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Inflation pressures

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Abstract. If the sum of the energy density ρ and thrice the pressure p is negative, then a Robertson–Walker universe can inflate or even oscillate. But we show at the one-loop level that in a generic grandly unified theory $\rho + 3p$ cannot be negative at the absolute minimum of the finite-temperature effective potential. We discuss the implications of this result for inflation in two grandly unified theories and speculate on the effects of gravitational friction and anti-friction.

1. The inflation pressure

In the early universe, the sum of the energy density ρ and three times the pressure p is a crucial quantity. In a homogeneous and isotropic universe, the sum $\rho + 3p$ determines the acceleration $\ddot{R}$ of the scale factor R

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p)\,. \tag{1}$$

Inflation, which is characterized by an exponentially increasing scale factor, can occur only when $\rho+3p$ is negative. If $\rho+3p$, which might be called the *inflation pressure*, is negative and constant, then the scale factor R may inflate [1–4] as

$$R(t) = \mathrm{e}^{Ht} R_0 \tag{2}$$

with $H^2 = -4\pi G(\rho + 3p)/3$. But when the inflation pressure $\rho + 3p$ is positive, the universe cannot inflate.

The inflation pressure also determines whether a uniform universe has had an initial singularity. If $\rho + 3p$ has always been positive, then $\ddot{R}$ has always been negative and the velocity $\dot{R}$ of the scale factor must always have been greater than at present. Thus at some finite time in the past the scale factor R must have vanished: there was an initial singularity.

The opposite scenario can occur in a contracting universe if the inflation pressure is negative and the Higgs field ϕ lies at a local minimum ϕ_0 of the effective potential with free energy $V(\phi_0, T) \approx V_0 - \pi^2 N T^4/90$. In this case the scale factor may vary inversely

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with the temperature, $R = a/T$; and the universe may exhibit deflation followed by inflation according to the non-singular solution [5]

$$R(t) = R_1 \left[1 + (1 - x^2)^{1/2} \cosh \frac{\sqrt{2}t}{R_1} \right]^{1/2} . \tag{3}$$

For typical values of V_0, the constants $R_1 = \sqrt{3/(16\pi G V_0)}$ and $x = 2(2\pi a/3)^2 \sqrt{6 N V_0 / 5}$ are such that the minimum radius of this bounce solution exceeds the Planck length by a factor of a million or more.

In this paper we shall discuss the inflation pressure $\rho + 3p$ in terms of the one-loop, finite-temperature effective potential [6] for a generic grandly unified model [7]. The effective potential $V(\phi, T)$ is the Gibbs free energy of the particle theory; it is a function of the Higgs field and the temperature. We shall show that whenever the Higgs field is at the absolute minimum of $V(\phi, T)$, then the inflation pressure $\rho + 3p$ cannot be negative, which implies that $\ddot{R} < 0$. This result is a constraint upon inflation and particularly upon oscillation.

We then numerically describe the behaviour of the inflation pressure in two simple, grandly unified models and the implications of that behaviour for inflation. For the $SU(5)$ model of Albrecht and Steinhardt [4, 8], inflation begins when the temperature has dropped below $T_i = 0.0755\sigma$, where σ is the scale of grand unification. In a generic supersymmetric grandly unified model, inflation cannot start until the temperature is below $T_i = 0.123\sigma$ and then not unless the Higgs field lies in the region of negative inflation pressure.

Finally we argue that the effects of gravitational friction and anti-friction may allow some portions of a locally non-uniform, contracting universe to avoid a singularity and re-expand.

2. The finite-temperature effective potential

We shall assume that the breaking and restoration of symmetry can be described by a finite-temperature Higgs mechanism [6] in a generic model of particle physics with grand unification [7] at an energy scale σ. For simplicity we shall consider only the modulus of the mean value $\phi_c \equiv \phi$ of the Higgs field. We shall also restrict ourselves to the case of weak Higgs self-coupling so that we can ignore the contributions of the scalar loops, which require special treatment [9]. If the Higgs field is described at the one-loop level by the zero-temperature effective potential $V_1(\phi)$, then for finite temperatures the one-loop effective potential $V(\phi, T)$ is of the form

$$V(\phi, T) = V_1(\phi) + T^4 I(\phi^2/T^2) . \tag{4}$$

Here the one-loop, zero-temperature effective potential $V_1(\phi)$ is the sum of the classical potential $V_0(\phi)$ and the one-loop contributions

$$V_1(\phi) = V_0(\phi) + \sum_i \frac{(-1)^{2j_i}(2j_i + 1)}{64\pi^2} m_i^4(\phi) \log(\phi^2/\sigma^2) \tag{5}$$

one for each fermion and gauge boson of spin j_i and mass $m_i(\phi) = c_i \phi$, with Dirac fermions being counted as two two-component spinors. The function $I(\phi^2/T^2)$ is a similar sum of integrals

$$I(\phi^2/T^2) = \sum_i \frac{(-1)^{2j_i}(2j_i + 1)}{2\pi^2} \int_0^\infty x^2 \log\left(1 - (-1)^{2j_i}\, \mathrm{e}^{-(x^2 + (m_i(\phi)/T)^2)^{1/2}}\right) \mathrm{d}x . \tag{6}$$

It is strictly negative for all ϕ and T,

$$I(\phi^2/T^2) < 0 . \tag{7}$$

The product $T^4 I(\phi^2/T^2)$ contains all the temperature dependence of the one-loop, finite-temperature effective potential $V(\phi, T)$.

If b is the number of gauge bosons and f the number of fermions (with electrons and positrons counted together as one), then at very high temperatures, $T \gg m_i(\phi)$ for all particles i, we may expand the effective potential as

$$V(\phi, T) \approx V_1(\phi) - \frac{\pi^2 N}{90} T^4 + \frac{n\bar{m}(\phi)^2 T^2}{24} \tag{8}$$

where N is the effective number of degrees of freedom $N = 3b + 7f/2$, $n = 3b + 2f$, and $\bar{m}(\phi) = \bar{c}\phi$ is an average mass. At such high temperatures, the absolute minimum of the effective potential is at $\phi = 0$, and the symmetry of the Lagrangian is restored.

At low temperatures, the effective potential is approximately

$$V(\phi, T) \approx V_1(\phi) - \sum_i (2j_i + 1) T^4 \left(\frac{m_i(\phi)}{2\pi T} \right)^{3/2} \mathrm{e}^{-m_i(\phi)/T} \tag{9}$$

and its minima are those of $V_1(\phi)$ which spoil the symmetry of the Lagrangian. We shall take these minima to lie at $\phi = \sigma$, the scale of grand unification.

3. The minimum of the effective potential

We shall now show that at the absolute minimum of the one-loop, finite-temperature effective potential

$$V(\phi, T) = V_1(\phi) + T^4 I(\phi^2/T^2), \tag{10}$$

which is the Gibbs free energy, the inflation pressure $\rho + 3p$ cannot be negative. Since in a homogeneous isotropic universe, a positive inflation pressure rules out inflation, this result is a constraint upon inflationary models. We shall assume that in our generic grandly unified model, the ground-state or vacuum energy vanishes so that the one-loop, zero-temperature effective potential $V_1(\phi)$ is positive except at its global minimum $\phi = \sigma$ where it is zero.

Now, the function $I(\phi^2/T^2)$ is negative as noted in (7), and so

$$V(\phi, T) \leqslant V(\phi, 0) . \tag{11}$$

Also, as its argument ϕ^2/T^2 goes to infinity, the function $I(\phi^2/T^2)$ tends monotonically to zero from below. Thus its derivative $I' = I'(\phi^2/T^2)$ with respect to its argument ϕ^2/T^2 is positive,

$$I' \geqslant 0 . \tag{12}$$

In terms of $V_1 = V_1(\phi)$, I, and I', the energy density

$$\rho = V(\phi, T) - T \partial V(\phi, T)/\partial T \tag{13}$$

is

$$\rho = V_1 - 3T^4 I + 2\phi^2 T^2 I' \tag{14}$$

and the pressure $p = -V(\phi, T)$ is

$$p = -V_1 - T^4 I \, . \tag{15}$$

So the inflation pressure $\rho + 3p$ is the sum of three terms

$$\rho + 3p = -2V_1 - 6T^4 I + 2\phi^2 T^2 I' \, . \tag{16}$$

Of these terms only the first is negative.

Now, if the field ϕ is at the absolute minimum of the effective potential $V(\phi, T)$, then we must have

$$V(\phi, T) = V_1(\phi) + T^4 I(\phi^2/T^2) \leqslant V(\sigma, T) \leqslant V(\sigma, 0) = 0 \tag{17}$$

where the second inequality follows from (11). Hence

$$-T^4 I \geqslant V_1 \geqslant 0 \tag{18}$$

and so the inflation pressure $\rho + 3p$ exceeds the sum of two non-negative terms

$$\rho + 3p \geqslant 4V_1 + 2\phi^2 T^2 I' \geqslant 0 \tag{19}$$

and therefore is non-negative.

Thus if the Higgs field remains in equilibrium at the absolute minimum of the effective potential, then

$$\ddot{R} = -4\pi G(\rho + 3p)R/3 \leqslant 0 \, . \tag{20}$$

The acceleration $\ddot{R}$ of the scale factor is then negative or zero, and neither inflation nor a bounce is possible.

4. Implications for inflation

Obviously this result is a constraint upon inflation. At the very high temperatures of an early, post-big-bang, Robertson–Walker universe, the absolute minimum of the effective potential is at $\phi = 0$, and the symmetry of the Lagrangian is restored. At such high temperatures, the inflation pressure $\rho + 3p$ is typically positive for many models of particle physics, and inflation cannot begin until the temperature has dropped considerably. For two generic, grandly unified models, we have numerically integrated the equations (4)–(6) and (14) which determine the one-loop inflation pressure. In the $SU(5)$ model of Albrecht and Steinhardt [4, 8], inflation begins when the temperature has dropped below $T_i = 0.0755\sigma$, where $\sigma = 2 \times 10^{15}$ GeV is the scale of grand unification. In a supersymmetric model, we found that inflation cannot start until the temperature has dropped below $T_i = 0.123\sigma$, where $\sigma = 10^{16}$ GeV is the scale of grand unification, and then not unless the Higgs field is in the region of negative inflation pressure when $T < T_i$.

The first model is the $SU(5)$ model of 'new inflation' discussed by Albrecht and Steinhardt [4, 8] in which there are twelve heavy gauge bosons. The one-loop zero-temperature effective potential $V_1(\phi)$ is of the Coleman–Weinberg form

$$V_1(\phi) = \frac{25\alpha_{\rm GUT}^2}{16}\left[\frac{\sigma^4}{2} + \phi^4\left(\log\frac{\phi^2}{\sigma^2} - \frac{1}{2}\right)\right] \tag{21}$$

with $\alpha_{\rm GUT} = 1/45$ at $\sigma = 2 \times 10^{15}$ GeV. The heavy gauge bosons have a common mass $m(\phi) = \sqrt{5\pi\alpha_{\rm GUT}/3}\,\phi$, and the function $I(\phi^2/T^2)$ of (6) is

$$I(\phi^2/T^2) = \frac{18}{\pi^2}\int_0^\infty x^2 \log\left(1 - {\rm e}^{-(x^2+(5\pi\alpha_{\rm GUT}/3)(\phi/T)^2)^{1/2}}\right){\rm d}x\,. \tag{22}$$

In this model twelve gauge bosons and all the fermions are light; these light particles have a negligible effect upon the effective potential. We ignore the effects of these light particles as well as those of loops of Higgs bosons.

At temperatures $T > \sigma$ above the scale of unification, the global minimum of the effective potential $V(\phi, T)$ is at $\phi = 0$ where $V(0, T) \ll 0$, and the inflation pressure $\rho + 3p$ is large and positive. At $T = \sigma$, the global minimum of $V(\phi, \sigma)$ is still at $\phi = 0$ where $V(0, \sigma) = -10\,232 V_1(0)$ as compared with its value at $\phi = \sigma$ of $V(\sigma, \sigma) = -9864 V_1(0)$. (Here and throughout our discussion of these models, the number of significant figures quoted is subjective and does not reflect the errors due to the use of the one-loop approximation.) The inflation pressure is hugely positive with a value of $\rho + 3p = 61\,395 V_1(0)$ at $\phi = 0$. When the temperature T has dropped to $T = 0.127\sigma$, a second minimum begins to develop near $\phi = 0.80\sigma$. The critical temperature $T_{\rm c}$ is 0.1065σ. The two local minima are at $\phi = 0$ and at $\phi = 0.95\sigma$ and have the equal values $V(0, T_{\rm c}) = V(0.95\sigma, T_{\rm c}) = -0.31\sigma$. These minima are separated by a maximum of $V(0.5\sigma, T_{\rm c}) = -0.016\sigma$ at $\phi = 0.5\sigma$, and so the phase transition is first order.

The inflation pressure initially turns negative when the temperature falls to $T = 0.082\sigma$, attaining the value of $\rho + 3p = -0.01 V_1(0)$ near $\phi = 0.5\sigma$. But inflation probably does not start until the temperature has dropped further to $T_{\rm i} = 0.0755\sigma$. At this temperature the inflation pressure is negative for $\phi \leqslant 0.75\sigma$. In particular the inflation pressure is negative, $\rho + 3p = -0.005 V_1(0)$, near $\phi = 0$ which is now a high local minimum of the effective potential, $V(0, T_{\rm i}) = 0.67 V_1(0)$. At $T \leqslant T_{\rm i}$, the global minimum is at $\phi = \sigma$ where $V(\sigma, T_{\rm i}) = -0.029 V_1(0)$. In this theory the point $\phi = 0$ is always a local minimum of the one-loop effective potential $V(\phi, T)$. Thus the Higgs field remains trapped at $\phi = 0$ until either the one-loop approximation loses its validity or the field ϕ tunnels through the potential barrier, which becomes ever thinner and shallower as the temperature drops. Below $T = 0.05\sigma$, the inflation pressure is negative for $|\phi - \sigma| \geqslant 0.03\sigma$. (Throughout this paper, we are using ϕ to mean the length $|\phi|$ of the vector of Higgs fields.)

In this model inflation begins at $T = T_{\rm i}$ and continues until the Higgs field enters a region in which the inflation pressure is positive or until this semi-classical picture is no longer adequate.

The second model is a nearly supersymmetric, grandly unified model with $b = 100$ kinds of gauge bosons and $f = 75$ kinds of fermions. For simplicity, we set the Yukawa constants $c_i = \frac{1}{2}$ for all i. In this case all the gauge bosons and fermions have the same mass $m_i(\phi) = \phi/2$, and the one-loop, zero-temperature effective potential reduces to the tree-level potential which we took to be $V_0(\phi) = (\lambda/4!)(\phi^2 - \sigma^2)^2$ with $\lambda = 1$ and $\sigma = 10^{16}$ GeV.

At a temperature ten times the unification scale $T = 10\sigma$, the minimum of the effective potential $V(\phi, T)$ is at $\phi = 0$, but this minimum is very shallow: $V(0, 10\sigma)$ is lower than $V(\sigma, 10\sigma)$ by less than one part in a thousand; both are large and negative, about $-1.5 \times 10^7 V_0(0)$. At this temperature the inflation pressure (13) is enormously positive, being $8.9 \times 10^7 V_0(0)$ at $\phi = 0$. At a temperature equal to the unification scale, $T = \sigma$, the inflation pressure is still huge: $\rho + 3p = 8881 V_0(0)$ at $\phi = 0$, which is the absolute minimum of the effective potential. At this temperature, the effective potential at $\phi = 0$ is $-1479 V_0(0)$ which is 7% lower than at $\phi = \sigma$. By $T = 0.2\sigma$, the inflation pressure has dropped to $12 V_0(0)$ at $\phi = 0$. The minimum of $V(\phi, 0.2\sigma)$ is still at $\phi = 0$, where it is now $-1.37 V_0(0)$ which is 54% lower than its value $V(\sigma, 0.2\sigma) = -0.89 V_0(0)$ at $\phi = \sigma$. At $T = 0.189\sigma$, a second minimum develops near $\phi = 0.65\sigma$. The critical temperature is $T_c = 0.18\sigma$, and the phase transition is first order since the minima $V(0, T_c) = V(0.8\sigma, T_c) = -0.55 V_0(0)$ are separated by a potential barrier with a maximum of -0.49σ at $\phi = 0.4\sigma$. By $T = 0.133\sigma$, the local minimum at $\phi = 0$ has disappeared, and the value there of the effective potential has risen to $V(0, 0.133\sigma) = 0.54 V_0(0)$ which is considerably greater than its minimum value of $-0.08 V_0(0)$ at $\phi = 0.97\sigma$. The Higgs field can roll down to $\phi = \sigma$. But the inflation pressure is positive for $\phi < 1.2\sigma$.

Inflation cannot start until the inflation pressure becomes negative which happens for $0.15\sigma \leqslant \phi \leqslant 0.40\sigma$ when the temperature has fallen to $T = T_i = 0.123\sigma$. If the Higgs field is still in this region of negative inflation pressure when $T \leqslant T_i$, then inflation will occur in this model. At $T = 0.122\sigma$, the inflation pressure is negative for $\phi \leqslant 0.45\sigma$. At $T = 0.122\sigma$, the symmetric point $\phi = 0$ is a local maximum of the effective potential $V(0, 0.12\sigma) = 0.67 V_0(0)$, the absolute minimum being at $\phi = \sigma$ where it is slightly negative, $V(\sigma, 0.12\sigma) = -0.04 V_0(0)$. At $T = \sigma/10$, the inflation pressure runs from $-1.11 V_0(0)$ at $\phi = 0$, where $V(0, 0.1\sigma) = 0.85 V_0(0)$, through zero at $\phi = 0.85\sigma$ to $0.09 V_0(0)$ at $\phi = \sigma$, where $V(\sigma, 0.1\sigma) = -0.01 V_0(0)$. At temperatures lower than $T = 0.05\sigma$, the absolute minima of the effective potential are near $\phi = \sigma$, and the inflation pressure is negative for $|\phi - \sigma| > 0.005\sigma$.

In this model, inflation may not occur at all. But if as the temperature drops below $T_i = 0.123\sigma$, the region of negative $\rho + 3p$ grows faster than ϕ, which is slowed by gravitational friction, then there could be substantial inflation.

5. Gravitational friction and anti-friction

In a Robertson–Walker universe, the field equation for the evolution of the mean value ϕ of the Higgs field is

$$\ddot{\phi} = -3\dot{R}\dot{\phi}/R - V'(\phi, T) \tag{23}$$

in which spatial derivatives have been suppressed and $V'(\phi, T) = \partial V(\phi, T)/\partial\phi$. The term $-3\dot{R}\dot{\phi}$ arises from the coupling of the Higgs field ϕ to the gravitational field and can force ϕ from its equilibrium position when $\dot{R}$ and $\dot{\phi}$ are appreciable. As the universe contracts, $\dot{R}$ is negative, and so $\ddot{\phi} \propto \dot{\phi}$. Thus for $\dot{R} < 0$, any motion of the Higgs field ϕ is accelerated: there is *anti-friction* or *negative* friction. As the universe expands, $\dot{R}$ is positive, and so $\ddot{\phi} \propto -\dot{\phi}$: any motion of ϕ is slowed by gravitational friction.

At the one-loop level, since $I' > 0$ for all ϕ and T, the force $-V'(\phi, T)$ due to the temperature-dependent part of the effective potential is negative where $V_0'(\phi) \approx 0$, which is true near $\phi = 0$ and $\phi = \sigma$. These regions of negative $-V'(\phi, T)$ grow with the temperature T and merge when symmetry is restored. While the field ϕ is in these regions, it is forced toward $\phi = 0$ and its speed $-\dot{\phi}$ increases. Thus, although between these regions (and before symmetry restoration) the force $-V'(\phi, T)$ is positive, it may be possible during a collapse for negative gravitational friction to push the Higgs field into a region of negative inflation pressure (closer to the symmetric point $\phi = 0$), and then for gravitational friction to keep it there during a subsequent expansion. By this mechanism it may be possible for some parts of a collapsing universe to avoid an initial singularity.

Our numerical computations of the effective potential $V(\phi, T)$ and of the inflation pressure $\rho + 3p$ for the two models studied in the last section suggest various possibilities. One scenario applies only to the $SU(5)$ model of Albrecht and Steinhardt [4, 8] and other theories in which the symmetric point $\phi = 0$ can be a local minimum of the effective potential with negative inflation pressure. In that model the inflation pressure is negative near $\phi = 0$ when $T \leqslant T_i = 0.0755\sigma$ and also negative for $|\phi - \sigma| > 0.03\sigma$ when $T \leqslant \sigma/20$. The regions of negative $-V'(\phi, T)$ grow with T and merge at $T = 0.128\sigma$ when the symmetry is restored. In this scenario the Higgs field arrives at the lip of the cup of the local minimum at $\phi = 0$ where $\rho + 3p < 0$ with so slow a speed $\dot{\phi}$ and at such a time that it is still within the cup as the universe begins to expand. The chances of such lucky timing may be very small [10]. But in a locally non-uniform universe, there are many somewhat different sets of initial conditions, and so although the timing required for a bounce may be exquisitely precise, it is plausible for the required timing to occur in some small regions. These regions then inflate.

In another scenario, which applies to both the $SU(5)$ model and the SUSY-GUT model, the Higgs field is driven by negative friction and so oscillates about the minimum of the effective potential with increasing amplitude. In such models the field ϕ may spend most of its time in regions of negative inflation pressure when $T \leqslant \sigma/20$, since at such temperatures $\rho + 3p$ is negative for $|\phi - \sigma| > 0.03\sigma$ in the $SU(5)$ model and for $|\phi - \sigma| > 0.005\sigma$ in the SUSY-GUT model. If the contraction of the universe reverses while the Higgs field is in such a region, then gravitational friction may hold it there long enough for the universe to inflate substantially.

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References

[1] Guth A H 1981 *Phys. Rev.* D **23** 347
[2] Linde A D 1982 *Phys. Lett.* **108B** 389; 1983 *Phys. Lett.* **129B** 177
[3] Linde A D 1990 *Inflation and Quantum Cosmology* (New York: Academic)
[4] Albrecht A and Steinhardt P 1982 *Phys. Rev. Lett.* **48** 1220
[5] Harrison E R 1967 *Mon. Not. R. Astronom. Soc.* **137** 69
[6] Dolan L and Jackiw R 1974 *Phys. Rev.* D **9** 3320
Weinberg S *Phys. Rev.* D **9** 3357
[7] Georgi H and Glashow S L 1974 *Phys. Rev. Lett.* **32** 438
[8] Kolb E W and Turner M S 1990 *The Early Universe* (Reading, MA: Addison-Wesley) p 291
[9] Cahill K 1993 A more-effective potential *Preprint* available from hep-ph@xxx.lanl.gov as paper 9301294
[10] Barrow J D and Matzner R 1980 *Phys. Rev.* D **21** 336