pi T_{ab}$

FIELD EQNS.

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho \quad [G_{nn}]$$

$$D^j(K_{ij} - \gamma_{ij}K) = 8\pi j_a \quad [G_{ni}]$$

CONSTRAINTS

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}.$$

$$\partial_t K_{ij} = -D_iD_j\alpha + \alpha\left[R_{ij} - 2K_i{}^k K_{jk} + K K_{ij}\right] + \mathcal{L}_\beta \gamma_{ij}$$
$$- 8\pi\alpha\left[S_{ij} - \tfrac{1}{2}\gamma_{ij}S + \tfrac{1}{2}\gamma_{ij}\rho\right] \qquad [R_{ij}]$$

CAUCHY PROBLEM: CHOOSE γ_{ij}, K_{ij} CONSTRAINT SATISFYING.
WHAT HAPPENS LATER? [CLOSURE OF CONSTRAINTS]

FREE-EVOLUTION FORMULATIONS OF GR AND WP OF THE IVP [LECTURE II].

LAST TIME WE SAW THAT THE EES NATURALLY DECOMPOSE AS:

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$
$$D^j(K_{ij} - \gamma_{ij}K) = 8\pi j_a$$

CONSTRAINTS

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}.$$

$$\partial_t K_{ij} = -D_iD_j\alpha + \alpha\left[R_{ij} - 2K_i{}^k K_{jk} + K K_{ij}\right] + \mathcal{L}_\beta \gamma_{ij}$$
$$- 8\pi\alpha\left[S_{ij} - \tfrac{1}{2}\gamma_{ij}S + \tfrac{1}{2}\gamma_{ij}\rho\right]$$

EVOLUTION EQUATIONS.

COMPARE W.E:
$$\partial_t \phi = -\pi$$
$$\partial_t \pi = -\Delta\phi$$

COMPARE MAXWELL:
$$\partial_t A_i = -E_i - \partial_i\phi \qquad M = \partial_i E^i = 0$$
$$\partial_t E_i = -\Delta A_i + \partial_i\partial^j A_j$$

RECALL FREE-EVOLUTION FROM MAXWELL: $\partial_t M = 0$.

WHAT ABOUT GR?

RECALL $$\nabla^b [\underbrace{G_{ab}}_{\text{VANISHES AS A GEOMETRIC IDENTITY.}} - 8\pi T_{ab}] = 0$$

$$\begin{cases} G_{nn} - 8\pi T_{nn} = \frac{1}{2} H \\ G_{in} - 8\pi T_{in} = M_i \end{cases}$$

USING THIS AND TURNING THE CRANK, WE GET

$$\begin{cases} \partial_t H = -2\alpha D^i M_i - 4 M^i D_i \alpha + 2\alpha K H + \mathcal{L}_\beta H \\ \partial_t M_i = -\frac{1}{2}\alpha D_i H + \alpha K M_i - (D_i \alpha) H + \mathcal{L}_\beta M_i \end{cases}$$

SO, "FREE-EVOLUTION" COULD WORK AS IN MAXWELL.

THESE DAYS IN NR THERE ARE ESSENTIALLY TWO CLASSES OF FREE-EVOLUTION FORMULATIONS IN USE:

- GHG "GENERALIZED HARMONIC GAUGE"
- "CONFORMAL DECOMPOSITIONS" [CONFORMAL Z4/ BSSNOK]

I'LL EXPLAIN THIS BECAUSE IT IS FASTER. PRINCIPLES THE SAME FOR CONFORMAL CHOICES.

THE GHG FORMULATION: LET'S DO AWAY WITH THE 3+1 DECOMPOSITION, AND WORK IN VACUUM. TRACE REVERSED EEs READ

$$0 = R_{\mu\nu} = \underbrace{-\tfrac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu}}_{\text{"WAVE-OPERATOR"}} + \partial_{(\mu} \Gamma_{\nu)} + (\Gamma\Gamma - \Gamma\Gamma)_{\mu\nu}$$

$\Gamma_\nu = g^{\alpha\beta}\Gamma_{\nu\alpha\beta}$ SECOND DERIVATIVES.

LOWER DERIVATIVES

OBSERVE: $\Gamma_\mu = -g_{\mu\nu} \nabla_\alpha \nabla^\alpha x^\nu = -g_{\mu\nu} \Box x^\nu$

CLEVER TRICK: CHOOSE $\Box x^\mu = H^\mu(x, g)$

"HARMONIC COORDINATES"
"WAVE COORDINATES"

FORMALLY, DEFINE CONSTRAINT: $Z_\mu = -\frac{1}{2}\Gamma_\mu - \frac{1}{2}H_\mu$

SOLVE "REDUCED EEs":

$$0 = \mathcal{R}_{\mu\nu} \simeq R_{\mu\nu} + 2\partial_{(\mu} Z_{\nu)} \simeq -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + \dots$$

THINK: $\mathcal{G}_{\mu\nu} \simeq G_{\mu\nu} + 2\partial_{(\mu} Z_{\nu)} - g_{\mu\nu} \partial^\alpha Z_\alpha$

$\Rightarrow \quad \partial_t Z_\mu$ "=" HAMILTONIAN + MOMENTUM
+ $\partial_i Z_\mu$ TERMS.

$\nabla^\nu \mathcal{G}_{\mu\nu} \simeq \Box Z_\mu +$ CONSTRAINT TERMS.

IT FOLLOWS THAT IF $Z_\mu = 0$, AND CONSTRAINTS OF GR SATISFIED THEN SOLUTION IS A SOLUTION TO GR.

RECIPE FOR FREE-EVOLUTION FORMULATIONS

i). GAUGE CHOICE $\quad$ [$\Box x^\mu = H^\mu$ IN GHG
$\Rightarrow \partial_t \alpha = -\alpha^2 K + \mathcal{L}_\beta \alpha$
$\partial_t \beta^i = \alpha^2 \Gamma^i - \alpha D^i \alpha + \beta^j \partial_j \beta^i$]

ii). DEFINE NEW CONSTRAINTS WITH EOMS.

iii). ADD (DERIVATIVES OF) CONSTRAINTS TO EVOLUTION EQUATIONS.

ALL FREE-EVOLUTION FORMULATIONS ARE BUILT LIKE THIS: NOR, ADM, BSSNOK, (Z4, Z4C, CCZ4), EC, KST ...

THIS JUST LOOKS DIFFERENT WITH 3+1 VARIABLES.

WELL-POSEDNESS: SO FAR WE HAVE JUST RELIED ON HEURISTIC SENSE THAT WAVE EQUATIONS ARE GOOD. TIME TO UNPACK THAT:

BASIC PDE CLASSIFICATION: (HIGH-SCHOOL VERSION). CONSIDER A 2ND ORDER, LINEAR PDE WITH (REAL) CONSTANT COEFFICIENTS.

$$\begin{cases} a\, u_{,xx} + 2b\, u_{,xy} + c\, u_{,yy} \\ + \text{"LOWER DERIVATIVES"} = 0 \end{cases}$$

ELLIPTIC: $b^2 - ac < 0$.
- NO INTRINSIC TIME
- GOOD BVP. PROTOTYPE IS LAPLACE.

IN NR: CONSTRAINTS.

PARABOLIC: $b^2 = ac$
- INTRINSIC TIME. GOOD IVP (ONE WAY). INFINITE PROPAGATION SPEEDS. HEAT EQUATION

(APPARENT HORIZONS, SOME METHODS)

HYPERBOLIC: $b^2 - ac > 0$.
- TIME AND CAUSALITY: FINITE PROPAGATION SPEEDS. FUNDAMENTAL IN RELATIVISTIC CONTEXT.
- GOOD IVP.

(ADM EVOLUTION EQNS)

LIFE IS COMPLICATED! MODELS OF NATURE ARISE WITH ALL TYPES - ESPECIALLY IN THEORIES WITH "GAUGE FREEDOM", LIKE E&M & GR, ENCOUNTER ALL THREE.

OBSERVE, TYPE OF PROBLEM [IVP, BVP, IBVP, CIVP] THAT IS "GOOD" IS DETERMINED BY CLASSIFICATION.

WHAT DOES "GOOD" MEAN?

- WELL-POSEDNESS: A PDE PROBLEM [DOMAIN, EQN, DATA] IS CALLED WP IF IT ADMITS A UNIQUE SOLUTION THAT DEPENDS CONTINUOUSLY ON THE GIVEN DATA (IN SOME NORM).

HYPERBOLIC PDE SYSTEMS IN FIRST ORDER FORM

(KREISS - BUSENHART).

CONSIDER A SYSTEM WHICH IS LINEAR WITH CONSTANT COEFFICIENTS:

$$\partial_t u = A^p \partial_p u + S \quad (*)$$

CAUCHY / IVP: SPECIFY $u(t=0, x^i)$. WHAT IS THE SOLUTION $u(t, x^i)$?

PDE PROBLEM WP IF THERE IS A NORM $\|\cdot\|$ S.T

$$\|u(t,\cdot)\| \leq K e^{\alpha T} \|u(0,\cdot)\|,$$

K, α INDEP. OF ID. [THINK: $\|\cdot\| \sim L^2$ FOR NOW]

EXAMPLE OF ILL-POSED IVP:

(TERRIBLE) 2D LAPLACE EQN

$$\partial_t^2 \phi = -\partial_x^2 \phi$$

FIRST ORDER REDUCTION

$$\partial_t \phi = u_1, \quad \partial_x \phi = u_2$$

$$\left.\begin{aligned} \partial_t u_1 &= -\partial_x u_2 \\ \partial_t u_2 &= \partial_x u_1 \end{aligned}\right\}$$

CHOOSE ID $\phi(0,x) = e^{ikx} \phi_0$

SOLUTION: $\phi(t,x) = \phi_0 e^{kt+ikx}$

EXPONENTIAL GROWTH, BUT DEPENDENT ON ID.

$$\left\{\begin{aligned} u_1 &= k\phi_0 e^{kt+ikx} \\ u_2 &= ik\phi_0 e^{kt+ikx} \end{aligned}\right.$$

EXAMPLE: WEAKLY HYPERBOLIC SYSTEM

$$\partial_t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$u = (u_1, u_2)^T$

$u(0,x) = (Be^{ikx}, Ae^{ikx})^T$

$$\Rightarrow \quad u_1(t,x) = (ikAt + B)e^{ik(t+x)}$$

$$u_2(t,x) = Ae^{ik(t+x)}$$

u_2 FINE.

u_1 ? LINEAR GROWTH, BUT FREQ. DEPENDENT.

IVP ILL-POSED IN L^2.

SO WHAT DOES WORK?

[LECTURE IV]: LAST TIME WE CONSIDERED (*) WITH... EVERYTHING THAT DIDN'T WORK.

WEAK, STRONG AND SYMMETRIC HYPERBOLICITY

CONSIDER IVP FOR (*). TAKE ARBITRARY UNIT SPATIAL COVECTOR S_p. THE PRINCIPAL SYMBOL IN THE S_p DIRECT. IS

$$A^S \equiv A^p S_p$$

DEFN: IF $\forall S_p$ THE PRINCIPAL SYMBOL HAS REAL EIGENVALUES, THE SYSTEM IS CALLED WEAKLY HYPERBOLIC.

DEFN: IF THE SYSTEM IS WEAKLY HYPERBOLIC AND $\forall S_p$ A^S HAS A COMPLETE SET OF EIGENVECTORS AND

$$|T_S| + |T_S^{-1}| \leq K$$

[UNIFORMITY CONDITION]

$\left\{ \begin{array}{l} K \text{ INDEP. OF } S_p \\ T_S \text{ HAS E'VECS OF } A^S \\ \text{AS COLUMNS} \end{array} \right.$

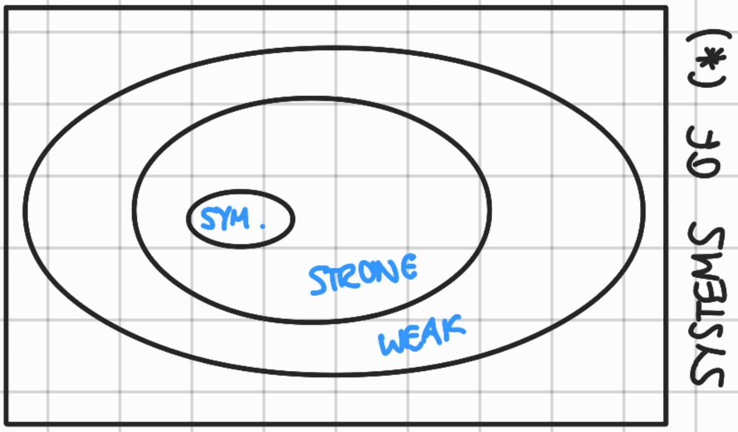
THEN THE SYSTEM IS CALLED STRONGLY HYPERBOLIC.

DEFN: IF THERE EXISTS A SYMMETRIC, POSITIVE DEF. H (INDEP. OF S_p), CALLED A SYMMETRIZER, S.T:

$$HA^p \text{ IS SYMMETRIC } \forall p,$$

THEN THE SYSTEM IS CALLED SYMMETRIC HYPERBOLIC.

DIAGRAM



INTUITIVE SUMMARY

- SYM. HYP: GOOD IBVP [DEPENDING ON BCS]
- STRONG HYP: GOOD NP. IBVP HARDER.
- WEAK: NOTHING GOOD IN L^2.

$$\boxed{\hat{f} = \int_{-\infty}^{\infty} f e^{2\pi i \omega \cdot x} dx}$$

THM: IVP FOR (*) WP (IN L^2) I.F.F STRONGLY HYP.

(PART OF) PROOF: FOURIER TRANSFORM IN SPACE

$$\partial_t \hat{u} = i|\omega| A^{\hat{\omega}} \hat{u}$$ STRONG HYP $\Rightarrow$

$\exists$ SIMILARITY TRANSF. WITH

$$T_{\hat{\omega}} A^{\hat{\omega}} T_{\hat{\omega}}^{-1} = \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

[$T_{\hat{\omega}}^{-1}$ MATRIX OF EVECS.]

LET $\hat{H}(\hat{\omega}) = T_{\hat{\omega}}^{+} T_{\hat{\omega}}$

$\hat{H}(\hat{\omega})$ IS SYM. P.D. [LAW OF INERTIA]

AND $\hat{H}(\hat{\omega}) A^{\hat{\omega}} = (\hat{H} A^{\hat{\omega}})^{+}$ [EXERCISE].

CONSIDER THE NORM

$$\| u \|_H^2 = \int \hat{u}^+ H \hat{u} \, d\omega$$ [UNIFORMITY GUARANTEES EQUIV. WITH L^2]

COMPUTE TIME-DERIVATIVE:

$$\partial_t \|u\|_H^2 = \int \hat{u}^\dagger H A^{\hat{\omega}} i|\omega| \hat{u} - i (A^{\hat{\omega}} \hat{u})^\dagger H \hat{u} |\omega| \, d\omega$$

$$= \int i|\omega| \, \hat{u}^\dagger \{ H A^{\hat{\omega}} - (H A^{\hat{\omega}})^\dagger \} \hat{u} \, d\omega$$

$$= 0$$ => NORM IS CONSERVED! => SYSTEM IS WP.

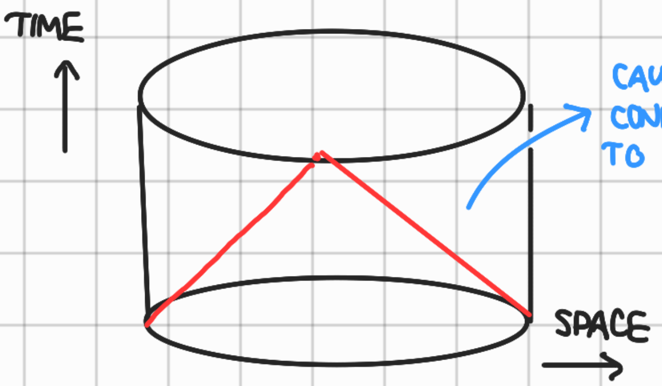
NOTES: • SOURCE TERMS DO NOT BREAK ESTIMATE, JUST INDUCE EXPONENTIAL GROWTH. [GRÖNWALL'S LEMMA]

THIS IS A LONG WAY FROM GR. → COULD MAKE FIRST ORDER REDUCTION. THEN LINEARIZE @ ARBITRARY SOLUTION. THEN (WITH ADDITIONAL SMOOTHNESS REQUIREMENTS) THIS TYPE OF RESULT CARRIES OVER LOCALLY IN TIME.

[TECHNICAL: FIRST ORDER].

TAKE HOME MESSAGE: WHATEVER MANIPULATION WE MAKE TO THE EES IN BUILDING OUR FORMULATION, WE NEED TO RENDER THE EOMS AT LEAST STRONGLY HYPERBOLIC. IS THIS GOOD ENOUGH?

NO! ACTUALLY SOLVE IBVP:



CAUSALLY CONNECTED TO BOUNDARY } ACTUALLY NEED WP IBVP WITH "CPBCS" } THEN CAN THINK ABOUT MODELLING.

[SEE SARBACH - TIGLIO LIVING REVIEW]

- LOTS OF STRONGLY HYPERBOLIC FORMULATIONS. INTERESTING RESULT: STR. HYP. GAUGE NECESSARY FOR STR. HYP FORMULATION [HILDITCH-RICHTER 2015]
- SYMMETRIC HYPERBOLIC IN USE: GHG. KNOWN TO HAVE A GOOD IBVP WITH RADIATION CONTROLLING CPBCs.

PDE REFS: KREISS - LORENZ
GUSTAFSSON - KREISS - OLIGER } BOOKS
KREISS - BUSENHART } LECTURE NOTES
SARBACH - TIGLIO } LIVING REVIEW.

IDEA OF "CONFORMAL FORMULATIONS":

- TAKE A STRONGLY HYPERBOLIC FORMULATION
- CHANGE DEPENDENT VARIABLES TO MATCH WELL "EASY" BH INITIAL DATA.

RECALL SCHWARZSCHILD IN ISOTROPIC COORDINATES:

$$\begin{cases} \gamma_{ij} = \psi^4 \delta_{ij} \qquad \text{WITH} \qquad \psi = \left(1 + \frac{M}{2r}\right). \\ K_{ij} = 0. \end{cases}$$

- SO MUCH MORE TO SAY!