LORENTZ BOOSTS IN DE SITTER AND ANTI-DE SITTER SPACE-TIMES

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Symmetries of de Sitter and anti-de Sitter space-times are discussed. In particular, Lorentz boosts, which appear to be given previously only in the infinitesimal form, are presented here in closed finite forms.

1. Introduction

It is well-known that the maximal number of isometries of a four-dimensional space-time is 10 corresponding to the same number of independent Killing vectors (infinitesimal generators of the symmetry transformations). Such maximally symmetric space-times have a constant curvature R (see, e.g.[1]). According to the sign of R, there are three possibilities: one gets flat Minkowski space-time ($R = 0$), de Sitter space-time ($R > 0$) and anti-de Sitter space-time ($R < 0$). These space-times are the only conformally flat vacuum solutions of Einstein's field equations with cosmological constant λ. The constant curvature R is related to λ by $R = 4\lambda$.

2. Symmetries of de Sitter space-time

De Sitter manifold can be visualized as a hyperboloid

$$-z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = \alpha^2, \tag{1}$$

$\alpha = \sqrt{\frac{\lambda}{3}}$, embedded in a flat five-dimensional space-time

$$ds^2 = -dz_0^2 + dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 \,. \tag{2}$$

We use coordinates (η, x, y, z) covering the entire hyperboloid given by

$$\begin{aligned} z_0 &= \frac{1}{2\eta}(\alpha^2 + s) \\ z_1 &= \alpha\frac{x}{\eta} \\ z_2 &= \alpha\frac{y}{\eta} \\ z_3 &= \alpha\frac{z}{\eta} \\ z_4 &= \frac{1}{2\eta}(\alpha^2 - s) \end{aligned} \tag{3}$$

where $s = -\eta^2 + x^2 + y^2 + z^2$ and $\eta, x, y, z \in (-\infty, +\infty)$. In the global coordinates (3) de Sitter metric can be written as

$$ds^2 = \frac{\alpha^2}{\eta^2}(-d\eta^2 + dx^2 + dy^2 + dz^2)\,. \tag{4}$$

This conformally flat metric (see, e.g. [3]) is related to the 'standard' form of de Sitter metric

$$ds^2 = -dt^2 + \exp{(2\frac{t}{\alpha})}(dx^2 + dy^2 + dz^2) \tag{5}$$

by a simple transformation $\eta = \alpha \exp{(-t/\alpha)}$. However, the metric (5) covers only one half $(z_0 + z_4 > 0)$ of the manifold (1). Thus, one needs another coordinate chart of the form (5) to cover also $z_0 + z_4 < 0$; such a chart is given by $\eta = -\alpha \exp{(-t^*/\alpha)}$. We prefer coordinates (3) since they do not only cover the entire manifold but the symmetries can be seen most easily just in these coordinates.

Killing vectors are (see [2],[3], where, however, the expressions are given in 'standard' coordinates used in (5) and thus have more complicated forms):

$$\begin{aligned} \xi^\mu_{(1)} &= (0, 1, 0, 0) \\ \xi^\mu_{(2)} &= (0, 0, 1, 0) \\ \xi^\mu_{(3)} &= (0, 0, 0, 1) \\ \xi^\mu_{(4)} &= (0, y, -x, 0) \\ \xi^\mu_{(5)} &= (0, z, 0, -x) \\ \xi^\mu_{(6)} &= (0, 0, z, -y) \\ \xi^\mu_{(7)} &= (\eta, x, y, z) \\ \xi^\mu_{(8)} &= -2x(\eta, x - \frac{s}{2x}, y, z) \\ \xi^\mu_{(9)} &= -2y(\eta, x, y - \frac{s}{2y}, z) \\ \xi^\mu_{(10)} &= -2z(\eta, x, y, z - \frac{s}{2z})\,. \end{aligned} \tag{6}$$

By integrating differential equations [2] of the forms

$$\left.\frac{\partial x'^{\mu}}{\partial \varepsilon}\right|_{\varepsilon=0} = \xi^{\mu}(x^{\alpha}) \tag{7}$$

we find finite symmetry transformations. Clearly, there are 3 spatial translations and 3 spatial rotations of the form exactly the same as in the flat Minkowski space-time. There is one time translation accompanied by a spatial dilatation given by

$$\begin{aligned} \eta \rightarrow \eta' &= (A+1)\eta \\ x \rightarrow x' &= (A+1)x \\ y \rightarrow y' &= (A+1)y \\ z \rightarrow z' &= (A+1)z \end{aligned} \tag{8}$$

where $A = const.$ Remaining 3 symmetries are the analogues of Lorentz boosts. The boost in x-direction is given by

$$\begin{aligned} \eta \rightarrow \eta' &= \frac{\eta}{sB^2 + 2xB + 1} \\ x \rightarrow x' &= \frac{sB + x}{sB^2 + 2xB + 1} \\ y \rightarrow y' &= \frac{y}{sB^2 + 2xB + 1} \\ z \rightarrow z' &= \frac{z}{sB^2 + 2xB + 1} \end{aligned} \tag{9}$$

where $B = const.$ The boosts in y , z-directions can be obtained from (9) simply by a cyclic permutation $x \rightarrow y \rightarrow z \rightarrow x$.

3. Symmetries of anti-de Sitter space-time

The anti-de Sitter manifold can be visualized as a hyperboloid

$$-z_0^2 - z_1^2 + z_2^2 + z_3^2 + z_4^2 = -\beta^2, \tag{10}$$

$\beta = \sqrt{-\frac{\lambda}{3}}$, embedded in a flat five-dimensional space-time

$$ds^2 = -dz_0^2 - dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 \,. \tag{11}$$

We use coordinates (η, x, y, z) covering the entire hyperboloid given by

$$\begin{aligned} z_0 &= \frac{1}{2x}(\beta^2 + s) \\ z_1 &= \beta\frac{\eta}{x} \\ z_2 &= \beta\frac{y}{x} \\ z_3 &= \beta\frac{z}{x} \\ z_4 &= \frac{1}{2x}(\beta^2 - s) \end{aligned} \tag{12}$$

where $\eta, x, y, z \in (-\infty, +\infty)$. In the global coordintes (12) anti-de Sitter metric can be written as

$$ds^2 = \frac{\beta^2}{x^2}(-d\eta^2 + dx^2 + dy^2 + dz^2)\,. \tag{13}$$

This conformally flat metric is related to the 'standard' form (see, e.g. [1]) of anti-de Sitter metric

$$ds^2 = -\cosh^2 r\, dt^2 + \beta^2 dr^2 + \beta^2 \sinh^2 r\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \tag{14}$$

by a transformation

$$\begin{aligned}
\eta &= X \cosh r\ \cos(\frac{t}{\beta}) \\
x &= X \\
y &= X \sinh r\ \cos\vartheta \\
z &= X \sinh r\ \sin\vartheta\ \cos\varphi
\end{aligned} \tag{15}$$

where $X = \beta/(\cosh r\ \sin(\frac{t}{\beta}) + \sinh r\ \sin\vartheta\ \sin\varphi)$.

In the coordinates (12) Killing vectors have the following form

$$\begin{aligned}
\xi^{\mu}_{(1)} &= (1, 0, 0, 0) \\
\xi^{\mu}_{(4)} &= (y, 0, \eta, 0) \\
\xi^{\mu}_{(5)} &= (z, 0, 0, \eta) \\
\xi^{\mu}_{(8)} &= 2\eta(\eta + \frac{s}{2\eta}, x, y, z),
\end{aligned} \tag{16}$$

six remaining vectors having the same form as in (6). Clearly, there is one time translation (in η direction), two spatial translations (in y , z-directions), two 'Minkowskian' boosts (in $\eta - y$ and $\eta - z$ planes) and one spatial rotation (in $y - z$ plane). There is also the symmetry given by (8). Remaining 3 symmetries are given by

$$\begin{aligned}
\eta \to \eta' &= \frac{sC + \eta}{-sC^2 - 2\eta C + 1} \\
x \to x' &= \frac{x}{-sC^2 - 2\eta C + 1} \\
y \to y' &= \frac{y}{-sC^2 - 2\eta C + 1} \\
z \to z' &= \frac{z}{-sC^2 - 2\eta C + 1}
\end{aligned} \tag{17}$$

where $C = const$,

$$\begin{aligned}
\eta \rightarrow \eta' &= \frac{\eta}{sD^2 + 2yD + 1} \\
x \rightarrow x' &= \frac{x}{sD^2 + 2yD + 1} \\
y \rightarrow y' &= \frac{sD + y}{sD^2 + 2yD + 1} \\
z \rightarrow z' &= \frac{z}{sD^2 + 2yD + 1}
\end{aligned} \qquad (18)$$

where $D = const$ and analogously (18) with $y \leftrightarrow z$. This completes the list of all isometries of de Sitter and anti-de Sitter space-times.

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References

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