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# Null multipole particles as sources of *pp*-waves

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**Abstract**

We describe a class of impulsive gravitational *pp*-waves generated by null particles, each of which have an arbitrary multipole structure. The nature of each point source is described for all possible modes. © 1997 Elsevier Science B.V.

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The widely known class of *pp*-waves [1] (plane-fronted gravitational waves with parallel rays) is defined by the property that the space-times admit a covariantly constant null vector field. The metric can be written in the form

$$ds^2 = 2du\,dr + 2H(u,\zeta,\bar{\zeta})du^2 - d\zeta d\bar{\zeta}, \qquad (1)$$

where the complex coordinate $\zeta$ spans the plane wave surfaces. The field equations for aligned null radiation reduce to the two-dimensional Poisson equation

$$\Delta H = 4H_{\zeta\bar{\zeta}} = 8\pi T_{uu}. \qquad (2)$$

In the vacuum case, the general solution can be written in the form $H = f(u,\zeta) + \bar{f}(u,\bar{\zeta})$, where $f$ is an arbitrary function of $u$ and $\zeta$, holomorphic in $\zeta$.

Of particular interest is the case given by Aichelburg and Sexl [2] in which $f = \mu \log\zeta\,\delta(u)$, where $\mu$ is a real constant. Introducing polar coordinates, $\zeta = \rho e^{i\phi}$, we see that, in this case,

$$H = 2\mu \log\rho\,\delta(u). \qquad (3)$$

Substituting this into Eq. (2) indicates that

$$T_{uu} = \tfrac{1}{2}\,\mu\,\delta(\rho)\,\delta(u),$$

where $\delta(\rho)$ represents the two-dimensional delta function. Thus, it can be seen that this solution describes an impulsive gravitational wave generated by a single null particle located at the origin of polar coordinates in the wave surface. It was originally obtained by boosting a Schwarzschild black hole to the speed of light while its mass is reduced to zero in an appropriate way [2]. Some more complicated solutions have also been obtained [3–6] by boosting other members of the Kerr–Newman family of solutions in given directions.

We consider here a more general class of impulsive waves for which $H = H(\rho,\phi)\delta(u)$, or

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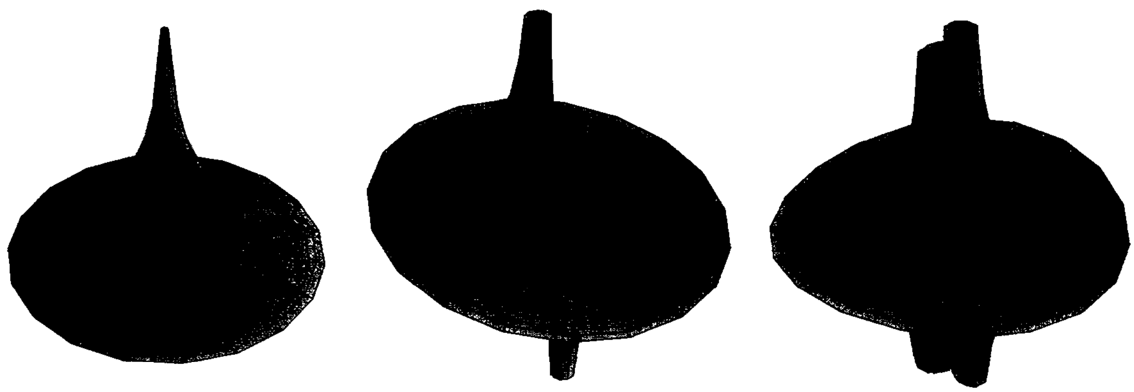

Fig. 1. Monopole, dipole and quadrupole modes showing the dependence of the functions $H_0$, $H_1$ and $H_2$ near the singular point representing the source of the impulsive waves.

$f = f(\zeta)\delta(u)$, and $T_{uu} = J(\rho,\phi)\delta(u)$. First we rewrite Eq. (2) in the form

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial H}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 H}{\partial\phi^2} = 8\pi J(\rho,\phi). \tag{4}$$

We then consider solutions that can be separated in the form $H_m = h(\rho)e^{-im\phi}$, where $m$ is a non-negative integer. For situations which are vacuum everywhere, except possibly at the origin, Eq. (4) becomes $h'' + (1/\rho)h' - (m^2/\rho^2)h = 0$ for which solutions are given by

$$\begin{aligned} h &= a_0 - b_0 \log\rho \qquad &&\text{for } m = 0, \\ &= a_m\rho^m + b_m\rho^{-m} \qquad &&\text{for } m \geqslant 1, \end{aligned}$$

where $a_m$ and $b_m$ are constants. The components $a_0$ and $a_1\rho$ can be removed by a simple transformation, while the $a_2\rho^2$ component describes a plane gravitational wave of arbitrary polarization. The component $-b_0\log\rho$ is the solution Eq. (3) above with $-b_0 = 2\mu$ which describes the impulsive gravitational wave generated by a single monopole particle. The components $a_m\rho^m$ for $m > 2$ are unbounded at infinity and will not be considered further here.

In this Letter, we concentrate on the remaining asymptotically flat cases in which $h = b_m\rho^{-m}$ for $m \geqslant 1$. These correspond to $f(\zeta) = \frac{1}{2}\beta_m\zeta^{-m}$, where $\beta_m$ is a complex constant. Putting $\beta_m = b_m e^{im\phi_m}$, the arbitrary real constants $b_m$ and $\phi_m$ describe the amplitude and phase of each component. These can then be written in the form

$$H_m = b_m\rho^{-m}\cos[m(\phi - \phi_m)].$$

It can be seen that each of these components describes a multipole solution of order $m$, with the monopole solution $H_0$ given by Eq. (3). The multipole character of the first three of these modes is clearly illustrated in Fig. 1. These terms each have the appropriate $\phi$ dependence. Further, by substituting into Eq. (4), it can be shown that the source of the $m$th mode is proportional to the $m$th derivative of the delta function $\delta(\rho)$ with respect to $\rho$.

In order to prove this statement, let us consider a sequence $h_m(\rho)$ satisfying the recurrence relation $h_{m+1} = -c_m h'_m$, where $c_0 = 1$ and $c_m = 1/m$ for $m \geqslant 1$. We then define another sequence $J_m(\rho)$ by the relation

$$h_m'' + \frac{1}{\rho}h'_m - \frac{m^2}{\rho^2}h_m = 8\pi J_m. \tag{5}$$

By differentiating Eq. (5) we obtain the recurrence relation for $J_m$,

$$J_{m+1} = -c_m J'_m - \frac{m}{4\pi\rho^2}\left(h_{m+1} - \frac{1}{\rho}h_m\right).$$

Thus, $J_1 = -J'_0$, and $J_{m+1} = -(1/m)J'_m$ for $m \geqslant 1$ provided $h_{m+1} = (1/\rho)h_m$. The required conditions are satisfied for the sequence of solutions $h_m = \rho^{-m}$, with $h_0 = -\log\rho$, introduced above. Then, since we know that $J_0 = -\frac{1}{4}\delta(\rho)$, we obtain $J_m = -\frac{1}{4}[(-1)^m/(m-1)!]\delta^{(m)}(\rho)$, and this may be considered to represent the source of the $m$th mode.

The general solution for a single particle with an arbitrary multipole structure can thus be written as a sum of all the above components,

$$H = \left( - b_0 \log \rho + \sum_{m=1}^{\infty} b_m \rho^{-m} \cos[m(\phi - \phi_m)] \right) \delta(u),$$

in which the source in the impulsive wavefront $u = 0$ is given by

$$J = -\frac{b_0}{4}\delta(\rho) - \sum_{m=1}^{\infty} \frac{b_m}{4} \frac{(-1)^m}{(m-1)!} \delta^{(m)}(\rho) \times \cos[m(\phi - \phi_m)].$$

Finally, we may observe that since Eq. (2) is linear, it follows that solutions can be constructed which contain an arbitrary number of such arbitrary multipole particles distributed arbitrarily over the wave surface. In addition, solutions are not restricted to one impulsive wave in the same space-time, and so further solutions with exotic sources may be constructed.

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