Black holes and other spherical solutions in quadratic gravity with a cosmological constant

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Abstract

We study static spherically symmetric solutions to the vacuum field equations of quadratic gravity in the presence of a cosmological constant Λ. Motivated by the trace no-hair theorem, we assume the Ricci scalar to be constant throughout a spacetime. Furthermore, we employ the conformal-to-Kundt metric ansatz that is valid for all static spherically symmetric spacetimes and leads to a considerable simplification of the field equations. We arrive at a set of two ordinary differential equations and study its solutions using the Frobenius-like approach of (infinite) power series expansions. While the indicial equations considerably restrict the set of possible leading powers, careful analysis of higher-order terms is necessary to establish the existence of the corresponding classes of solutions. We thus obtain various non-Einstein generalizations of the Schwarzschild, (anti-)de Sitter [or (A)dS for short], Nariai, and Plebański–Hacyan spacetimes. Interestingly, some classes of solutions allow for an arbitrary value of Λ, while other classes admit only discrete values of Λ. For most of these classes, we give recurrent formulas for all series coefficients. We determine which classes contain the Schwarzschild–(A)dS black hole as a special case and briefly discuss the physical interpretation of the spacetimes. In the discussion of physical properties, we naturally focus on the generalization of the Schwarzschild–(A)dS black hole, namely the Schwarzschild–Bach–(A)dS black hole, which possesses one additional Bach parameter. We also study its basic thermodynamical properties and observable effects on test particles caused by the presence of the Bach tensor. This work is a considerable extension of our letter [Phys. Rev. Lett., **121** 231104, 2018].

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1 Introduction

Despite great successes of Einstein's theory of gravity in giving predictions of various new physical phenomena, such as black holes and gravitational waves, one should keep in mind that so far this theory has been mostly tested in the weak-field regime and that tests of strong gravity have started to appear only recently [1–3]. Furthermore, even in the weak-field regime, there are attempts to incorporate the effects of dark matter and dark energy to the gravitational side of the equations by modifying the gravitational Lagrangian [4]. Perhaps more importantly, there are also strong theoretical reasons to consider higher-order corrections to the Einstein–Hilbert action to address the non-renormalizability of Einstein's gravity (see e.g. [5–8]).

Thus it is of considerable interest to study vacuum solutions, and, in particular, black hole solutions, appearing in theories of gravity with higher-order corrections. In this paper, we will focus on static spherically symmetric solutions of quadratic gravity (QG), for which quadratic terms in the curvature are added to the Einstein–Hilbert action,

$$S = \int \mathrm{d}^4x \, \sqrt{-g} \left(\gamma \, (R - 2\Lambda) + \beta \, R^2 - \alpha \, C_{abcd} \, C^{abcd} \right) , \tag{1}$$

with $\gamma = 1/G$ (G is the Newtonian constant), the cosmological constant Λ, and additional constant parameters α, β of the theory.

It is straightforward to show that all Einstein spaces $R_{ab} = \Lambda g_{ab}$ solve the corresponding field equations. Thus, in particular, the Schwarzschild–(A)dS black hole is a vacuum solution to QG. It is perhaps natural to expect that appropriate non-Einstein generalizations of Einstein spacetimes solving vacuum QG field equations might exist.[1] Indeed, it has been recently demonstrated (in the case of vanishing Λ) using in part numerical methods that another spherically symmetric black hole "over and above" the Schwarzschild solution exists in QG [9, 10]. The uniqueness of the Schwarzschild black hole in this theory is thus lost. Nevertheless, there exist some no-hair type theorems forcing black holes in QG to share certain properties with the Schwarzschild black hole. In particular, the trace no-hair theorem of [10, 11] implies that static, asymptotically flat solutions of QG with a horizon have $R = 0$ throughout the spacetime. More generally, for static spacetimes with R sufficiently quickly approaching a constant, $R = 4\Lambda$ throughout the spacetime [10]. These results lead to a considerable simplification of the field equations which, assuming $R = 4\Lambda$, take the form

$$R_{ab} - \Lambda \, g_{ab} = 4k \, B_{ab} \, , \qquad \text{with} \qquad k \equiv \frac{\alpha}{\gamma + 8\beta\Lambda} \, , \tag{2}$$

where B_{ab} is the Bach tensor defined in Eq. (4). Thus, all non-Einsteinian terms appearing in the fourth-order field equations of QG are combined into the Bach tensor.

This observation can be employed to simplify the field equations of QG for static spherically symmetric spacetimes. In particular, it has been pointed out that these spacetimes are conformal to Kundt spacetimes (in fact this applies to a larger class of the Robinson–Trautman spacetimes [12]). Furthermore, the Bach tensor is well-behaved under a conformal transformation, see (6). Therefore, it is convenient to express the Bach tensor in an appropriate Kundt background determined only by one free function, and then simply rescale it to obtain its expression in a spherically symmetric spacetime. We have employed this approach leading to a considerable simplification of the field equations in [13–15]. In the present work, we study the case with a nonvanishing cosmological constant Λ in detail[2] using the Frobenius-like approach of infinite power series expansions. Indicial equations significantly restrict

[1] In this paper, we arrive at such generalizations of the Schwarzschild, (A)dS, Nariai, and Plebański–Hacyan spacetimes.

[2] Thus extending considerably the letter [14].

the set of possible leading powers. However, careful analysis of the corresponding higher-order terms shows that some of the classes compatible with indicial equations are considerably restricted or empty. Taking all higher orders into account, we arrive at eight classes of solutions in the Kundt coordinates[3] allowing for an arbitrary value of Λ, and several additional classes of solutions allowing only for a single non-zero value or a discrete set of values of Λ. For most classes, we derive recurrent formulas for all series coefficients and briefly discuss their physical interpretation. Interestingly, several classes contain various black hole solutions. Three of them (namely the classes $[-1,3]^\infty$, $[0,1]$, and $[0,0]$, corresponding to power series expansions in the vicinity of the origin, the horizon, and an arbitrary finite point, respectively) describe (possibly distinct) generalizations of the Schwarzschild–(A)dS black hole with a nonvanishing Bach tensor and an *arbitrary* cosmological constant Λ. We call them the Schwarzschild–Bach–(A)dS black holes. Furthermore, two other classes contain three QG generalizations of the Schwarzschild–(A)dS black hole with a nonvanishing Bach tensor and *discrete* values of Λ, called the higher-order discrete Schwarzschild–Bach–(A)dS black holes (class $[-1,0]$), the extreme higher-order discrete Schwarzschild–Bach–dS black holes (a subcase of class $[0,2]$), and the extreme Bachian–dS black hole (subcase of class $[0,2]$). All the above solutions admit the Schwarzschild–(A)dS limit. In contrast, several additional classes, e.g. Bachian singularity $[1,0]$ and Bachian vacuum $[-1,2]^\infty$, do *not* contain the Schwarzschild–(A)dS as a special case. Note also that some of the solutions found in this paper, such as the Nariai spacetime, its Bachian generalization $[0,2]^\infty$ and Plebański–Hacyan solutions $[0,0]^\infty$, $[0,1]^\infty$, are Kundt spacetimes which cannot be transformed into the standard static spherically symmetric coordinates.

Physical properties of the Schwarzschild–Bach–(A)dS black hole, which is a generalization of the classic Schwarzschild–(A)dS black hole possessing one additional (Bach) parameter, are discussed in more detail. In particular, we study basic thermodynamical properties of this black hole and observable effects on test particles caused by the presence of the Bach tensor.

Our paper is organized as follows. Section 2 focuses on the preliminary material, in particular the discussion of the field equations of QG, the conformal-to-Kundt ansatz for static spherical spacetimes, and curvature invariants for these spacetimes. Field equations of QG in the conformal-to-Kundt coordinates are presented in Section 3. In addition, the classes compatible with indicial equations of the Frobenius-like analysis are summarized therein. In Section 4, these classes are derived for the expansion in powers of $\Delta \equiv r - r_0$, around a fixed point r_0, and in Section 5, they are analysed in detail. Similarly, Sections 6 and 7 are devoted to the derivation and discussion of the classes obtained by the expansion in powers of r^{-1} as $r \to \infty$, respectively. In Section 8, all solutions found in this paper that can be transformed into the standard spherically symmetric coordinates are classified in the notation used in the literature. In concluding Sec. 9, we give lists of all solutions sorted by (both Kundt and physical) regions in which the solutions are expanded. Finally, we review the derivation of the field equations in Appendix A.

2 Preliminaries: quadratic gravity, conformal-to-Kundt metric ansatz, and invariants

2.1 Quadratic gravity

Quadratic gravity is a generalization of Einstein's theory whose action contains additional terms which are quadratic in curvature. Since in four dimensions the Gauss–Bonnet term does not contribute to

[3]Out of these eight classes, seven (cf. Tables 8 and 9) can be transformed to the standard spherically symmetric coordinates and lead to twelve (cf. Table 10) distinct classes of solutions in these coordinates with a continuous value of Λ.

the field equations,[4] the action of QG in vacuum can be expressed in full generality by (1). The corresponding vacuum field equations are

$$\gamma\left(R_{ab}-\tfrac{1}{2}R\,g_{ab}+\Lambda\,g_{ab}\right)-4\alpha\,B_{ab}+2\beta\left(R_{ab}-\tfrac{1}{4}R\,g_{ab}+g_{ab}\,\Box-\nabla_b\nabla_a\right)R=0\,, \tag{3}$$

where B_{ab} is the Bach tensor

$$B_{ab}\equiv\left(\nabla^c\nabla^d+\tfrac{1}{2}R^{cd}\right)C_{acbd}\,, \tag{4}$$

or equivalently

$$B_{ab}=\tfrac{1}{2}\Box R_{ab}-\tfrac{1}{6}\left(\nabla_a\nabla_b+\tfrac{1}{2}g_{ab}\Box\right)R-\tfrac{1}{3}RR_{ab}+R_{acbd}\,R^{cd}+\tfrac{1}{4}\left(\tfrac{1}{3}R^2-R_{cd}R^{cd}\right)g_{ab}\,. \tag{5}$$

It is traceless, symmetric, conserved, and well behaved under a conformal transformation $g_{ab}=\Omega^2\tilde{g}_{ab}$:

$$g^{ab}B_{ab}=0\,,\qquad B_{ab}=B_{ba}\,,\qquad \nabla^b B_{ab}=0\,,\qquad B_{ab}=\Omega^{-2}\tilde{B}_{ab}\,. \tag{6}$$

Furthermore, from (5) it can be seen that the Bach tensor vanishes for Einstein spacetimes, i.e., for spacetimes obeying $R_{ab}=\frac{1}{4}R\,g_{ab}$, where R is constant. Consequently, in four dimensions, all vacuum solutions to the Einstein theory (including solutions with a cosmological constant) solve also the vacuum equations of QG (3).

The trace no-hair theorem of [10] implies that for static spacetimes with R sufficiently quickly approaching a constant,

$$R=4\Lambda \tag{7}$$

throughout the spacetime. Then the vacuum QG field equations (3) simplify to (2). For $k=0$, the field equations (2) reduce to vacuum Einstein's field equations of general relativity. Another special case $k\to\infty$, i.e., $\gamma=-8\beta\Lambda$, was discussed in [12].

2.2 Static spherically symmetric metrics

For our study of static spherically symmetric solutions to QG, instead of employing the standard metric

$$\mathrm{d}s^2=-h(\bar{r})\,\mathrm{d}t^2+\frac{\mathrm{d}\bar{r}^2}{f(\bar{r})}+\bar{r}^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{8}$$

we use the conformal-to-Kundt ansatz [13–15]

$$\mathrm{d}s^2\equiv\Omega^2(r)\,\mathrm{d}s^2_{\text{Kundt}}=\Omega^2(r)\Big[\,\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2-2\,\mathrm{d}u\,\mathrm{d}r+\mathcal{H}(r)\,\mathrm{d}u^2\,\Big]\,, \tag{9}$$

related to (8) by the transformation

$$\bar{r}=\Omega(r)\,,\qquad t=u-\int\frac{\mathrm{d}r}{\mathcal{H}(r)}\,, \tag{10}$$

with

$$h=-\Omega^2\,\mathcal{H}\,,\qquad f=-\left(\frac{\Omega'}{\Omega}\right)^2\mathcal{H}\,. \tag{11}$$

Prime denotes the derivative with respect to r, and the argument r of both Ω and $\mathcal{H}$ must be expressed in terms of $\bar{r}$ using the inverse of the relation $\bar{r}=\Omega(r)$.

[4]However, let us remark that recently the specific regularization method taking the Gauss–Bonnet term into account even in four dimensions has been proposed in [16]. This has been immediately followed by various explicit examples of such an approach as well as many doubts about its mathematical and physical relevance. However, here we stay on a classic level considering the Gauss–Bonnet term being irrelevant in a four-dimensional theory.

In (9), the seed metric $\mathrm{d}s^2_{\text{Kundt}}$ is of Petrov type D (consequently, also the full metric (9) is of type D) and it is a direct product [12, 17] of two 2-spaces. It belongs to the Kundt class, admitting nonexpanding, shear-free, and twist-free null congruence, see [17,18]. The first spacelike part, spanned by θ, ϕ, is a round 2-sphere of Gaussian curvature $K = 1$, while the second part, spanned by u, r, is a two-dimensional Lorentzian spacetime. Using the stereographic representation of a 2-sphere given by $x + \mathrm{i}\, y = 2\tan(\theta/2)\exp(\mathrm{i}\phi)$, the Kundt seed metric takes the form

$$\mathrm{d}s^2_{\text{Kundt}} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\left(1 + \frac{1}{4}(x^2 + y^2)\right)^2} - 2\,\mathrm{d}u\,\mathrm{d}r + \mathcal{H}(r)\,\mathrm{d}u^2 \,. \tag{12}$$

The metric (9) admits a *gauge freedom* given by a constant rescaling and a shift of r,

$$r \to \lambda\, r + \upsilon\,, \qquad u \to \lambda^{-1}\, u\,. \tag{13}$$

Note that for the classic Schwarzschild–(A)dS metric

$$f(\bar{r}) = h(\bar{r}) = 1 - \frac{2m}{\bar{r}} - \frac{\Lambda}{3}\bar{r}^2\,, \tag{14}$$

the relations (10), (11) imply

$$\bar{r} = \Omega(r) = -\frac{1}{r}\,, \qquad \mathcal{H}(r) = \frac{\Lambda}{3} - r^2 - 2m\, r^3\,. \tag{15}$$

The horizon is located at the zeros of the metric function $\mathcal{H}$, where $h(\bar{r})$ and $f(\bar{r})$ also vanish by (11).

Similarly, in a general case, a horizon can be defined as the Killing horizon associated with the vector field ∂_u, which coincides with ∂_t, i.e., located at $r = r_h$ satisfying

$$\mathcal{H}\big|_{r=r_h} = 0\,, \tag{16}$$

since Ω is everywhere nonvanishing. Using (11), this condition corresponds to $h(\bar{r}_h) = 0 = f(\bar{r}_h)$.

Note that a time-scaling freedom of the metric (8)

$$t \to t/\sigma\,, \tag{17}$$

where $\sigma \neq 0$ is any constant, can be used to adjust a value of h at a chosen radius $\bar{r}$ since $h \to h\,\sigma^2$.

2.3 Curvature invariants and geometric classification

In [15], we have observed that for a geometrical and physical interpretation of solutions to the QG field equations, the scalar curvature invariants constructed from the Ricci, Bach, and Weyl tensors play an important role.

For the static spherically symmetric metric (9), we get

$$R_{ab}\, R^{ab} = 4\Lambda^2 + 16k^2\, B_{ab}B^{ab}\,, \tag{18}$$

$$B_{ab}\, B^{ab} = \tfrac{1}{72}\,\Omega^{-8}\left[(\mathcal{B}_1)^2 + 2(\mathcal{B}_1 + \mathcal{B}_2)^2\right], \tag{19}$$

$$C_{abcd}\, C^{abcd} = \tfrac{1}{3}\,\Omega^{-4}\left(\mathcal{H}'' + 2\right)^2, \tag{20}$$

where the functions $\mathcal{B}_1(r)$ and $\mathcal{B}_2(r)$ denote two independent components of the Bach tensor,

$$\mathcal{B}_1 \equiv \mathcal{H}\mathcal{H}''''\,, \tag{21}$$

$$\mathcal{B}_2 \equiv \mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2\,. \tag{22}$$

To derive (18)–(22), we have used expressions for the Ricci, Bach, and Weyl tensors given in Appendix A and Appendix B of [15] and the invariance of the Weyl tensor under conformal transformations, $C_{abcd}\,C^{abcd} = \Omega^{-4}\,C^{\text{Kundt}}_{abcd}\,C^{abcd}_{\text{Kundt}}$.

Note that the Bach component $\mathcal{B}_1 = \mathcal{H}\mathcal{H}''''$ vanishes on the horizon where $\mathcal{H}=0$, see (16). Similarly as in [15], one can show that

$$B_{ab} = 0 \quad \text{if, and only if,} \quad B_{ab}\,B^{ab} = 0\,, \tag{23}$$

and

$$C_{abcd}\,C^{abcd} = 0 \quad \text{implies} \quad B_{ab} = 0\,. \tag{24}$$

There are two geometrically distinct classes of solutions to QG field equations, depending on the Bach tensor B_{ab}, namely a simple case corresponding to $B_{ab}=0$, and a case with $B_{ab}\neq 0$, not allowed in general relativity. Later we will see that the Bach tensor influences various physical aspects of the solutions, such as the geodesic deviation equation for test particles and entropy of black holes.

3 The field equations

To derive an explicit form of the field equations, we proceed as in [15] using the conformal-to-Kundt metric ansatz (9), with the Ricci and Bach tensors for the Kundt seed metric g^{Kundt}_{ab} and a spherically symmetric metric (9) given in Appendices A and B of [15], respectively. Employing also the Bianchi identities, the QG field equations (2) reduce to an autonomous system of two compact ordinary differential equations for the two metric functions $\Omega(r)$ and $\mathcal{H}(r)$, see Appendix A here,

$$\Omega\Omega'' - 2\Omega'^2 = \tfrac{1}{3}k\,\mathcal{B}_1\mathcal{H}^{-1}\,, \tag{25}$$

$$\Omega\Omega'\mathcal{H}' + 3\Omega'^2\mathcal{H} + \Omega^2 - \Lambda\Omega^4 = \tfrac{1}{3}k\,\mathcal{B}_2\,, \tag{26}$$

where the functions $\mathcal{B}_1(r)$ and $\mathcal{B}_2(r)$ denote two independent components of the Bach tensor, (21) and (22), respectively.

The trace (7) of the field equations (2) takes the form

$$\mathcal{H}\Omega'' + \mathcal{H}'\Omega' + \tfrac{1}{6}(\mathcal{H}'' + 2)\Omega = \tfrac{2}{3}\Lambda\Omega^3\,. \tag{27}$$

In fact, (27) can be obtained by subtracting (25) multiplied by $\mathcal{H}'$ from the derivative of (26) and dividing the result by $6\Omega'$.

Note that similarly as in [15], vanishing of the Bach tensor implies the Schwarzschild–(A)dS solution (15) with the following scalar invariants (18)–(20)

$$R_{ab}\,R^{ab} = 4\Lambda^2\,, \qquad B_{ab}\,B^{ab} = 0\,, \qquad C_{abcd}\,C^{abcd} = 48\,m^2\,r^6\,. \tag{28}$$

For $m \neq 0$, there is a curvature singularity at $r\to\infty$ corresponding to $\bar{r} = \Omega(r) = 0$.[5]

In the rest of this paper, we concentrate on *solutions with a nontrivial Bach tensor*. In this case, the system (25), (26) is coupled in a complicated way and it seems hopeless to find explicit solutions in a closed form. Thus we focus on studying these solutions in terms of (infinite) power series. Since the system (25), (26) is autonomous, there are only two natural possibilities — the expansion in powers of the parameter $\Delta \equiv r - r_0$, which expresses the solution around any finite value r_0, and the expansion in powers of r^{-1}, which is applicable for sufficiently large values of r.

[5]For brevity, in this paper the symbol $r\to\infty$ means $|r|\to\infty$, unless the sign of r is explicitly specified.

3.1 Expansion in powers of $\Delta \equiv r - r_0$

We search for solutions of (25), (26) in the form of an expansion in powers[6] of $r - r_0$ around any fixed value r_0,

$$\Omega(r) = \Delta^n \sum_{i=0}^{\infty} a_i \, \Delta^i \,, \tag{29}$$

$$\mathcal{H}(r) = \Delta^p \sum_{i=0}^{\infty} c_i \, \Delta^i \,, \tag{30}$$

where

$$\Delta \equiv r - r_0 \,, \tag{31}$$

r_0 is a real constant, and $i = 0, 1, 2, \ldots$ are integers, so that the metric functions are expanded in integer steps of $\Delta = r - r_0$. On the other hand, the dominant real powers n and p in the expansions (29) and (30) need not be integers. We only assume that $a_0 \neq 0$ and $c_0 \neq 0$, so that the coefficients n and p are the leading powers.

By inserting (29)–(31) into the field equations (25) and (26), we will show in Section 4 that only the following *eight classes of solutions* are compatible with the leading orders of the expansion in Δ:

$$[n, p] = [-1, 2]\,,\ [0, 1]\,,\ [0, 0]\,,\ [0, 2]\,,\ [-1, 0]\,,\ [1, 0]\,,\ [0, >2]\,,\ [<0, 2(n+1) < 0]\,. \tag{32}$$

Section 5 contains more detailed analysis of these solutions, namely

- Sec. 5.1: class $[-1, 2]$, already discussed in [15], contains only the Schwarzchild black hole for which the Bach tensor vanishes;
- Sec. 5.2: class $[0, 1]$ contains the Schwarzschild–Bach–(anti-)de Sitter black hole (abbreviated as Schwarzschild–Bach–(A)dS, or even shorter as Schwa–Bach–(A)dS) with a nonvanishing Bach tensor;
- Sec. 5.4: class $[0, 0]$ describes all other discussed solutions at generic points;
- Sec. 5.5: class $[0, 2]$, apart from the extreme Schwarzschild–dS solution with a generic Λ, includes also the extreme higher-order (discrete) Schwarzschild–Bach–dS black holes with restricted values of Λ, and the extreme Bachian black hole with $\Lambda = 3/(8k)$;
- Sec. 5.6: class $[-1, 0]$, apart from the Schwarzschild–(A)dS black hole for a generic Λ, contains also the higher-order (discrete) Schwarzschild–Bach–(A)dS black holes admitting only special values of Λ;
- Sec. 5.7: class $[1, 0]$ describes a Bachian singularity at the origin;
- Sec. 5.8: class $[0, >2]$ is, in fact, empty;
- Sec. 5.9: classes $[<0, 2n+2]$, requiring discrete values of n and Λ, describe asymptotic regions of solutions with a strictly nonvanishing Bach tensor.

[6]Note that other solutions containing for example $\log(r - r_0)$ terms may also exist.

3.2 Expansion in powers of r^{-1}

Analogously, we may study and classify all possible solutions to the vacuum QG field equations for an asymptotic expansion as $r \to \infty$. Instead of (29), (30) with (31), for very large r we can assume that the metric functions $\Omega(r)$, $\mathcal{H}(r)$ are expanded in *negative powers* of r as

$$\Omega(r) = r^N \sum_{i=0}^{\infty} A_i \, r^{-i} \,, \tag{33}$$

$$\mathcal{H}(r) = r^P \sum_{i=0}^{\infty} C_i \, r^{-i} \,. \tag{34}$$

By inserting the series (33), (34) into the field equations (25), (26), it can be shown that the following five classes of solutions are compatible with the leading orders of the expansion in r^{-1}:

$$[N, P] = [-1, 3]^{\infty}\,,\ [-1, 2]^{\infty}\,,\ [0, 2]^{\infty}\,,\ [0, < 2]^{\infty}\,,\ [> 0, 2N + 2]^{\infty}\,, \tag{35}$$

see Section 6. Subsequent Section 7 contains more detailed analysis of the above solutions, namely

- Sec. 7.1: class $[-1, 3]^{\infty}$ describes the Schwarzschild–Bach–(A)dS black hole near the singularity at the origin;
- Sec. 7.2: class $[-1, 2]^{\infty}$ describes Bachian–(A)dS vacuum near the origin — a specific Bachian generalization of the (A)dS space;
- Sec. 7.3: class $[0, 2]^{\infty}$ contains the exact Nariai spacetime with arbitrary Λ, the spherically symmetric higher-order discrete Nariai–Bach solutions near a finite point with a nonvanishing Bach tensor and discrete spectrum of Λ, and another Bachian generalization of the Nariai spacetime belonging to the Kundt class;
- Sec. 7.4: classes $[0, < 2]^{\infty}$ contain only exact Plebański–Hacyan solutions in the Kundt class $[0, 1]^{\infty}$ and $[0, 0]^{\infty}$ with $\Lambda = \frac{3}{8k}$;
- Sec. 7.5: classes $[N > 0, P = 2N + 2]^{\infty}$ contain solutions with regular Bachinan infinity, which require discrete values of N and Λ and describe asymptotic regions of solutions with a strictly nonvanishing Bach tensor.

4 Discussion of solutions using the expansion in powers of Δ

The series (29), (30) together with the first field equation (25) yield

$$\sum_{l=2n-2}^{\infty} \Delta^l \sum_{i=0}^{l-2n+2} a_i\, a_{l-i-2n+2}\,(l-i-n+2)(l-3i-3n+1) = \tfrac{1}{3}k \sum_{l=p-4}^{\infty} \Delta^l\, c_{l-p+4}\,(l+4)(l+3)(l+2)(l+1)\,, \tag{36}$$

while the second field equation (26) gives

$$\sum_{l=2n+p-2}^{\infty} \Delta^l \sum_{j=0}^{l-2n-p+2} \sum_{i=0}^{j} a_i\, a_{j-i}\, c_{l-j-2n-p+2}\,(j-i+n)(l-j+3i+n+2) + \sum_{l=2n}^{\infty} \Delta^l \sum_{i=0}^{l-2n} a_i\, a_{l-i-2n} - \Lambda \sum_{l=4n}^{\infty} \Delta^l \sum_{m=0}^{l-4n} \Big(\sum_{i=0}^{m} a_i\, a_{m-i}\Big)\Big(\sum_{j=0}^{l-m-4n} a_j\, a_{l-m-j-4n}\Big)$$
$$= \tfrac{1}{3}k\Big[2 + \sum_{l=2p-4}^{\infty} \Delta^l \sum_{i=0}^{l-2p+4} c_i\, c_{l-i-2p+4}\,(i+p)(l-i-p+4)(l-i-p+3)(l-\tfrac{3}{2}i-\tfrac{3}{2}p+\tfrac{5}{2})\Big]\,. \tag{37}$$

It is also useful to consider considerably simpler constraints following from the trace equation (27)

$$\sum_{l=n+p-2}^{\infty} \Delta^l \sum_{i=0}^{l-n-p+2} c_i\, a_{l-i-n-p+2}\,\big[(l-i-p+2)(l+1) + \tfrac{1}{6}(i+p)(i+p-1)\big] + \tfrac{1}{3}\sum_{l=n}^{\infty} \Delta^l\, a_{l-n} = \tfrac{2}{3}\Lambda \sum_{l=3n}^{\infty} \Delta^l \sum_{j=0}^{l-3n} \sum_{i=0}^{j} a_i\, a_{j-i}\, a_{l-j-3n}\,. \tag{38}$$

Coefficients of the same powers of Δ^l in Eq. (36) give expressions for the coefficients c_j in terms of a_j. Since the lowest orders on the left and right sides are $l = 2n-2$ and $l = p-4$, respectively, there are three cases to be considered

- Case I: $2n-2 < p-4\,,$ i.e., $p > 2n+2\,,$
- Case II: $2n-2 > p-4\,,$ i.e., $p < 2n+2\,,$
- Case III: $2n-2 = p-4\,,$ i.e., $p = 2n+2\,.$

Notice that Eq. (36) does not depend on the cosmological constant Λ and thus the above cases do not differ from the $\Lambda = 0$ cases discussed systematically in [15].

In what follows, we study various solutions in these three cases.

4.1 Case I

In Case I, the lowest order in (36) is on the left hand side (Δ^{2n-2}) and therefore, since $a_0 \neq 0$,

$$n(n+1) = 0\,, \tag{39}$$

leading to two possible cases $n = 0$ and $n = -1$. The lowest orders of Eq. (38) are

$$\big[6n(n+p-1) + p(p-1)\big]c_0\, \Delta^{n+p-2} + \cdots + 2\,\Delta^n + \cdots - 4\Lambda a_0^2 \Delta^{3n} + \cdots = 0\,. \tag{40}$$

For $n = 0$, these powers are Δ^{p-2}, Δ^0, and Δ^0, respectively, but in Case I, $p-2 > 2n = 0$. The lowest order $(2 - 4\Lambda a_0^2)\Delta^0$ thus leads to

$$2\Lambda a_0^2 = 1\,, \tag{41}$$

and Eq. (37) then gives

$$\Lambda = \frac{3}{8k}\,. \tag{42}$$

Thus such class exists only for nonvanishing Λ of this special value.

For $n = -1$, Eq. (40) reduces to

$$(p-3)(p-4)c_0\,\Delta^{p-3} + \cdots + 2\,\Delta^{-1} + \cdots - 4\Lambda a_0^2\Delta^{-3} + \cdots = 0\,. \tag{43}$$

Since $c_0 \neq 0 \neq a_0$, the only possibility is $p = 2$, $\Lambda = 0$, with $c_0 = -1$.

To summarize: The only possible classes of solutions in Case I are

$$[n,p] = [-1,2] \qquad \text{with} \quad \Lambda = 0\,, \quad c_0 = -1\,, \tag{44}$$

$$[n,p] = [0, p>2] \quad \text{with} \quad \Lambda = \frac{3}{8k}\,, \quad a_0^2 = \frac{1}{2\Lambda}\,. \tag{45}$$

4.2 Case II

In Case II, $2n - 2 > p - 4$, the lowest order in (36) is Δ^{p-4}, implying

$$p(p-1)(p-2)(p-3) = 0\,. \tag{46}$$

Therefore, as in [15], there are four possible cases $p = 0$, $p = 1$, $p = 2$, and $p = 3$. Equation (38) has the following lowest orders (see (40))

$$\text{for} \quad p=0: \quad \big[6n(n-1)\big]c_0\,\Delta^{n-2} + \cdots = -2\,\Delta^n + \cdots + 4\Lambda a_0^2\Delta^{3n} + \cdots \qquad \text{necessarily} \quad n = 0, 1\,, \tag{47}$$

$$\text{for} \quad p=1: \quad \big[6n^2\big]c_0\,\Delta^{n-1} + \cdots = -2\,\Delta^n + \cdots + 4\Lambda a_0^2\Delta^{3n} + \cdots \qquad \text{necessarily} \quad n = 0\,, \tag{48}$$

$$\text{for} \quad p=2: \quad \big[6n(n+1)+2\big]c_0\,\Delta^{n} + \cdots = -2\,\Delta^n + \cdots + 4\Lambda a_0^2\Delta^{3n} + \cdots \qquad (3n^2+3n+1)c_0 = -1\,, \tag{49}$$

$$\text{for} \quad p=3: \quad \big[6n(n+2)+6\big]c_0\,\Delta^{n+1} + \cdots = -2\,\Delta^n + \cdots + 4\Lambda a_0^2\Delta^{3n} + \cdots \quad \text{not compatible}\,. \tag{50}$$

The lowest orders in (37) for $p = 2$, implying $n > 0$, are

$$3a_0^2\,[n(3n+2)c_0 + 1]\,\Delta^{2n} + 2k(c_0^2 - 1) - 3\Lambda a_0^4\Delta^{4n} + \cdots = 0\,, \tag{51}$$

and thus $c_0 = \pm 1$, however, the constraint (49), $3n^2 + 3n + 1 = \pm 1$, is in a contradiction with $n > 0$ and therefore, the $p = 2$ case is not allowed.

To summarize: The only three possible classes of solutions in Case II are given by

$$[n,p] = [0,1]\,, \qquad [n,p] = [0,0]\,, \qquad [n,p] = [1,0]\,. \tag{52}$$

4.3 Case III

In Case III, $2n-2=p-4$, i.e., $p=2n+2$. Then the lowest order Δ^{p-4} in (36) implies

$$p(p-2)\big[3a_0^2+4kc_0(p-1)(p-3)\big]=0\,. \tag{53}$$

Therefore, as in [15], there are three subcases $p=0$, $p=2$, and $3a_0^2=-4kc_0(p-1)(p-3)$ with $p\neq 0,1,2,3$, corresponding to $n=-1$, $n=0$, and $3a_0^2=-4kc_0(4n^2-1)$ with $n\neq -1,-1/2,0,1/2$, respectively. The leading orders of (38) read

$$(11n^2+6n+1)c_0\,\Delta^{3n}+\cdots\;=\;-\,\Delta^n+\cdots+2\Lambda a_0^2\Delta^{3n}\cdots\,, \tag{54}$$

which implies $n\leq 0$ and

$$\begin{aligned}
&\text{for}\quad n=-1\Leftrightarrow p=0: && 6c_0\,\Delta^{-3}+\cdots=-\,\Delta^{-1}+\cdots+2\Lambda a_0^2\Delta^{-3}+\cdots \quad\Rightarrow c_0=\tfrac{\Lambda}{3}a_0^2\,, && (55)\\
&\text{for}\quad n=0\Leftrightarrow p=2: && c_0+\cdots=-1+\cdots+2\Lambda a_0^2\cdots \quad\Rightarrow c_0=2\Lambda a_0^2-1\,, && (56)\\
&\text{for}\quad 3a_0^2=4kc_0(1-4n^2): && (11n^2+6n+1)c_0+\cdots=2\Lambda a_0^2+\cdots\,. && (57)
\end{aligned}$$

In the case $[0,2]$, Eq. (37) implies

$$3a_0^2(1-\Lambda a_0^2)+2k(c_0^2-1)=0\,, \tag{58}$$

that gives either

$$\Lambda a_0^2=1\;\;\Rightarrow\;\;c_0=1\,, \tag{59}$$

or

$$\Lambda=\frac{3}{8k}\;\;\Rightarrow\;\;c_0=\frac{3}{4k}a_0^2-1\,. \tag{60}$$

In the case $3a_0^2=4kc_0(1-4n^2)$, with $p\neq 0,1,2,3$, and thus $n\neq -1,-1/2,0,1/2$, respectively, Eq. (57) with $p\neq 0,1$ ($n\neq -1,-1/2$, respectively, $p,n<0$), using $a_0^2=\frac{4}{3}kc_0(1-4n^2)$, gives

$$\Lambda=\frac{3}{8k}\frac{11n^2+6n+1}{1-4n^2}\qquad\Rightarrow\qquad c_0=\frac{3}{4k}\frac{a_0^2}{1-4n^2}\,. \tag{61}$$

Equation (37) is then identically satisfied.

To summarize: In Case III, there are three possible classes of solutions

$$[n,p]\;=\;[-1,0]\quad\text{with}\quad c_0=\frac{\Lambda}{3}a_0^2\,, \tag{62}$$

$$[n,p]\;=\;[0,2]\quad\text{with}\quad\text{either}\quad \Lambda a_0^2=1\,,\;c_0=1\,,\quad\text{or}\quad \Lambda=\frac{3}{8k}\,,\;\;c_0=\frac{3}{4k}a_0^2-1\,, \tag{63}$$

$$[n,p]\;=\;[n<0,2n+2<0]\quad\text{with}\quad \Lambda=\frac{3}{8k}\frac{11n^2+6n+1}{1-4n^2}\,,\;c_0=\frac{3}{4k}\frac{a_0^2}{1-4n^2}\,. \tag{64}$$

5 Description and study of all possible solutions in powers of Δ

In this section, we will investigate all the solutions contained in Cases I, II, and III, namely eight classes (44), (45) (52), (62), (63), and (64).

5.1 Uniqueness of the Schwarzschild black hole in the class $[n,p]=[-1,2]$

The class $[n,p]=[-1,2]$ in Case I, see (44), necessarily has $\Lambda=0$ and therefore it has been already studied in detail in [15]. Therein, we have shown that the only solution in this class is the Schwarzschild solution given by

$$\Omega(r) = -\frac{1}{r}\,, \qquad \mathcal{H}(r) = -r^2 - 2m\,r^3\,, \tag{65}$$

see (15).

5.2 Schwarzschild–Bach–(A)dS black hole in the class $[n,p]=[0,1]$: near the horizon

In this section, we will present a detailed derivation of the metric of the class $[0,1]$ that represents a spherically symmetric non-Schwarzschild solution to QG, the Schwarzschild–Bach–(A)dS black hole with a nonvanishing Bach tensor and a cosmological constant, as we pointed out already in [14]. The first three terms in the expansion of such a solution read

$$\Omega(r) = -\frac{1}{r} + b\frac{(r-r_h)}{\rho\, r_h^2} - \frac{b}{r_h}\left[2 - \left(\frac{7}{3}\Lambda - \frac{1}{8k}\right)\frac{1}{r_h^2} + b\right]\left(\frac{r-r_h}{\rho\, r_h}\right)^2 + \cdots\,, \tag{66}$$

$$\begin{aligned}\mathcal{H}(r) = (r-r_h)\bigg\{&\frac{r^2}{r_h} - \frac{\Lambda}{3\,r_h^3}\left(r^2 + r r_h + r_h^2\right)\\ &+ 3b\,\rho\, r_h\left[\left(\frac{r-r_h}{\rho\, r_h}\right) + \frac{1}{3}\left[4 - \frac{1}{r_h^2}\left(2\Lambda + \frac{1}{2k}\right) + 3b\right]\left(\frac{r-r_h}{\rho\, r_h}\right)^2 + \cdots\right]\bigg\}\,,\end{aligned} \tag{67}$$

where

$$\rho \equiv 1 - \frac{\Lambda}{r_h^2}\,, \tag{68}$$

and

$$r_0 \equiv r_h \tag{69}$$

is the *black hole horizon* since $\mathcal{H}(r_h)=0$. For a vanishing *"Bach parameter"*, $b=0$, the metric functions (66), (67) immediately reduce to the Schwarzschild–(A)dS metric functions (15) with a vanishing Bach tensor.

Let us derive the complete analytical form of this black hole leading to (66) and (67). For $[n,p]=[0,1]$, Eq. (36) reduces to

$$\sum_{i=0}^{l+1} a_i\, a_{l+2-i}\,(l+2-i)(l+1-3i) = \tfrac{1}{3}k\, c_{l+3}\,(l+4)(l+3)(l+2)(l+1)\,, \tag{70}$$

where $l \geq 0$. Relabeling $l \to l-1$, Eq. (70) gives

$$c_{l+2} = \frac{3}{k\,(l+3)(l+2)(l+1)l}\sum_{i=0}^{l} a_i\, a_{l+1-i}(l+1-i)(l-3i) \qquad \forall\; l \geq 1\,, \tag{71}$$

and thus all coefficients c_{l+2}, starting from c_3, can be expressed in terms of $a_0, \dots, a_{l+1}$.

The lowest nontrivial order $l=0$ of the "trace equation" (38) gives

$$a_1 = \frac{a_0}{3c_0}\left[2\Lambda a_0^2 - (1+c_1)\right], \tag{72}$$

and higher orders $l = 1, 2, \dots$ read

$$(l+1)^2 c_0 a_{l+1} = \tfrac{2}{3}\Lambda \sum_{j=0}^{l}\sum_{i=0}^{j} a_i a_{j-i} a_{l-j} - \tfrac{1}{3}\, a_l - \sum_{i=1}^{l+1} c_i a_{l-i+1}\left[(l-i+1)(l+1) + \tfrac{1}{6} i(i+1)\right]. \tag{73}$$

This leads (after relabeling $l \to l-1$) to

$$a_l = \frac{1}{l^2 c_0}\Big[\tfrac{2}{3}\Lambda \sum_{j=0}^{l-1}\sum_{i=0}^{j} a_i\, a_{j-i}\, a_{l-1-j} - \tfrac{1}{3}\, a_{l-1} - \sum_{i=1}^{l} c_i\, a_{l-i}\left(l(l-i) + \tfrac{1}{6} i(i+1)\right)\Big] \quad \forall\, l \geq 2, \tag{74}$$

which gives all the coefficients a_l in terms of $a_0, \dots, a_{l-1}$ and $c_0, \dots, c_l$.

In addition, the lowest nontrivial order $l=0$ of (37) gives the constraint

$$6kc_0c_2 = 3a_0[a_1c_0 + a_0(1-\Lambda a_0^2)] + 2k(c_1^2 - 1), \tag{75}$$

which, using (72), becomes

$$c_2 = \frac{1}{6kc_0}\left[2k(c_1^2-1) + a_0^2(2 - c_1 - \Lambda a_0^2)\right]. \tag{76}$$

Therefore, this class of solutions has three free parameters a_0, c_0, and c_1. Then a_1, c_2 are determined by (72), (76), respectively, and all other coefficients a_{l+1}, c_{l+2} for all $l = 1, 2, \dots$ can be obtained using the recurrent relations (74), (71), respectively.

The Bach and Weyl invariants (19), (20) at $r = r_h \equiv r_0$ read

$$B_{ab}\, B^{ab}(r_h) = \left(\frac{1 - c_1^2 + 3c_0c_2}{3a_0^4}\right)^2 = \left(\frac{c_1 - 2 + \Lambda a_0^2}{6ka_0^2}\right)^2 = \frac{b^2}{4k^2a_0^4}, \tag{77}$$

$$C_{abcd}\, C^{abcd}(r_h) = \frac{4}{3a_0^4}(1+c_1)^2, \tag{78}$$

where we have introduced a key Bach parameter b by

$$b \equiv \frac{1}{3}(c_1 - 2 + \Lambda a_0^2). \tag{79}$$

The Bach tensor B_{ab} is in general nonvanishing. Using the recurrent relations (74) and (71), the first few coefficients read

$$
\begin{aligned}
a_1 &= -\frac{a_0}{c_0}\Big((1-\Lambda a_0^2)+b\Big)\,, \qquad (80)\\
a_2 &= +\frac{a_0}{c_0^2}\Big((1-\Lambda a_0^2)^2+\Big[2-\Big(\frac{7}{3}\Lambda-\frac{1}{8k}\Big)a_0^2\Big]b+b^2\Big)\,,\\
a_3 &= -\frac{a_0}{c_0^3}\Big((1-\Lambda a_0^2)^3+\frac{1}{9}\Big[25-\Big(\frac{179}{3}\Lambda-\frac{29}{8k}\Big)a_0^2+\Big(\frac{298}{9}\Lambda^2-\frac{77}{24k}\Lambda+\frac{1}{16k^2}\Big)a_0^4\Big]b\\
&\qquad\qquad +\frac{1}{9}\Big[23-\Big(\frac{104}{3}\Lambda-\frac{35}{8k}\Big)a_0^2\Big]b^2+\frac{7}{9}b^3\Big)\,,\dots\,,\\
c_1 &= 2-\Lambda a_0^2+3b\,, \qquad (81)\\
c_2 &= \frac{1}{3c_0}\Big((1-\Lambda a_0^2)(3-\Lambda a_0^2)+3\Big[4-\Big(2\Lambda+\frac{1}{2k}\Big)a_0^2\Big]b+9b^2\Big)\,,\\
c_3 &= \frac{a_0^4}{2kc_0^2}\,b\Big(-\frac{1}{6}\Lambda+\frac{1}{16k}\Big)\,,\\
c_4 &= \frac{a_0^2}{270kc_0^3}\,b\Big(9+\Big(12\Lambda-\frac{45}{4k}\Big)a_0^2-\Big(14\Lambda^2-\frac{75}{8k}\Lambda+\frac{9}{32k^2}\Big)a_0^4+3\Big[6+\Big(7\Lambda-\frac{39}{8k}\Big)a_0^2\Big]b+9b^2\Big)\,,\dots\,,
\end{aligned}
$$

where a_0, c_0, and b are free parameters.

5.2.1 Identification of the Schwarzschild–(A)dS black hole

To identify the Schwarzschild–(A)dS black hole, we must set $B_{ab}=0$, i.e., we choose $b=0$. Then the expansion coefficients (80), (81) simplify to

$$a_i = a_0\left(-\frac{1-\Lambda a_0^2}{c_0}\right)^i \quad \text{for all } i\geq 0\,, \qquad (82)$$

$$c_1 = 2-\Lambda a_0^2\,, \quad c_2 = \frac{1}{3c_0}(1-\Lambda a_0^2)(3-\Lambda a_0^2)\,, \quad c_i = 0 \quad \text{for all } i\geq 3\,, \qquad (83)$$

where the first sequence is a geometrical series and the second series is truncated to a polynomial of the 3rd order. Thus the metric functions take the explicit closed form

$$
\begin{aligned}
\Omega(r) &= a_0\sum_{i=0}^{\infty}\Big(-(1-\Lambda a_0^2)\frac{\Delta}{c_0}\Big)^i = \frac{a_0\,c_0}{c_0+(1-\Lambda a_0^2)\Delta} = \frac{a_0\,c_0}{(1-\Lambda a_0^2)(r-r_h)+c_0}\,, \qquad (84)\\
\mathcal{H}(r) &= c_0(r-r_h)+c_1(r-r_h)^2+c_2(r-r_h)^3\,. \qquad (85)
\end{aligned}
$$

Using the gauge freedom (13), we can always set

$$a_0 = -\frac{1}{r_h}\,, \qquad c_0 = r_h-\frac{\Lambda}{r_h}\,, \qquad (86)$$

so that the metric functions reduce to

$$\bar{r} = \Omega(r) = -\frac{1}{r}\,, \qquad \mathcal{H}(r) = \frac{\Lambda}{3}-r^2-\Big(\frac{\Lambda}{3}-r_h^2\Big)\frac{r^3}{r_h^3}\,. \qquad (87)$$

Thus for $b=0$ we recover the Schwarzschild–(A)dS solution (15), with the black hole horizon located at $r=r_h$, so that $2m=(\frac{\Lambda}{3}-r_h^2)/r_h^3$.

5.2.2 More general Schwarzschild–Bach–(A)dS black hole

For $b \neq 0$, the solution (74), (71), that is (80), (81), represents a generalization of the Schwarzschild–(A)dS black hole with a nontrivial Bach tensor with the Bach invariant $B_{ab}\,B^{ab}$ proportional to b^2 at the horizon (77). It reduces to the Schwarzschild–(A)dS solution (87) for $b \to 0$. After summing the "background" terms independent of b as in (84), and using the same gauge (86), one obtains the explicit form of the solution (66), (67) with $r = r_h$ still being the horizon.

As in [15], we rewrite this class of solutions in an alternative and more explicit form. Let us introduce coefficients α_i, γ_i as those parts of a_i, c_i, respectively, that *do not* involve the "$b = 0$" Schwarzschild–(A)dS background, i.e.,

$$a_i \equiv a_i(b=0) - \frac{b}{r_h}\,\frac{\alpha_i}{(-r_h\rho)^i}\,, \qquad \text{where} \quad a_i(b=0) \equiv \frac{1}{(-r_h)^{1+i}} \quad i \geq 0\,, \tag{88}$$

$$c_1 \equiv 2 - \frac{\Lambda}{r_h^2} + 3b\,\gamma_1, \quad c_2 \equiv \frac{r_h}{3(r_h^2-\Lambda)}\left(1-\frac{\Lambda}{r_h^2}\right)\left(3-\frac{\Lambda}{r_h^2}\right) + 3b\,\frac{r_h\gamma_2}{r_h^2-\Lambda},$$
$$c_i \equiv 3b\,\frac{\gamma_i\, r_h^{i-1}}{(r_h^2-\Lambda)^{i-1}} \quad i \geq 3. \tag{89}$$

Then Ω and $\mathcal{H}$ take the explicit form

$$\Omega(r) = -\frac{1}{r} - \frac{b}{r_h}\sum_{i=1}^{\infty}\alpha_i\Big(\frac{r_h-r}{\rho\, r_h}\Big)^i\,, \tag{90}$$

$$\mathcal{H}(r) = (r-r_h)\Big[\frac{r^2}{r_h} - \frac{\Lambda}{3r_h^3}\left(r^2+r r_h+r_h^2\right) + 3b\,\rho\, r_h\sum_{i=1}^{\infty}\gamma_i\Big(\frac{r-r_h}{\rho\, r_h}\Big)^i\Big]\,, \tag{91}$$

where ρ is given by (68) and

$$\alpha_1 \equiv 1\,, \quad \gamma_1 = 1\,, \quad \gamma_2 = \frac{1}{3}\Big[4 - \frac{1}{r_h^2}\Big(2\Lambda + \frac{1}{2k}\Big) + 3b\Big]\,. \tag{92}$$

Using (74) and (71), the remaining coefficients α_l, γ_{l+1} for $l \geq 2$ are given by the recurrent relations (defining $\alpha_0 = 0$)

$$\alpha_l = \frac{1}{l^2}\Big[-\frac{2\Lambda}{3r_h^2}\sum_{j=0}^{l-1}\sum_{i=0}^{j}\big[\alpha_{l-1-j}\rho^j + \big(\rho^{l-1-j}+b\,\alpha_{l-1-j}\big)\big(\alpha_i\rho^{j-i}+\alpha_{j-i}(\rho^i+b\,\alpha_i)\big)\big] - \frac{1}{3}\alpha_{l-2}(2+\rho)\rho(l-1)^2$$
$$+\,\alpha_{l-1}\big[\frac{1}{3}+(1+\rho)\big(l(l-1)+\frac{1}{3}\big)\big] - 3\sum_{i=1}^{l}(-1)^i\,\gamma_i\,(\rho^{l-i}+b\,\alpha_{l-i})\big(l(l-i)+\frac{1}{6}i(i+1)\big)\Big]\,,$$

$$\gamma_{l+1} = \frac{(-1)^l}{k r_h^2\,(l+2)(l+1)l(l-1)}\sum_{i=0}^{l-1}\big[\alpha_i\rho^{l-i}+\alpha_{l-i}\big(\rho^i+b\,\alpha_i\big)\big](l-i)(l-1-3i) \quad \text{for} \quad l \geq 2\,. \tag{93}$$

Then the first few terms read

$$\begin{aligned}
\alpha_2 &= 2-\left(\frac{7}{3}\Lambda-\frac{1}{8k}\right)\frac{1}{r_h^2}+b\,,\\
\alpha_3 &= \frac{1}{9}\left[25+\left(\frac{29}{8k}-\frac{179}{3}\Lambda\right)\frac{1}{r_h^2}+\left(\frac{1}{16k^2}-\frac{77}{24k}\Lambda+\frac{298}{9}\Lambda^2\right)\frac{1}{r_h^4}\right]\\
&\quad+\frac{1}{9}\left[23+\left(\frac{35}{8k}-\frac{104}{3}\Lambda\right)\frac{1}{r_h^2}\right]b+\frac{7}{9}b^2\,,\quad \ldots\,,
\end{aligned} \tag{94}$$

$$\begin{aligned}
\gamma_3 &= \frac{1}{96k^2r_h^4}\left(1-\frac{8k}{3}\Lambda\right),\\
\gamma_4 &= \frac{1}{18kr_h^2}\left[\frac{1}{5}+\left(-\frac{1}{4k}+\frac{4}{15}\Lambda\right)\frac{1}{r_h^2}-\frac{1}{160k^2r_h^4}-\frac{1}{45r_h^4}\left(14\Lambda^2-\frac{75}{8k}\Lambda\right)\right]\\
&\quad+\frac{1}{720kr_h^2}\left[16+\left(-\frac{13}{k}+\frac{56}{3}\Lambda\right)\frac{1}{r_h^2}\right]b+\frac{1}{90kr_h^2}b^2\,,\quad \ldots\,,
\end{aligned} \tag{95}$$

leading to (66), (67).

This spherically symmetric Schwarzschild–Bach–(A)dS black-hole spacetime (90), (91) in QG admits *three independent parameters*: the cosmological constant Λ, the horizon position r_h (r_h is a root of $\mathcal{H}(r)$ given by (91)), and the dimensionless Bach parameter b, chosen in such a way that it determines the value of the Bach tensor (21), (22) on the horizon r_h, namely

$$\mathcal{B}_1(r_h)=0\,,\qquad \mathcal{B}_2(r_h)=-\frac{3}{kr_h^2}\,b\,. \tag{96}$$

Then the invariants (19) and (20) reduce to

$$B_{ab}\,B^{ab}(r_h)=\frac{r_h^4}{4k^2}\,b^2\,,\qquad C_{abcd}\,C^{abcd}(r_h)=12\,\left[r_h^2\,(1+b)-\frac{\Lambda}{3}\right]^2 \tag{97}$$

on the horizon.

As $\Delta\equiv r-r_h\to 0$, the dominant terms of the two metric functions are $\Omega=a_0=-\frac{1}{r_h}$ and $\mathcal{H}=c_0\,\Delta=(r_h-\frac{\Lambda}{r_h})(r-r_h)$ (cf. (15)), in which case the relations (10), (11) give

$$\bar{r}=a_0+\cdots\;\to\;\text{const.}\,, \tag{98}$$

$$h\;\to\;\bar{\Delta}(\equiv(\bar{r}-a_0))\;\to\;0\,,\quad f\;\to\;\bar{\Delta}(\equiv(\bar{r}-a_0))\;\to\;0\,, \tag{99}$$

confirming the existence of the horizon at $r=r_h$, i.e., $\bar{r}_h=a_0=-\frac{1}{r_h}$.

Finally let us check that the $k=0$ limit (corresponding to the Einstein limit of QG) of this solution reduces to the Schwarzschild–(A)dS solution. The condition $k=0$ implies $a_1c_0+a_0(1-\Lambda a_0^2)=0$, see the constraint (75). This, substituted into (72), yields $c_1=2-\Lambda a_0^2$, and thus $b=0$ due to our definition (79).

To conclude: The class $[n,p]=[0,1]$, expressed in terms of the series (90), (91) around the horizon r_h, represents the spherically symmetric Schwarzschild–Bach–(A)dS black hole generalizing the Schwarzschild–(A)dS black hole.

5.3 Analysis of physical properties of the Schwarzschild–Bach–(A)dS black hole

In this section, we study various aspects of the class of black holes found in Sec. 5.2.

5.3.1 Behaviour of the metric functions

This part is devoted to the analysis of the Schwarzschild–Bach–(A)dS metric functions and their comparison with the classic Schwarzschild–(A)dS solution.

First, let us verify that the metric functions $\Omega(r)$ and $\mathcal{H}(r)$, represented by the power series (90) and (91), really solve the field equations (25) and (26) in some neighbourhood around the horizon r_h, with a reasonable precision depending on the assumed finite order n. To do so, the difference between the left and right hand sides of the field equations (25) and (26) is plotted in Fig. 1. It clearly approaches zero with growing order n of the polynomial approximation.

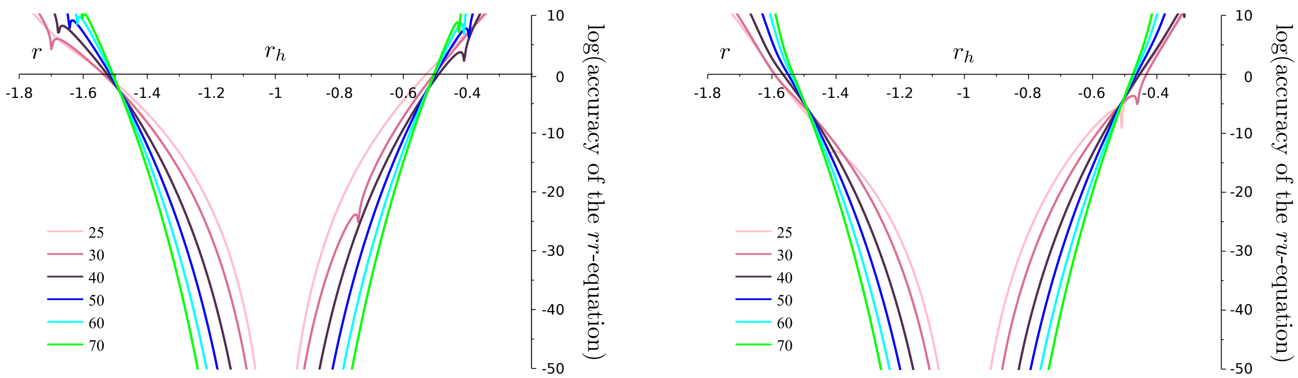

Figure 1: The difference between the left and right hand side of the field equation (25) (left) and (26) (right) for the metric functions Ω and $\mathcal{H}$ given by (90), (91). The plotted values approach zero in a certain range of the radial coordinate r around the horizon r_h with the growing order n. This indicates that the equations are satisfied with a reasonable precision, depending on the finite order of the polynomial approximation in some convergence radius. Here we set $r_h = -1$, $k = 0.5$, $b = 0.3$, and $\Lambda = 0.2$.

As a next natural step, we examine the *convergence radius* of the power series (90) and (91). These radii can be roughly deduced already from Fig. 1. However, to be more precise we employ the standard d'Alembert ratio test for two different sets of parameters. The ratio between two subsequent coefficients α_n/α_{n-1} and $-\gamma_n/\gamma_{n-1}$ is visualized in Fig. 2 (this figure and some of the following figures are taken from out letter [14]). The plots clearly indicate that such a ratio approaches a specific constant. We may thus conclude that the solution *asymptotically approaches a geometric series*, for which the radius of convergence can be simply estimated.

Now, let us plot the typical behaviour of the metric functions $\Omega(r)$ and $\mathcal{H}(r)$ near the black-hole horizon r_h, see Fig. 3. The qualitative behaviour of the function $\mathcal{H}$ depends on the sign of the cosmological constant Λ. For any value of Λ, the black-hole horizon separates static ($r > r_h$) and non-static ($r < r_h$) regions of the spacetime. However, for positive cosmological constant $\Lambda > 0$, an additional outer boundary of the static region appears, corresponding to the *cosmological horizon* given by the second root of the function $\mathcal{H}$, similarly as for the Schwarzschild–de Sitter black hole. For $\Lambda < 0$, the outer static region seems to be unbounded, as in the Einstein theory.

To give a complementary picture of the Schwarzschild–Bach–(A)dS metric behaviour, which can be more intuitively compared with the classic Schwarzschild–(A)dS case, we employ the standard spherically symmetric line element (8) and plot the metric function $f(\bar{r})$ in Fig. 4. Obviously, there

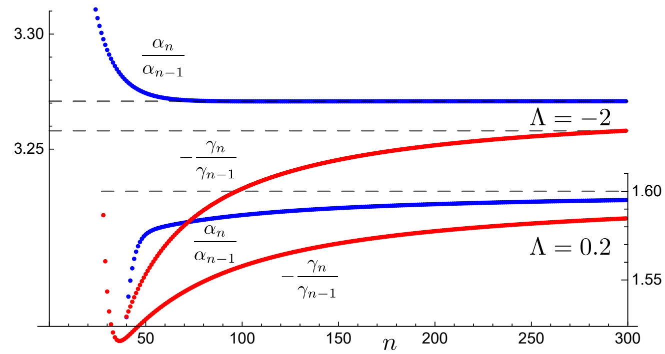

Figure 2: The convergence radius for the power series (90) and (91) representing the metric functions Ω and $\mathcal{H}$ can be estimated using the d'Alembert ratio test demonstrating the asymptotic behaviour similar to the geometric series. Here we assume $r_h = -1$, $k = 0.5$ with $b = 0.3$, $\Lambda = 0.2$ (bottom) and $b = 0.2$, $\Lambda = -2$ (top).

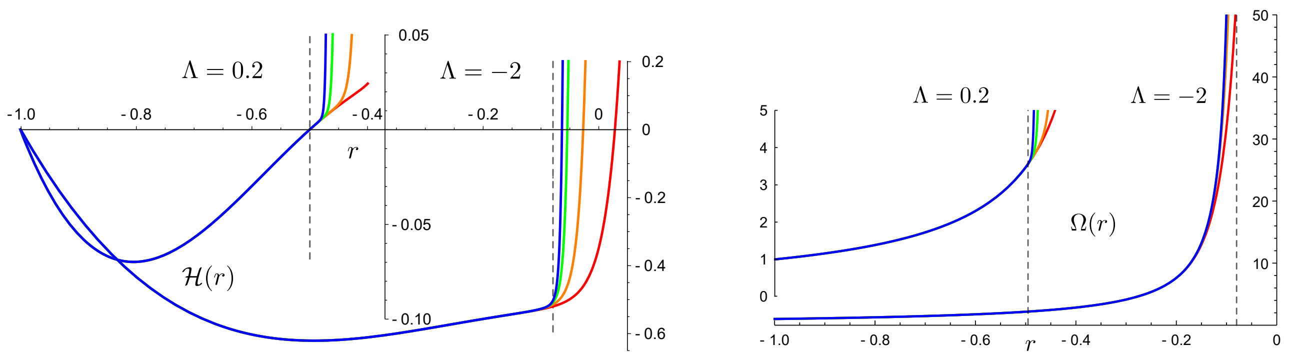

Figure 3: The functions $\mathcal{H}(r)$ given by (91) (left) and $\Omega(r)$ given by (90) (right) for two values of the cosmological constant Λ (with the same parameters as in Fig. 2). The black-hole horizon $r_h = -1$ is located at the centre of the radius of convergence. Here we plot the functions $\mathcal{H}(r)$ in the regions outside the black-hole horizon up to the outer radii of convergence indicated by dashed vertical lines. For $\Lambda > 0$, the function $\mathcal{H}(r)$ seems to have another root corresponding to the cosmological horizon, while for $\Lambda < 0$, it remains nonvanishing. First 50 (red), 100 (orange), 200 (green), 300 (blue) terms in the expansions are used. The results fully agree with the numerical solutions up to the dashed lines, where such simulations also fail. The function $\Omega(r)$ grows monotonously.

is not any significant qualitative difference with respect to the Einstein theory. The second root for the positive value of Λ again indicates the presence of the cosmological horizon, separating static and non-static regions.

To clarify the difference between the Schwarzschild–(A)dS and Schwarzschild–Bach–(A)dS solutions, in Fig. 5 we plot the *Bach invariant* (19) as a function of the Kundt coordinate r. Recall that this invariant identically vanishes for the Schwarzschild–(A)dS black hole. In addition, we also plot the metric functions $f(\bar{r})$ and $h(\bar{r})$ of standard spherically symmetric line element (8) corresponding to the solution (90), (91) via (11). In contrast with the Schwarzschild–(A)dS black hole, these functions are *not* identical.

Finally, we can also compare the Schwarzschild–Bach–(A)dS solution with its classic analogy on an invariant level of *spacetime geometry* by plotting the dependence of the area of a sphere with r fixed on the value of the expansion $\Theta = \frac{1}{2}k^a{}_{;a} = \Omega^{-3}\,\Omega_{,r}$ of the privileged null congruence $\mathbf{k} = \partial_r$ inducing

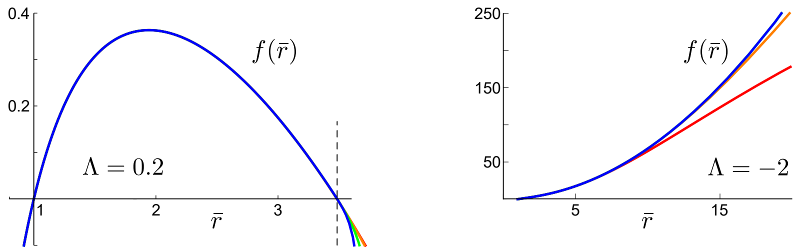

Figure 4: The function $f(\bar{r})$ of the standard metric form (8) related to the solution (90), (91) via the transformation (11). The *positive* Λ case (left) indicates the presence of the cosmological horizon at the boundary of the convergence interval (the dashed line). For *negative* Λ (right), the series converges in the whole plotted range, corresponding to a static region everywhere above the black-hole horizon. Here we use the same values of parameters as in Figs. 2, 3.

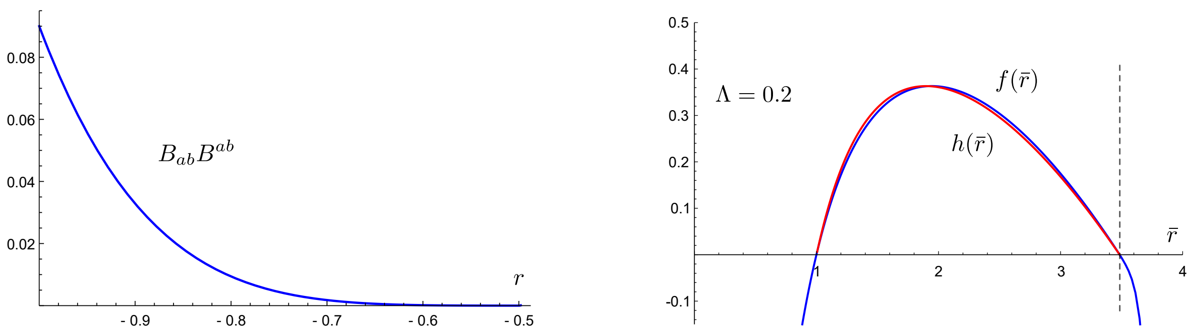

Figure 5: The Bach invariant (19) as a function of the Kundt coordinate r (left), and the metric functions $f(\bar{r})$ and $h(\bar{r})$ of standard line element (8) corresponding to the solution (90), (91) via (11) (right), with the same parameters as in Figs. 2, 3 for $\Lambda = 0.2$.

a spacetime foliation, see Fig. 6. Here we may identify specific quantitative differences. Namely, any observer following the geodesics generated by $\mathbf{k}$ will measure a larger spherical area when $b > 0$ (and smaller for $b < 0$) for any fixed value of the congruence expansion. Put it in a opposite way, on spheres of the same area, the expansion of $\mathbf{k}$ increases with b growing, and vice versa.

5.3.2 Thermodynamic properties: horizon area, temperature, entropy

In this section, let us study geometrical and thermodynamic properties of the Schwarzschild–Bach–(A)dS black hole, extending the concise discussion in the letter [14].

The horizon generated by the null Killing vector $\xi \equiv \sigma\partial_u = \sigma\partial_t$ (considering the time-scaling freedom (17) represented by the parameter σ) is given by vanishing of the norm of ξ, i.e., by $\mathcal{H}(r) = 0$ at $r = r_h$ (16), using (91). Integrating the angular coordinates of the metric (9), we obtain the horizon area

$$\mathcal{A} = 4\pi\,\Omega^2(r_h) = \frac{4\pi}{r_h^2} = 4\pi\,\bar{r}_h^2\,. \tag{100}$$

With the only nonvanishing derivatives of ξ being $\xi_{u;r} = -\xi_{r;u} = \frac{1}{2}\sigma(\Omega^2\mathcal{H})'$, $\xi^{\,r;u} = -\xi^{\,u;r} = \Omega^{-4}\xi_{u;r}$, using (91), the surface gravity $\kappa^2 \equiv -\frac{1}{2}\,\xi_{\mu;\nu}\,\xi^{\,\mu;\nu}$ [19] takes the form

$$\kappa/\sigma = -\frac{1}{2}(\mathcal{H}' + 2\mathcal{H}\,\Omega'/\Omega)_{|r=r_h} = -\frac{1}{2}\,\mathcal{H}'(r_h) = -\frac{1}{2}\rho\,r_h = \frac{1}{2}\,\bar{r}_h^{-1}(1 - \Lambda\bar{r}_h^2)\,. \tag{101}$$

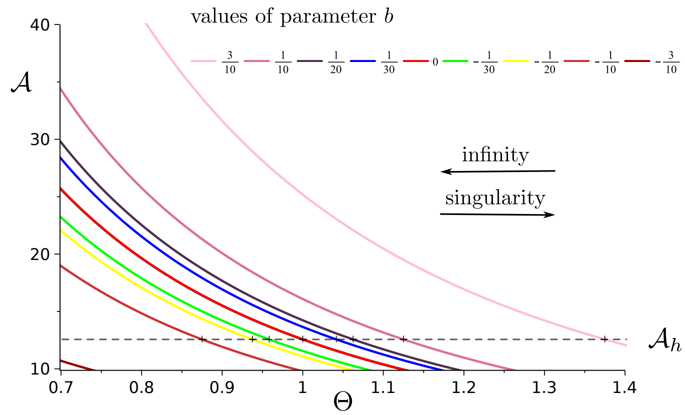

Figure 6: Relation between the sphere area $\mathcal{A}$ (of constant r) and the corresponding value of the expansion Θ of the privileged null vector field $\mathbf{k} = \partial_r$, for different values of the Bach parameter b. The dashed line indicates the horizon area $\mathcal{A}_h$ which is b-independent. Here $r_h = -1$, $k = 0.5$, $\Lambda = 0.2$.

Therefore, the temperature of the black hole horizon $T \equiv \kappa/(2\pi)$ [20] reads

$$T/\sigma = -\frac{1}{4\pi}\rho\, r_h = \frac{1}{4\pi}\,\bar{r}_h^{-1}(1-\Lambda\bar{r}_h^2)\,. \tag{102}$$

Note that the Bach parameter b does not enter the expressions (101) and (102), and thus they are same as for the Schwarzschild–(A)dS solution. Both are zero for $\bar{r} = 1/\sqrt{\Lambda}$ corresponding to the extreme Schwarzschild–dS solution with coinciding black-hole and cosmological horizons.

To determine the black-hole horizon entropy, we employ the generalized formula for higher-derivative theories derived by Wald [21, 22]

$$S = \frac{2\pi}{\kappa}\oint \mathbf{Q}\,, \tag{103}$$

with the Noether-charge 2-form $\mathbf{Q}$ on the horizon (corresponding to the Lagrangian $\mathcal{L}$ of the theory) being

$$\begin{aligned} \mathbf{Q} &= \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\, Q^{\mu\nu}\, \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta\,, \\ Q^{\mu\nu} &= 2X^{\mu\nu\rho\sigma}\,\xi_{\rho;\sigma} + 4X^{\mu\nu\rho\sigma}{}_{;\rho}\,\xi_\sigma \qquad \text{and} \qquad X^{\mu\nu\rho\sigma} \equiv \frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\,. \end{aligned} \tag{104}$$

For quadratic gravity (1), we obtain

$$X^{\mu\nu\rho\sigma} = \frac{1}{16\pi}\Big[\Big(\gamma + \frac{2}{3}(2\alpha+3\beta)R\Big)g^{\nu[\sigma}g^{\rho]\mu} - 4\alpha\, g^{\kappa[\nu}g^{\mu][\rho}g^{\sigma]\lambda}R_{\kappa\lambda}\Big]\,. \tag{105}$$

A lengthy calculation for the metric (9) then gives the Noether-charge 2-form on the horizon

$$\mathbf{Q} = -\frac{\Omega^2\,\mathcal{H}'}{16\pi}\left[\gamma + \frac{4}{3}\Lambda(\alpha+6\beta) + \frac{4}{3}k\alpha\,\frac{\mathcal{B}_1+\mathcal{B}_2}{\Omega^4}\right]\Bigg|_{r=r_h}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi\,. \tag{106}$$

Finally, using (100), (101), and (96), the Schwarzschild–Bach–(A)dS black hole horizon entropy (103) is given by a simple explicit formula (cf. Eq. (6.5) of [14])

$$S = \frac{1}{4}\mathcal{A}\left[\gamma + \frac{4}{3}\Lambda\,(\alpha+6\beta) - 4\alpha\,\frac{b}{\bar{r}_h^2}\right]\,. \tag{107}$$

For $\Lambda = 0$, it agrees with [9] (with the identification $k = \alpha$ and $b = \delta^*$), [13], and [15]. Therefore, the "non-Schwarzschild parameter" δ^* of [9] is the dimensionless Bach parameter b determining the value of the Bach tensor on the horizon r_h, see (96). The standard entropy expression for the Schwarzschild black hole, $S = \frac{1}{4G}\,\mathcal{A}$, is recovered either for $b = 0, \Lambda = 0$ (the Schwarzschild black hole in QG) or for $\alpha = 0 = \beta$ (the Schwarzschild black hole in Einstein's gravity). The result of [23], $S = \frac{1}{4G}\,\mathcal{A}\left(1 + \frac{4}{3}k\Lambda\right)$, for the Schwarzschild–(A)dS black hole in the Einstein–Weyl gravity is recovered by setting $b = 0$ and $\beta = 0$ in (107). The entropy (107) for the Schwarzschild–(A)dS black hole ($b = 0$) vanishes in critical gravity with $\beta = 0$, $\alpha = k\gamma$, $\Lambda = -\frac{3}{4k} < 0$ [23]. Note also that the deviations from $S = \frac{1}{4}\mathcal{A}\left[\gamma + \frac{4}{3}\Lambda(\alpha + 6\beta)\right]$ are larger for smaller black holes because they have smaller $\bar{r}_h^2$.

5.3.3 Specific motion of test particles caused by the Bach tensor

This section generalizes Sec. 13 of [15] to the case of a nonvanishing cosmological constant Λ, and extends Sec. V. and VI. of [14]. We will demonstrate that the effect of the Bach tensor parts $\mathcal{B}_1$, $\mathcal{B}_2$ given by (21), (22), entering the Bach invariant (19), can be directly observed through the relative motion of freely falling particles described by the equation of geodesic deviation.

A: Interpreting solutions in quadratic gravity using geodesic deviation
Projecting the equation of geodesic deviation onto an orthonormal frame $\{\boldsymbol{e}_{(0)}, \boldsymbol{e}_{(1)}, \boldsymbol{e}_{(2)}, \boldsymbol{e}_{(3)}\}$ with the time-like vector being an observer's 4-velocity $\boldsymbol{e}_{(0)} = \boldsymbol{u}$ and satisfying $\boldsymbol{e}_{(a)} \cdot \boldsymbol{e}_{(b)} = \eta_{ab}$, we obtain

$$\ddot{Z}^{(\mathrm{i})} = R^{(\mathrm{i})}{}_{(0)(0)(\mathrm{j})}\, Z^{(\mathrm{j})}\,, \qquad \mathrm{i, j} = 1, 2, 3\,, \tag{108}$$

where

$$\ddot{Z}^{(\mathrm{i})} \equiv e^{(\mathrm{i})}_a\, \frac{\mathrm{D}^2 Z^a}{\mathrm{d}\,\tau^2} = e^{(\mathrm{i})}_a\, Z^a{}_{;cd}\, u^c u^d\,, \qquad \text{and} \qquad R_{(\mathrm{i})(0)(0)(\mathrm{j})} \equiv R_{abcd}\, e^a_{(\mathrm{i})} u^b u^c e^d_{(\mathrm{j})}\,. \tag{109}$$

The decomposition of the Riemann tensor into the traceless Weyl tensor, the Ricci tensor, and the scalar curvature R gives

$$R_{(\mathrm{i})(0)(0)(\mathrm{j})} = C_{(\mathrm{i})(0)(0)(\mathrm{j})} + \tfrac{1}{2}\big(R_{(\mathrm{i})(\mathrm{j})} - \delta_{\mathrm{ij}}\, R_{(0)(0)}\big) - \tfrac{1}{6}\, R\, \delta_{\mathrm{ij}}\,. \tag{110}$$

Furthermore, employing the vacuum field equations (2) with (7) in (110), Eq. (108) takes the form

$$\ddot{Z}^{(\mathrm{i})} = \frac{\Lambda}{3}\, Z^{(\mathrm{i})} + C_{(\mathrm{i})(0)(0)(\mathrm{j})}\, Z^{(\mathrm{j})} + 2k\big(B_{(\mathrm{i})(\mathrm{j})}\, Z^{(\mathrm{j})} - B_{(0)(0)}\, Z^{(\mathrm{i})}\big)\,. \tag{111}$$

Applying the Newman–Penrose scalar decomposition (obtained in [24,25]) with respect to a (real) null frame $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}_i\}$ with two future-oriented null vectors $\boldsymbol{k}$ and $\boldsymbol{l}$ and two spatial vectors $\boldsymbol{m}_i$ orthogonal to them, normalized as $\boldsymbol{k} \cdot \boldsymbol{l} = -1$ and $\boldsymbol{m}_i \cdot \boldsymbol{m}_j = \delta_{ij}$, such that

$$\boldsymbol{k} = \tfrac{1}{\sqrt{2}}(\boldsymbol{u} + \boldsymbol{e}_{(1)})\,, \qquad \boldsymbol{l} = \tfrac{1}{\sqrt{2}}(\boldsymbol{u} - \boldsymbol{e}_{(1)})\,, \qquad \boldsymbol{m}_i = \boldsymbol{e}_{(i)} \quad \text{for} \quad i = 2, 3\,, \tag{112}$$

we arrive at the equation of geodesic deviation (111) in the QG theory in the form

$$\begin{aligned}
\ddot{Z}^{(1)} =& \frac{\Lambda}{3}\, Z^{(1)} + \Psi_{2S}\, Z^{(1)} + \tfrac{1}{\sqrt{2}}(\Psi_{1T^j} - \Psi_{3T^j})\, Z^{(j)} \\
&\qquad + 2k\left[(B_{(1)(1)} - B_{(0)(0)})\, Z^{(1)} + B_{(1)(j)}\, Z^{(j)}\right], \qquad (113)\\
\ddot{Z}^{(i)} =& \frac{\Lambda}{3}\, Z^{(i)} - \tfrac{1}{2}\Psi_{2S}\, Z^{(i)} + \tfrac{1}{\sqrt{2}}(\Psi_{1T^i} - \Psi_{3T^i})\, Z^{(1)} - \tfrac{1}{2}(\Psi_{0^{ij}} + \Psi_{4^{ij}})\, Z^{(j)} \\
&\qquad + 2k\left[\, B_{(i)(1)}\, Z^{(1)} + B_{(i)(j)}\, Z^{(j)} - B_{(0)(0)}\, Z^{(i)}\,\right], \qquad (114)
\end{aligned}$$

where $\Psi_{2T^{(ij)}} = \frac{1}{2}\Psi_{2S}\,\delta_{ij}$, see [25]. The system (113), (114) has a clear physical interpretation. Classical effects include the Newtonian tidal deformations, longitudinal motions, and the transverse effects of gravitational waves (propagating in the directions $\boldsymbol{e}_{(1)}, -\boldsymbol{e}_{(1)}$) caused by the scalar components Ψ_{2S}, $\{\Psi_{3T^i}, \Psi_{1T^i}\}$, and $\{\Psi_{4^{ij}}, \Psi_{0^{ij}}\}$, respectively, and isotropic radial motions of test particles caused by the cosmological constant Λ. Apart from these classical effects, there are additional specific effects caused by the nonvanishing Bach tensor, encoded in the frame components $B_{(a)(b)}$.

B: Geodesic deviation in the Schwarzschild–Bach–(A)dS black hole spacetime

The general results obtained in the previous section can now be applied to the spherically symmetric black hole metric (9). We choose an orthonormal frame associated with a radially falling observer, i.e., $\dot{x} = 0 = \dot{y}$, namely

$$\begin{aligned}
\boldsymbol{e}_{(0)} &\equiv \boldsymbol{u} = \dot{r}\,\partial_r + \dot{u}\,\partial_u\,,\\
\boldsymbol{e}_{(1)} &= \tfrac{1}{2}\big[(\Omega^2\dot{u})^{-1} - \mathcal{H}\dot{u}\big]\partial_r - \dot{u}\,\partial_u\,,\\
\boldsymbol{e}_{(i)} &= \Omega^{-1}\big[1 + \tfrac{1}{4}(x^2+y^2)\big]\partial_i\,,
\end{aligned} \tag{115}$$

where $\dot{r} = \frac{1}{2}\big[(\Omega^2\,\dot{u})^{-1} + \mathcal{H}\dot{u}\big]$ due to the normalisation of an observer's four-velocity $\boldsymbol{u}\cdot\boldsymbol{u} = -1$. Then the associated null interpretation frame (112) reads

$$\boldsymbol{k} = \frac{1}{\sqrt{2}\,\dot{u}\,\Omega^2}\,\partial_r\,, \qquad \boldsymbol{l} = \frac{\dot{u}\,\mathcal{H}}{\sqrt{2}}\,\partial_r + \sqrt{2}\dot{u}\,\partial_u\,, \qquad \boldsymbol{m}_i = \Omega^{-1}\big[1 + \tfrac{1}{4}(x^2+y^2)\big]\partial_i\,. \tag{116}$$

Since the spherically symmetric black hole metric (9) is of algebraic type D, there is only one nonvanishing Weyl-tensor component with respect to (116), namely

$$\Psi_{2S} \equiv C_{abcd}\,k^a\,l^b\,l^c\,k^d = \tfrac{1}{6}\,\Omega^{-2}(\mathcal{H}'' + 2)\,. \tag{117}$$

Using the Bach tensor projections with respect to the orthonormal frame (115),

$$B_{(0)(0)} = \frac{1}{24\,\Omega^6\dot{u}^2}\Big[-(1-\Omega^2\mathcal{H}\dot{u}^2)^2\,\mathcal{H}'''' + 2\Omega^2\dot{u}^2(\mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2)\Big]\,, \tag{118}$$

$$B_{(1)(1)} = \frac{1}{24\,\Omega^6\dot{u}^2}\Big[-(1+\Omega^2\mathcal{H}\dot{u}^2)^2\,\mathcal{H}'''' - 2\Omega^2\dot{u}^2(\mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2)\Big]\,, \tag{119}$$

$$B_{(0)(1)} = -\frac{1}{24\,\Omega^6\dot{u}^2}\,(1-\Omega^4\mathcal{H}^2\dot{u}^4)\,\mathcal{H}''''\,, \qquad B_{(0)(i)} = 0\,, \tag{120}$$

$$B_{(i)(j)} = \frac{\delta_{ij}}{12\,\Omega^4}(\mathcal{H}\mathcal{H}'''' + \mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2)\,, \qquad B_{(1)(i)} = 0\,, \tag{121}$$

the equations of geodesics deviation (113), (114) take the form

$$\ddot{Z}^{(1)} = \frac{\Lambda}{3}\,Z^{(1)} + \frac{1}{6}\,\Omega^{-2}\big(\mathcal{H}''+2\big)\,Z^{(1)} - \frac{1}{3}\,k\,\Omega^{-4}\big(\mathcal{H}\mathcal{H}'''' + \mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2\big)Z^{(1)}\,, \tag{122}$$

$$\ddot{Z}^{(i)} = \frac{\Lambda}{3}\,Z^{(i)} - \frac{1}{12}\,\Omega^{-2}\big(\mathcal{H}''+2\big)\,Z^{(i)} + \frac{1}{12}\,k\,\Omega^{-4}\big((\Omega^2\mathcal{H}\dot{u}^2)^{-1} + \Omega^2\mathcal{H}\dot{u}^2\big)\mathcal{H}\mathcal{H}''''\,Z^{(i)}\,. \tag{123}$$

Therefore, apart from the classical effects, namely the isotropic influence of the cosmological constant Λ and a classical tidal deformation caused by the Weyl curvature (117) proportional to $\Omega^{-2}(\mathcal{H}''+2)$, i.e., the square root of the invariant (20), there are two additional effects of quadratic gravity caused by the presence of a nonvanishing Bach tensor. They appear in the longitudinal (122) and transverse (123) components of the acceleration and are proportional (up to a constant) to the square roots of the two parts of the invariant (19), to the amplitudes $\mathcal{B}_1$, $\mathcal{B}_2$, see (21), (22).

When considering a test particle which is initially static ($\dot{r}=0$), the geodesic deviation equations simplify to

$$\ddot{Z}^{(1)} = \frac{\Lambda}{3}\,Z^{(1)} + \frac{1}{6}\,\Omega^{-2}\big(\mathcal{H}''+2\big)\,Z^{(1)} - \frac{1}{3}\,k\,\Omega^{-4}\big(\mathcal{B}_1+\mathcal{B}_2\big)Z^{(1)}\,, \tag{124}$$

$$\ddot{Z}^{(i)} = \frac{\Lambda}{3}\,Z^{(i)} - \frac{1}{12}\,\Omega^{-2}\big(\mathcal{H}''+2\big)\,Z^{(i)} - \frac{1}{6}\,k\,\Omega^{-4}\,\mathcal{B}_1\,Z^{(i)}\,, \tag{125}$$

thanks to the 4-velocity normalization $\Omega^2\mathcal{H}\,\dot{u}^2=-1$. Note that the first component $\mathcal{B}_1$ of the Bach tensor can be directly observed in the transverse components of the acceleration (125) along $\boldsymbol{e}_{(2)},\boldsymbol{e}_{(3)}$, i.e., ∂_x,∂_y (equivalent to $\partial_\theta,\partial_\phi$), while the second component $\mathcal{B}_2$ enters the radial component (124) along $\boldsymbol{e}_{(1)}=-\dot{u}\,(\partial_u+\mathcal{H}\,\partial_r)=-\mathcal{H}\,\Omega'\,\dot{u}\,\partial_{\bar{r}}$, proportional to $\partial_{\bar{r}}$. Note also that on the horizon, there is only the radial effect given by $\mathcal{B}_2(r_h)$ since $\mathcal{B}_1(r_h)=0$ due to (21) and (16), see also (96).

As has been already argued in [15], the effects of $\mathcal{B}_1,\mathcal{B}_2$ can be distinguished from the Newtonian tidal effect in the Schwarzschild solution. Thus, by observing a free fall of a set of test particles, one can distinguish the Schwarzschild–(A)dS black hole from the Schwarzschild–Bach–(A)dS black hole with a nonvanishing Bach tensor.

5.4 Schwarzschild–Bach–(A)dS black hole in the class $[n,p]=[0,0]$: near a generic point

This class of solutions has the *highest number of free parameters* (see Table 10) and for a special choice of these parameters, it represents the previously discussed Schwarzschild–Bach–(A)dS black hole near *any* regular point $r=r_0\neq r_h$. The first few terms of the expansion read

$$\Omega(r) \;=\; -\frac{1}{r} - \frac{b_1}{2r_0\big(r_0^2\nu+\frac{1}{3}\Lambda\big)}\,(r-r_0)^2+\ldots\,, \tag{126}$$

$$\begin{aligned}\mathcal{H}(r) \;=\; & \frac{\Lambda}{3} - r^2 - \Big(\frac{\Lambda}{3}-r_h^2\Big)\frac{r^3}{r_h^3} + (b_1-b_2)\,r_0(r-r_0) - 3b_2\,(r-r_0)^2 \\ & + \frac{(b_2-b_1)\big(1+\nu+\frac{1}{2kr_0^2}\big) - 2(2+3\nu)b_2 + 3b_2^2}{(1+3\nu+b_1-b_2)\,r_0}\,(r-r_0)^3\ldots\,,\end{aligned} \tag{127}$$

where

$$\nu \equiv \omega - 1 - \frac{\Lambda}{3}\frac{\omega^3}{r_0^2}\,, \qquad \omega \equiv \frac{r_0}{r_h}\,, \tag{128}$$

and where *two independent Bach parameters*, denoted as b_1 and b_2, are proportional to the values of the two components of the Bach tensor at r_0 (see (145), (146)). For $b_1=0=b_2$, the solution reduces to the Schwarzschild–(A)dS solution.

To find the coefficients a_i and c_i in (29)–(31), we start with Eqs. (36) and (38) that give (after relabeling $l\to l-1$) for $l\geq 1$

$$c_{l+3} \;=\; \frac{3}{k\,(l+3)(l+2)(l+1)l}\sum_{i=0}^{l} a_i\,a_{l+1-i}(l+1-i)(l-3i)\,, \tag{129}$$

$$a_{l+1} \;=\; \frac{1}{l(l+1)c_0}\bigg\{\tfrac{2}{3}\Lambda\sum_{j=0}^{l-1}\sum_{i=0}^{j} a_i a_{j-i} a_{l-j-1} - \tfrac{1}{3}\,a_{l-1} - \sum_{i=1}^{l+1} c_i a_{l-i+1}\big[l(l-i+1)+\tfrac{1}{6}i(i-1)\big]\bigg\}, \tag{130}$$

respectively. An additional constraint follows from the lowest nontrivial order $l=0$ of Eq. (37),

$$c_3 = \frac{1}{6kc_1}\big[9a_1^2c_0 + 2k(c_2^2-1) + 3a_0(a_0+a_1c_1-\Lambda a_0^3)\big]\,. \tag{131}$$

Therefore, this solution admits five free initial parameters a_0, a_1, c_0, c_1, c_2. The remaining coefficients a_{l+1}, c_{l+3} in (126) and (127), respectively, are given by the recurrent relations (130), (129), respectively,

$$a_2 = -\frac{a_0 + 3a_1c_1 + a_0c_2 - 2a_0^3\Lambda}{6c_0}, \dots, \tag{132}$$

$$c_4 = -\frac{6a_1^2c_0 + a_0(a_0 + 3a_1c_1 + a_0c_2 - 2a_0^3\Lambda)}{24c_0k}, \dots . \tag{133}$$

Now we will show that this class of solutions contains the Schwarzschild–(A)dS solution as a special subcase.

5.4.1 Identification of the Schwarzschild–(A)dS black hole

Let us employ the scalar invariant (19), with (21), (22), which at $r = r_0$ read

$$B_{ab}\,B^{ab}(r_0) = \frac{1}{72\,a_0^8}\left[(\mathcal{B}_1)^2 + 2(\mathcal{B}_1 + \mathcal{B}_2)^2\right],$$
$$\text{where} \quad \mathcal{B}_1(r_0) = 24c_0c_4\,, \qquad \mathcal{B}_2(r_0) = 2(3c_1c_3 - c_2^2 + 1)\,. \tag{134}$$

The Schwarzschild–(A)dS solution is uniquely identified by vanishing of the Bach tensor, $B_{ab} = 0 \Leftrightarrow \mathcal{B}_1 = 0 = \mathcal{B}_2$, which implies $c_4 = 0$ and $3c_1c_3 - c_2^2 + 1 = 0$. Together with (133), (131), this yields two necessary conditions

$$c_1 = -\frac{a_0}{a_1}\Big(1 - \Lambda a_0^2 + 3\frac{a_1^2}{a_0^2}c_0\Big)\,, \qquad c_2 = 2 - \Lambda a_0^2 + 3\frac{a_1^2}{a_0^2}c_0\,. \tag{135}$$

Then the recurrent relations (130), (129) reduce to

$$a_i = a_0\left(\frac{a_1}{a_0}\right)^i \quad \text{for all } i \geq 0\,, \qquad c_3 = -\frac{a_1}{a_0}\Big(1 + \frac{a_1^2}{a_0^2}c_0 - \tfrac{1}{3}\Lambda a_0^2\Big)\,, \quad c_i = 0 \quad \text{for all } i \geq 4\,, \tag{136}$$

where the first sequence is a geometrical series, while the second series is truncated to the 3rd-order polynomial. The metric functions thus take the following closed form

$$\Omega(r) \;=\; a_0\sum_{i=0}^{\infty}\Big(\frac{a_1}{a_0}\Delta\Big)^i = \frac{a_0^2}{a_0 - a_1\Delta} = \frac{a_0^2}{(a_0 + a_1r_0) - a_1r}\,, \tag{137}$$

$$\mathcal{H}(r) \;=\; c_0 + c_1(r - r_0) + c_2(r - r_0)^2 + c_3(r - r_0)^3\,. \tag{138}$$

Using the gauge (13), we can set

$$a_0 = -\frac{1}{r_0}\,, \qquad a_1 = \frac{1}{r_0^2}\,, \tag{139}$$

so that the metric functions read

$$\bar{r} = \Omega(r) = -\frac{1}{r}\,, \qquad \mathcal{H}(r) = \big(r - r_0\big)\frac{r^2}{r_0} + \frac{c_0 - \frac{1}{3}\Lambda}{r_0^3}\,r^3 + \frac{\Lambda}{3}\,. \tag{140}$$

Such $\mathcal{H}$ can be rewritten as

$$\mathcal{H}(r) = \frac{\Lambda}{3} - r^2 - \Big(\frac{\Lambda}{3} - r_h^2\Big)\frac{r^3}{r_h^3}\,, \tag{141}$$

and c_0, r_h are related by

$$c_0 = \frac{\Lambda}{3}(1 - \omega^3) + r_0^2(\omega - 1) = r_0^2\nu + \frac{\Lambda}{3}\,. \tag{142}$$

This is the Schwarzschild–(A)dS solution (87) (see also (15)) with the black hole horizon at $r = r_h$.

Therefore, for vanishing Bach tensor, the class $[n, p] = [0, 0]$ corresponds to the Schwarzschild–(A)dS black hole solution.

5.4.2 More general solution with nontrivial Bach tensor

Now, let us return to the generic case of $[n,p]=[0,0]$ with a nonvanishing Bach tensor (129)–(133) with the free parameters a_0, a_1, c_0, c_1, c_2. To simplify the two components $\mathcal{B}_1(r_0)$ and $\mathcal{B}_2(r_0)$ of the Bach tensor (21) and (22) evaluated at r_0, we introduce new more physical dimensionless Bach parameters b_1 and b_2, corresponding to $\mathcal{B}_1(r_0)$ and $\mathcal{B}_2(r_0)$, respectively,

$$b_1 \equiv \tfrac{1}{3}\Big(-1-6\nu-c_2+3\frac{c_1}{r_0}\Big)\,, \qquad b_2 \equiv \tfrac{1}{3}\big(2+3\nu-c_2\big)\,, \tag{143}$$

so that

$$c_1 = (1+3\nu+b_1-b_2)\,r_0\,, \qquad c_2 = 2+3\nu-3b_2\,, \tag{144}$$

where ν has been introduced in (128) and the gauge (139) has been used. Then using (133) and (131), $\mathcal{B}_1(r_0)$ and $\mathcal{B}_2(r_0)$ are related to the Bach parameters b_1, b_2 by

$$b_1 = \tfrac{1}{3}kr_0^2\,\mathcal{B}_1(r_0)\,, \qquad b_2 = \tfrac{1}{3}kr_0^2\,\big(\mathcal{B}_1(r_0)+\mathcal{B}_2(r_0)\big)\,, \tag{145}$$

and the Bach invariant at r_0 is

$$B_{ab}\,B^{ab}(r_0) = \frac{r_0^4}{8k^2}\,\big(b_1^2+2\,b_2^2\big)\,. \tag{146}$$

The coefficients a_i, c_i take now the form

$$a_0 = -\frac{1}{r_0}\,, \quad a_1 = \frac{1}{r_0^2}\,, \quad a_2 = -\frac{1}{r_0^3} - \frac{b_1}{2c_0 r_0}\,, \ldots\,, \tag{147}$$

$$\begin{aligned} &c_0 = \nu\, r_0^2 + \tfrac{1}{3}\Lambda\,, \quad c_1 = (1+3\nu)r_0 + (b_1-b_2)\,r_0\,, \quad c_2 = 2+3\nu-3b_2\,,\\ &c_3 = \frac{(1+\nu)(1+3\nu)-2(2+3\nu)b_2+3b_2^2+(b_2-b_1)\frac{1}{2kr_0^2}}{(1+3\nu+b_1-b_2)\,r_0}\,, \quad c_4 = \frac{b_1}{8kc_0r_0^2}\,, \ldots\,. \end{aligned} \tag{148}$$

For $b_1 = 0 = b_2$, the Bach tensor vanishes and thus the solution reduces to the Schwarzschild–(A)dS solution (see Sec. 5.4.1). Thus the more general solution (129) and (130) includes a modification of the Schwarzschild–(A)dS solution admitting a nonvanishing Bach tensor (see also (147), (148) or (126), (127)).

Notice that as $\Delta \to 0$, the functions $\bar{r}$, f, and h approach *constants* (cf. (10), (11)).

5.4.3 Identification of the Schwa–Bach–(A)dS black hole $[0,1]$ in the class $[0,0]$

Since the generic class $[0,0]$ contains also the Schwarzschild–Bach–(A)dS black hole solution (90), (91), expressed around the horizon r_h in the class $[0,1]$, one should be able to express the five free parameters a_0, a_1, c_0, c_1, c_2 of the $[0,0]$ class in terms of the free parameters of the $[0,1]$ class. This can be done (within the convergence radius of (90), (91)) by evaluating the functions (90), (91) and their derivatives at $r = r_0$, and comparing them with the expansions (29) and (30) with $n = 0, p = 0$:

$$a_0 = -\frac{1}{r_0} - \frac{b}{r_h}\sum_{i=1}^{\infty}\alpha_i\left(\frac{r_h - r_0}{\rho r_h}\right)^i, \tag{149}$$

$$a_1 = \frac{1}{r_0^2} + \frac{b}{\rho r_h^2}\sum_{i=1}^{\infty} i\,\alpha_i\left(\frac{r_h - r_0}{\rho r_h}\right)^{i-1}, \tag{150}$$

$$c_0 = (r_0 - r_h)\left[\frac{r_0^2}{r_h} - \frac{\Lambda}{3r_h^3}(r_0^2 + r_0 r_h + r_h^2) + 3b\rho\, r_h\sum_{i=1}^{\infty}\gamma_i\left(\frac{r_0 - r_h}{\rho r_h}\right)^i\right], \tag{151}$$

$$c_1 = (3r_0 - 2r_h)\frac{r_0}{r_h} - \frac{\Lambda r_0^2}{r_h^3} + 3b\rho\, r_h\sum_{i=1}^{\infty}(i+1)\,\gamma_i\left(\frac{r_0 - r_h}{\rho r_h}\right)^i, \tag{152}$$

$$c_2 = (3r_0 - r_h)\frac{1}{r_h} - \frac{\Lambda r_0}{r_h^3} + \frac{3}{2}b\sum_{i=1}^{\infty} i(i+1)\,\gamma_i\left(\frac{r_0 - r_h}{\rho r_h}\right)^{i-1}. \tag{153}$$

Thus we obtain an expansion of the Schwarzschild–Bach–(A)dS black hole *around any point* r_0 in terms of just one Bach parameter b (which determines the value of the Bach tensor on the horizon r_h) in the recurrent relations (129)–(131).

5.4.4 Formal limit $r_0 \to r_h$

Finally, let us perform a "consistency check" between the two series corresponding to the Schwarzschild–Bach–(A)dS black hole solution in the class $[0,1]$ (see (66), (67)) and in the class $[0,0]$ (see (126), (127)). Here we temporarily denote the coefficients in the class $[0,0]$ by $\hat{c}_i$ and $\hat{a}_i$. In the limit $r_0 \to r_h$ in (149)–(153), using also (86) and the first relations in (80), (81), we obtain the following relations

$$\hat{a}_0 \,=\, -\frac{1}{r_h} \equiv a_0\,, \qquad \hat{a}_1 = \frac{1}{r_h^2}\left(1 + \frac{b}{\rho}\right) \equiv a_1\,, \tag{154}$$

$$\hat{c}_0 \,=\, 0\,, \qquad \hat{c}_1 = r_h - \frac{\Lambda}{r_h} = r_h\rho \equiv c_0\,, \qquad \hat{c}_2 = 2 - \frac{\Lambda}{r_h^2} + 3b \equiv c_1\,. \tag{155}$$

Since $\hat{c}_{j+1}$ and c_j satisfy the same recurrent relations (cf. (71) and (129)), the functions $\mathcal{H}$ agree. In addition, the condition $\hat{c}_0 = 0$ in (130) implies

$$0 = \tfrac{1}{3}\,\hat{a}_{l-1} + l^2\,\hat{c}_1\,\hat{a}_l + \sum_{i=2}^{l+1}\hat{c}_i\,\hat{a}_{l+1-i}\left[l(l+1-i) + \tfrac{1}{6}i(i-1)\right] - \tfrac{2}{3}\Lambda\sum_{j=0}^{l-1}\sum_{i=0}^{j}a_i a_{j-i} a_{l-j-1}\,, \tag{156}$$

i.e.,

$$\hat{a}_l = -\frac{1}{l^2\,\hat{c}_1}\Big[\tfrac{1}{3}\,\hat{a}_{l-1} + \sum_{i=1}^{l}\hat{c}_{i+1}\,\hat{a}_{l-i}\left[l(l-i) + \tfrac{1}{6}i(i+1)\right] - \tfrac{2}{3}\Lambda\sum_{j=0}^{l-1}\sum_{i=0}^{j}a_i a_{j-i} a_{l-j-1}\Big], \tag{157}$$

which is the same as (74) for a_{l+1} (with the identification $\hat{c}_{i+1} = c_i$) and thus the functions Ω also agree. Therefore, in the limit $r_0 \to r_h$ we obtain

$$\hat{c}_0 \to 0\,, \quad \hat{c}_{j+1} \to c_j\,, \quad \hat{a}_j \to a_j \quad \text{for all } j \geq 0\,, \tag{158}$$

which shows the consistency of the two expressions for the Schwarzschild–Bach–(A)dS black hole in the $[0,0]$ and $[0,1]$ classes.

To conclude: The class $[0,0]$, expressed in terms of the series (126) and (127) around an arbitrary point r_0, represents the spherically symmetric Schwarzschild–Bach–(A)dS black hole. However, without using the specific choice of parameters (139), it may represent any of the other spherical solutions as well.

5.5 Solutions with extreme double horizon in the class $[0,2]$

The $[n,p]=[0,2]$ class of solutions corresponds to an expansion around an extreme (double degenerate) horizon r_0 since for $p=2$ the key metric function reads $\mathcal{H}(r)=(r-r_0)^2\Big[c_0+c_1\,(r-r_0)+\cdots\Big]$. For a generic Λ, it contains only the extreme Schwarzschild–de Sitter spacetime. However, for certain special values of Λ, other solutions with a nonvanishing Bach tensor also exist.

Indeed, for $l\geq 0$, Eq. (36) gives

$$\tfrac{1}{3}kc_l(l+2)(l+1)l(l-1)=(l-1)la_0a_l+\sum_{i=1}^{l-1}a_ia_{l-i}(l-i)(l-3i-1)\,, \tag{159}$$

while Eq. (38) yields

$$c_0=2\Lambda a_0^2-1\,,\qquad a_1=\frac{3a_0c_1}{2(3-4\Lambda a_0^2)}\,, \tag{160}$$

and

$$a_l\left[l(l+1)(2a_0^2\Lambda-1)-\tfrac{4}{3}a_0^2\Lambda\right]+\tfrac{1}{6}a_0c_l(l+1)(l+2)+\sum_{i=1}^{l-1}c_ia_{l-i}\left[(l-i)(l+1)+\tfrac{1}{6}(i+1)(i+2)\right]$$
$$=\tfrac{2}{3}\Lambda\Bigg[\sum_{i=1}^{l-1}a_0a_ia_{l-i}+\sum_{j=1}^{l-1}\sum_{i=0}^{j}a_ia_{j-i}a_{l-j}\Bigg]\,. \tag{161}$$

Interestingly, Eq. (37) implies

$$\text{either}\quad \text{a)}\ \ \Lambda a_0^2=1\,,\quad \Lambda>0\,, \tag{162}$$
$$\text{or}\quad \text{b)}\ \ \Lambda=\frac{3}{8k}\,. \tag{163}$$

Let us discuss these two distinct cases separately.

5.5.1 Case a) $\Lambda a_0^2=1$: extreme Schwarzschild(–Bach)–dS black hole

In this case, it follows from (160) that $c_0=1$ while c_1 is arbitrary. From (159) and (161), we infer that for the first coefficients $a_i=a_0\left(-\tfrac{3}{2}c_1\right)^i\equiv\pm\tfrac{1}{\sqrt{\Lambda}}\left(-\tfrac{3}{2}c_1\right)^i$ and $c_j=0$ for $j>1$. Let us thus assume that $a_i=a_0\left(-\tfrac{3}{2}c_1\right)^i\equiv\pm\tfrac{1}{\sqrt{\Lambda}}\left(-\tfrac{3}{2}c_1\right)^i$ for *all* $0\leq i\leq l-1$, and $c_j=0$ for $2\leq j\leq l-1$. Then Eq. (159) gives

$$a_l=\frac{k}{3a_0}\,c_l(l+2)(l+1)+a_0\left(-\tfrac{3}{2}c_1\right)^l\equiv\pm\Big(\tfrac{1}{3}k\sqrt{\Lambda}\,c_l(l+2)(l+1)+\frac{1}{\sqrt{\Lambda}}\left(-\tfrac{3}{2}c_1\right)^l\Big)\,, \tag{164}$$

and Eq. (161) implies (note that this relation is valid even for $l=2$)

$$c_l\big[2k\Lambda\left[3l(l+1)-4\right]+3\big]=0\,. \tag{165}$$

Now we have to distinguish two subcases:

• Using the mathematical induction, the only solution of this system with a *generic value* of $\Lambda > 0$ is

$$a_i = a_0 \left(-\tfrac{3}{2}c_1\right)^i \equiv \pm\frac{1}{\sqrt{\Lambda}}\left(-\tfrac{3}{2}c_1\right)^i , \quad \forall i \geq 0 , \qquad \text{and} \qquad c_j = 0 ,\ \forall j \geq 2 , \tag{166}$$

with a free parameter c_1. Thus

$$\mathcal{H}(r) = \Delta^2 + c_1\Delta^3 , \tag{167}$$

and Ω is a sum of a geometric series (166) which can be expressed in the closed form as

$$\Omega(r) = \frac{\pm 2}{\sqrt{\Lambda}\,(2 + 3c_1\Delta)} . \tag{168}$$

Using the gauge

$$c_1 = \mp\frac{2}{3\sqrt{\Lambda}} , \qquad r_0 = \mp\sqrt{\Lambda} , \tag{169}$$

the metric functions are simplified to

$$\Omega = -\frac{1}{r} , \qquad \mathcal{H} = \Delta^2(1 + c_1\Delta) \equiv \frac{(r \pm \sqrt{\Lambda})^2(\sqrt{\Lambda} \mp 2r)}{3\sqrt{\Lambda}} . \tag{170}$$

It is the extreme Schwarzschild–de Sitter black hole (15) characterised by the condition $9\Lambda m^2 = 1$ (see e.g., Section 9.4.2 in [17]). The double degenerate Killing horizon is located at $r_h \equiv \mp\sqrt{\Lambda}$, that is at $\bar{r}_h \equiv \Omega(r_h) = \pm 1/\sqrt{\Lambda} = 3m$.

To conclude: The class $[0, 2]$ with $\Lambda a_0^2 = 1$ and arbitrary $\Lambda > 0$, with the metric functions expressed in terms of (170), represents the extreme Schwarzschild–de Sitter black hole solution.

• The second branch of the $[0, 2]$ class of solutions obeying (162) exists only for *discrete values* of the cosmological constant $\Lambda > 0$, restricted by (165) as

$$\Lambda = -\frac{3}{2k\,[3L(L+1) - 4]} , \qquad \text{where} \quad L \in \mathbb{N} , \quad L \geq 2 , \quad k < 0 , \tag{171}$$

or equivalently

$$\Lambda = -\frac{3\gamma}{[6L(L+1) - 8]\alpha + 24\beta} . \tag{172}$$

In this class, all c_i vanish for $2 \leq i \leq L-1$, while a_j are determined by (166) for $0 \leq j \leq L-1$. Eq. (159) implies

$$a_L = \pm\Big(\tfrac{1}{3}k\sqrt{\Lambda}\,\tilde{b}_L(L+1)(L+2) + \frac{1}{\sqrt{\Lambda}}\left(-\tfrac{3}{2}c_1\right)^L\Big) , \tag{173}$$

cf. (164), where

$$\tilde{b}_L \equiv c_L \tag{174}$$

is *a new free "Bach" parameter*. The remaining coefficients c_i, a_i (for $i > L$) are obtained from (159) and (161) by the recurrent relations[7]

$$\begin{aligned} c_l &= 6\frac{\left[3l(l+1)(2a_0^2\Lambda - 1) - 4a_0^2\Lambda\right]U_{l-1} - 3a_0 l(l-1)V_{l-1}}{(l-1)l(l+1)(l+2)[3a_0^2 - 8ka_0^2\Lambda + 6kl(l+1)(2a_0^2\Lambda - 1)]} , \\ a_l &= -3\frac{a_0U_{l-1} + 2kl(l-1)V_{l-1}}{(l-1)l[3a_0^2 - 8ka_0^2\Lambda + 6kl(l+1)(2a_0^2\Lambda - 1)]} , \end{aligned} \tag{175}$$

[7]Since these relations are valid also in the Case b), see Sec. 5.5.2, we do not use (162) to simplify them.

where

$$U_{l-1} = \sum_{i=1}^{l-1} a_i a_{l-i}(l-i)(l-3i-1)\,, \tag{176}$$

$$V_{l-1} = \sum_{i=1}^{l-1} c_i a_{l-i}\left[(l-i)(l+1)+\tfrac{1}{6}(i+1)(i+2)\right] - \tfrac{2}{3}\Lambda\Big(\sum_{i=1}^{l-1} a_0 a_i a_{l-i} + \sum_{j=1}^{l-1}\sum_{i=0}^{j} a_i a_{j-i} a_{l-j}\Big)\,.$$

Recall that in this case

$$a_0 = \pm\frac{1}{\sqrt{\Lambda}}\,, \qquad c_0 = 1\,, \tag{177}$$

see (162), (160), respectively. Therefore, in general the metric functions take the form

$$\Omega(r) = \sum_{i=0}^{\infty} a_i\,(r-r_0)^i \equiv \frac{a_0}{1+\frac{3}{2}c_1(r-r_0)} + \sum_{i=L}^{\infty}\Big[a_i - a_0\left(-\tfrac{3}{2}c_1\right)^i\Big](r-r_0)^i\,, \tag{178}$$

$$\mathcal{H}(r) = (r-r_0)^2\Big[1 + c_1(r-r_0) + \sum_{i=L}^{\infty} c_i\,(r-r_0)^i\Big]\,. \tag{179}$$

The corresponding leading terms in the Bach and Weyl invariants (19), (20) read

$$B_{ab}\,B^{ab} = \frac{(3L^2+8L+8)(L-1)^2(L+1)^2(L+2)^2}{72a_0^8}\;\tilde{b}_L^2\,(r-r_0)^{2L} + \cdots\,, \tag{180}$$

$$C_{abcd}\,C^{abcd} = \frac{16}{3a_0^4} + \cdots\,. \tag{181}$$

Interestingly, unlike for the Schwarzschild–Bach–(A)dS black hole discussed in Sec. 5.2, for these black holes the (in general nonzero) *Bach tensor vanishes on the extreme horizon* localized ar r_0 since $(r-r_0)^{2L} \to 0$.

Close to this extreme horizon, that is for $r \to r_0$, Eqs. (10), (11) imply that in the standard spherically symmetric coordinates

$$\bar{r} \;\to\; a_0\,, \qquad h \;\sim\; (\bar{r}-a_0)^2\,, \qquad f \;\sim\; (\bar{r}-a_0)^2\,, \tag{182}$$

where $a_0 = \pm 1/\sqrt{\Lambda} \equiv \bar{r}_h$ denotes the position of the extreme horizon.

These solutions exist only for $k<0$ and $\Lambda>0$, coupled by (171), and represent a *new family of black holes* with an extreme horizon and nonvanishing Bach tensor, with the physical Bach parameter $\tilde{b}_L$ and two gauge parameters c_1, r_0. The Bach tensor vanishes only in the case of the extreme Schwarzschild–de Sitter black hole, see (170), which can be obtained by setting $\tilde{b}_L = 0$ (in the $L \to \infty$ limit, the solution approaches Schwarzschild with $\Lambda = 0$). Therefore, we may call these solutions *extreme higher-order (discrete) Schwarzschild–Bach–de Sitter black holes*. Typical behaviour of the metric functions is shown in Fig. 7

The first representative of this new class is given by $L=2$. Its initial coefficients in (178) and (179) read

$$a_2 - a_0\left(-\tfrac{3}{2}c_1\right)^2 = \pm 6\sqrt{\frac{|k|}{21}}\,\tilde{b}_2\,, \quad a_3 - a_0\left(-\tfrac{3}{2}c_1\right)^3 = \pm 29\sqrt{\frac{|k|}{21}}\,c_1\tilde{b}_2\,,\ldots\,,$$

$$c_2 = \tilde{b}_2\,, \quad c_3 = -2c_1\tilde{b}_2\,,\ldots\,. \tag{183}$$

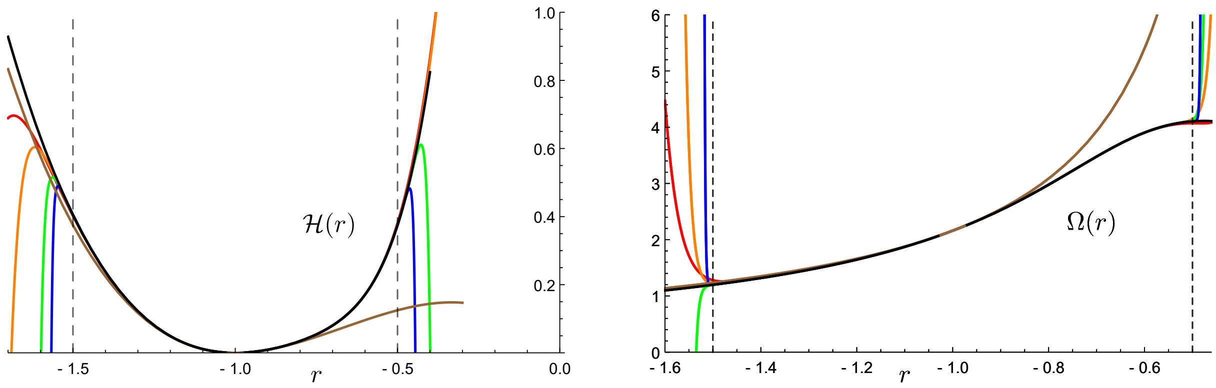

Figure 7: The metric functions $\mathcal{H}(r)$ (left) and $\Omega(r)$ (right) for the extreme higher-order (discrete) Schwarzschild–Bach–de Sitter solution $[0,2]$ with $L=2$, $k=-1/2$, $\Lambda=3/14$, $c_1=-1$, $\tilde{b}_2=1$, $r_0=-1$ and $a_0=1/\sqrt{\Lambda}$. First 10 (red), 20 (orange), 50 (green), and 100 (blue) terms in the expansions are used. Furthermore, the corresponding Schwarzschild–de Sitter solution obtained by setting $\tilde{b}_2=0$ is shown (brown). A numerical solution with initial values taken from the first 100 terms of the metric functions at $r=-1.15$ and $r=-0.85$ is also given (black). The numerics breaks down on the extreme horizon. Vertical dashed lines indicate the radius of the convergence obtained using the same method as in Fig. 3.

With the natural gauge choice (169), the explicit solution takes the form

$$\begin{aligned}\Omega(r) &= -\frac{1}{r}+\tilde{b}_2(r\pm\sqrt{\Lambda})^2\Big[\pm 2\sqrt{\frac{3}{7}|k|}+\frac{116}{9}k\,(r\pm\sqrt{\Lambda})+\cdots\Big],\\ \mathcal{H}(r) &= (r\pm\sqrt{\Lambda})^2\Big[1\mp\frac{4}{3}\sqrt{\frac{7}{3}|k|}\,(r\pm\sqrt{\Lambda})+\tilde{b}_2(r\pm\sqrt{\Lambda})^2\Big(1\pm\frac{8}{3}\sqrt{\frac{7}{3}|k|}\,(r\pm\sqrt{\Lambda})\cdots\Big)\Big].\end{aligned} \tag{184}$$

It reduces to the extreme Schwarzschild–de Sitter black hole (170) when the Bach tensor vanishes, that is for $\tilde{b}_2=0$.

To conclude: The class $[0,2]$ with $\Lambda a_0^2=1$ and the discrete values of $\Lambda>0$ given by (171), with the metric functions expressed in terms of the series (178), (179) around the horizon at r_0, represents the extreme higher-order (discrete) Schwarzschild–Bach–de Sitter black holes with vanishing Bach tensor on the horizon.

5.5.2 Case b) $\Lambda=\frac{3}{8k}\equiv\frac{3\gamma}{8(\alpha-3\beta)}$: another extreme Bachian–dS black hole

In a given QG theory with fixed values of the parameters α, β, and γ, there exists a *unique* value of the cosmological constant such that $\Lambda=\frac{3}{8k}$, that is $\Lambda=\frac{3\gamma}{8(\alpha-3\beta)}$.[8]

In this case, a_0, c_1 are free parameters. The coefficients c_0, a_1 are given by (160), while the remaining coefficients c_l, a_l for all $l\geq 2$ are again determined by the recurrent relations (175) with (176).

[8]In the particular theory with $\alpha=3\beta$, necessarily $\gamma=0$ but Λ remains arbitrary.

The first few such coefficients explicitly read

$$c_0 = \frac{3a_0^2}{4k} - 1\,, \quad c_2 = \frac{a_0^2 c_1^2 (3a_0^2 - 8k)k}{24(3a_0^2 - 4k)(a_0^2 - 2k)^2}\,, \quad c_3 = -\frac{a_0^2 c_1^3 (7a_0^2 - 12k)(3a_0^2 - 8k)k^2}{60(3a_0^2 - 4k)^2(a_0^2 - 2k)^3}\,, \ldots\,, \tag{185}$$

$$a_1 = -\frac{a_0 c_1 k}{a_0^2 - 2k}\,, \quad a_2 = \frac{a_0 c_1^2 (21a_0^2 - 32k)k^2}{6(3a_0^2 - 4k)(a_0^2 - 2k)^2}\,, \quad a_3 = -\frac{a_0 c_1^3 (231a_0^4 - 724a_0^2 k + 576k^2)k^3}{18(3a_0^2 - 4k)^2(a_0^2 - 2k)^3}\,, \ldots\,. \tag{186}$$

The metric functions thus take the form

$$\Omega(r) \,=\, a_0\bigg[1 - \frac{c_1 k}{a_0^2 - 2k}\,(r - r_0) + \Big(\frac{c_1 k}{a_0^2 - 2k}\Big)^2 \Big(1 + \frac{3a_0^2 - 8k}{18a_0^2 - 24k}\Big)\,(r - r_0)^2 + \cdots\,, \tag{187}$$

$$\mathcal{H}(r) \,=\, (r - r_0)^2 \bigg[2a_0^2\Lambda - 1 + c_1(r - r_0) + \frac{a_0^2 c_1^2 k(3a_0^2 - 8k)}{24(a_0^2 - 2k)^2(3a_0^2 - 4k)}(r - r_0)^2 + \cdots\bigg]\,. \tag{188}$$

Note that for a special case $a_1 = 0$, it follows that $c_1 = 0$ and subsequently a_2, a_3, c_2, c_3, U_1, U_2, U_3, V_1, V_2, V_3 also vanish (see (185), (186), (176)). The relations (175) with (176) then determine $a_4 = c_4 = 0$ which implies $U_4 = V_4 = 0$, etc. Thus, all $a_i = 0$ and $c_i = 0$, $i \geq 1$, and the metric functions reduce to $\Omega = a_0$ and $\mathcal{H} = (r - r_0)^2(2\Lambda a_0^2 - 1)$. This Kundt metric is a Bachian generalization of the Nariai spacetime (342), cf. (358), with an extreme horizon.

In the limit $r \to r_0$, the relations (182) hold, and the values of the Bach and Weyl invariants (19), (20) on the horizon are

$$B_{ab}\,B^{ab}(r_0) = \frac{1}{4k^2}\Big(\frac{3}{8k} - \frac{1}{a_0^2}\Big)^2\,, \qquad C_{abcd}\,C^{abcd}(r_0) = \frac{3}{4k^2}\,. \tag{189}$$

Therefore, the *Bach tensor is nonvanishing on the horizon*, unless $a_0^2 = \frac{8}{3}k$.

The *extreme Schwarzschild–de Sitter black hole* (170) is recovered for $B_{ab}\,B^{ab} = 0$, i.e., for

$$a_0^2 = \frac{8k}{3} = \frac{1}{\Lambda} > 0\,. \tag{190}$$

Thus $a_0^2\Lambda = 1$ as in Case a), cf. (162). Moreover, $c_0 = 1$, $c_i = 0$ for all $i \geq 2$, and $a_i = (-\frac{3}{2}c_1)^i a_0$, so that Eqs. (166) hold, leading to the solution (170).

In general, it is natural to introduce a dimensionless Bach parameter b_e by

$$b_e \equiv \frac{3a_0^2}{8k} - 1\,, \qquad \text{so that} \quad a_0^2 = \frac{8k}{3}(b_e + 1) \quad \text{and} \quad B_{ab}\,B^{ab}(r_0) = \Big(\frac{3}{16k^2}\frac{b_e}{b_e + 1}\Big)^2\,, \tag{191}$$

which vanishes for the extreme Schwarzschild–de Sitter black hole. The metric function Ω can then be rewritten as

$$\Omega(r) = a_0 \sum_{i=0}^{\infty} \Big(- \frac{c_1 k}{a_0^2 - 2k}\,(r - r_0)\Big)^i + \Big(\frac{c_1 k}{a_0^2 - 2k}\Big)^2 \frac{8k a_0}{18a_0^2 - 24k}\,b_e\,(r - r_0)^2 + \cdots\,. \tag{192}$$

The first term in $\Omega(r)$ is a geometric series which can by summed up to

$$a_0 \sum_{i=0}^{\infty} \Big(- \frac{c_1 k}{a_0^2 - 2k}\,(r - r_0)\Big)^i = \frac{a_0(a_0^2 - 2k)}{(a_0^2 - 2k - c_1 k\, r_0) + c_1 k\, r} = -\frac{1}{r}\,, \tag{193}$$

when the unique gauge

$$a_0 = -\frac{1}{r_0}\,, \qquad c_1 = \frac{1 - 2kr_0^2}{kr_0^3} \tag{194}$$

is used, so that

$$b_e = \frac{3}{8kr_0^2} - 1 \equiv \frac{\Lambda}{r_0^2} - 1\,. \tag{195}$$

The explicit solution thus reads

$$\Omega(r) \;=\; -\frac{1}{r} - \frac{4k\,b_e}{3r_0(3-4kr_0^2)}(r-r_0)^2 + \dots\,, \tag{196}$$

$$\mathcal{H}(r) \;=\; (r-r_0)^2\bigg[(1+2b_e) + \frac{1}{3r_0}(2+8b_e)(r-r_0) + \frac{b_e}{3r_0^2(3-4kr_0^2)}(r-r_0)^2 + \dots\bigg]\,. \tag{197}$$

Note that this solution has only one free parameter since r_0 and b_e are related via (195).

This class of spacetimes describes a black hole with the (double degenerate) *extreme horizon located at* $r = r_0$ *and a nonvanishing Bach tensor.* Its value on the horizon is given by the invariant (191),

$$B_{ab}\,B^{ab}(r_0) = \Big(\frac{r_0^2}{2k}\,b_e\Big)^2\,. \tag{198}$$

For $b_e = 0$, i.e., $r_0 = \mp\sqrt{\Lambda} \equiv \mp\sqrt{\frac{3}{8k}}$, this solution reduces to the extreme Schwarzschild–de Sitter solution (170).

To conclude: The class $[0,2]$ with $\Lambda = 3/(8k)$, expressed in terms of the series (196), (197) around the double degenerate horizon at r_0, represents another extreme Bachian–de Sitter black hole generalizing the extreme Schwarzschild–dS black hole with nonvanishing Bach tensor on the horizon.

5.6 Higher-order Schwarzschild–Bach–(A)dS black holes in the class $[-1,0]$

We will now prove that for a *generic* value of the cosmological constant Λ, the class $[n,p] = [-1,0]$ admits only the Schwarzschild–(A)dS solution. However, for *specific* values $\Lambda = -\frac{3}{2k(L+3)(L+2)}$, where $L \in \mathbb{N}$, $L \geq 0$, it represents "higher–order discrete Schwarzschild–Bach–(A)dS black holes".

5.6.1 Uniqueness of the Schwarzschild–(A)dS black hole for a generic value of Λ

For $l \geq 0$, Eq. (36) gives

$$\tfrac{1}{3}kc_{l+4}(l+4)(l+3)(l+2)(l+1) = (l+4)(l+5)a_0a_{l+4} + \sum_{i=1}^{l+3} a_ia_{l-i+4}(l-i+3)(l-3i+4)\,, \tag{199}$$

and

$$a_1 = a_2 = a_3 = 0\,. \tag{200}$$

Eq. (38) specifies

$$c_0 = \tfrac{1}{3}\Lambda a_0^2\,, \quad c_1 = 0\,, \quad c_2 = -1\,, \tag{201}$$

and

$$a_{l+4}[c_0(l+2)(l+3) - 2\Lambda a_0^2] + \tfrac{1}{6}a_0c_{l+4}l(l+1) + \tfrac{1}{3}a_{l+2} + \sum_{i=1}^{l+3} c_ia_{l+4-i}[(l+3-i)(l+2) + \tfrac{1}{6}i(i-1)]$$

$$= \tfrac{2}{3}\Lambda\bigg[\sum_{i=1}^{l+3} a_0a_ia_{l+4-i} + \sum_{j=1}^{l+3}\sum_{i=0}^{j} a_ia_{j-i}a_{l+4-j}\bigg]\,. \tag{202}$$

Finally, Eq. (37) implies

$$
\begin{aligned}
&a_0^2 c_{l+4}(l+1) + 2\Lambda a_0^3 a_{l+4}(l+5) = \sum_{j=1}^{l+3}\sum_{i=0}^{j} a_i a_{j-i} c_{l-j+4}(j-i-1)(l-j+3i+1) \\
&+ \sum_{i=0}^{l+2} a_i a_{l-i+2} + 3c_0 \sum_{i=1}^{l+3} a_i a_{l+4-i}(l+3-i)(i-1) - \tfrac{1}{3}k \sum_{i=1}^{l+3} c_i c_{l-i+4} i(l+4-i)(l+3-i)\left(l+\tfrac{1}{2}(5-3i)\right) \\
&- \Lambda\Bigg[2a_0^2 \sum_{i=1}^{l+3} a_i a_{l-i+4} + \sum_{m=1}^{l+3}\Big(\sum_{i=0}^{m} a_i a_{m-i}\Big)\Big(\sum_{j=0}^{l-m+4} a_j a_{l-m-j+4}\Big)\Bigg], \quad l \geq 1\,. \qquad (203)
\end{aligned}
$$

Therefore, the functions $\Omega(r)$ and $\mathcal{H}(r)$ have the form

$$
\Omega = \frac{a_0}{\Delta} + a_{\ell+1}\Delta^\ell + \cdots\,, \qquad \mathcal{H} = \frac{\Lambda}{3}a_0^2 - \Delta^2 + c_3\,\Delta^3 + c_m\,\Delta^m + \cdots\,, \qquad (204)
$$

where $\ell \equiv l+3 \geq 3$, $m \geq 4$.

Assuming that there exists l such that $a_i = 0$ for all $1 \leq i \leq l+3$ and $c_j = 0$ for all $4 \leq j \leq l+3$ (the following equations hold also for $l = 0$), equations (199) and (203) simplify to

$$
\begin{aligned}
2\Lambda(l+5)\,a_0\,a_{l+4} &= -(l+1)\,c_{l+4}\,, \qquad (205)\\
(l+5)\,a_0\,a_{l+4} &= \tfrac{1}{3}k\,(l+3)(l+2)(l+1)\,c_{l+4}\,. \qquad (206)
\end{aligned}
$$

For *generic values* of Λ (i.e., other than those given by condition (208)), necessarily $a_{l+4} = 0 = c_{l+4}$ and thus $a_i = 0 = c_{i+3}$ *for all* $i \geq 1$ (otherwise, the two equations (205), (206) are not compatible). Thus, for a generic Λ, the only free parameters are a_0 and c_3, and the metric functions become

$$
\Omega = \frac{a_0}{\Delta}\,, \qquad \mathcal{H} = \frac{\Lambda}{3}a_0^2 - \Delta^2 + c_3\,\Delta^3\,. \qquad (207)
$$

Using the remaining coordinate freedom, we can set $a_0 = -1$ and $r_0 = 0$ (implying $\Delta = r$), which gives the Schwarzschild–(A)dS solution (15) with the identification $c_3 = -2m$.

To conclude: The class $[-1, 0]$ with an arbitrary value of Λ, expressed in terms of the metric functions (207), represents the spherically symmetric Schwarzschild–(A)dS black hole.

5.6.2 Higher-order Schwarzschild–Bach–(A)dS black holes for special values of Λ

Interestingly, for *special* (discrete) values of the cosmological constant Λ given by

$$
\Lambda = -\frac{3}{2k(L+2)(L+3)}\,, \qquad \text{where} \quad L \in \mathbb{N}_0 \qquad (208)
$$

or equivalently, using (2), by

$$
\Lambda = -\frac{3\gamma}{24\beta + 2\alpha(L+2)(L+3)}\,, \qquad (209)
$$

the system of field equations (205), (206) also admits another class of solutions in the form

$$
a_{L+4} = k\,\frac{(L+1)(L+2)(L+3)}{3(L+5)\,a_0}\,b_L\,, \qquad L \in \mathbb{N}_0 \qquad (210)
$$

where

$$b_L \equiv c_{L+4} \tag{211}$$

is *an additional free "Bach" parameter.* All the other coefficients for $l > L$ are determined by (199) and (202) as

$$\tfrac{1}{3}kc_{l+4}(l+4)(l+3)(l+2)(l+1) = (l+4)(l+5)a_0a_{l+4} + \sum_{i=1}^{l+3} a_i a_{l-i+4}(l-i+3)(l-3i+4)\,, \tag{212}$$

and

$$\begin{aligned} a_{l+4}\frac{a_0^2 l(l+5)}{6(l+2)(l+3)}&\Big[2\Lambda(l+2)(l+3)+\frac{3}{k}\Big] \\ &= -\frac{a_0 l}{2k(l+4)(l+3)(l+2)}\sum_{i=1}^{l+3} a_i a_{l-i+4}(l-i+3)(l-3i+4) - \tfrac{1}{3}a_{l+2} \\ &\quad - \sum_{i=1}^{l+3} c_i a_{l+4-i}[(l+3-i)(l+2)+\tfrac{1}{6}i(i-1)] + \tfrac{2}{3}\Lambda\Big[\sum_{i=1}^{l+3} a_0 a_i a_{l+4-i} + \sum_{j=1}^{l+3}\sum_{i=0}^{j} a_i a_{j-i} a_{l+4-j}\Big], \end{aligned} \tag{213}$$

where Λ is given by (208), so that (213) gives a_{l+4} and then (212) c_{l+4}.

In the limit $\Delta \to 0$, the dominant terms of the metric functions become $\Omega = a_0\,\Delta^{-1} + \cdots$ and $\mathcal{H} = c_0 + \cdots = \frac{1}{3}\Lambda a_0^2 + \cdots$, similarly as in (15), in which case the relations (10), (11) give

$$\bar{r} = \frac{a_0}{\Delta} \;\to\; \infty \qquad (\text{so that } r \;\to\; \frac{a_0}{\bar{r}} + r_0)\,, \tag{214}$$

$$h \;\to\; -\frac{\Lambda}{3}a_0^2\,\bar{r}^2\,, \qquad f \;\to\; -\frac{\Lambda}{3}\,\bar{r}^2\,. \tag{215}$$

With the natural gauge choice $a_0 = -1$, this is exactly the asymptotic behavior of the Schwarzschild–(A)dS solution in the canonical form (8), (14) with $h = f = 1 - 2m/\bar{r} - \frac{\Lambda}{3}\,\bar{r}^2$ as $\bar{r} \to \infty$. Therefore, *these solutions asymptotically approach the Schwarzschild–(A)dS solution.*

In this case, the Bach and Weyl invariants (19) and (20) read

$$B_{ab}\,B^{ab} \;=\; \frac{[\,(L+1)(L+2)(L+3)(L+4)\,\Lambda\,]^2}{216a_0^4}\,b_L^2\,\Delta^{2L+8} + \cdots\,, \tag{216}$$

$$C_{abcd}\,C^{abcd} \;=\; \frac{12}{{a_0}^4}\,c_3^2\,\Delta^6 + \cdots\,, \tag{217}$$

respectively. For $b_L \neq 0 \neq c_3$, they are *both nonvanishing*, but for large values of $\bar{r}$, the invariants *approach zero asymptotically* for $\bar{r} \to \infty$ (that is for $\Delta \to 0$).

Note that the new free (non-Schwarzschild) *Bach parameter* b_L can be chosen to be *arbitrarily small* and thus (assuming analyticity) these solutions include arbitrarily small perturbations of the Schwarzschild–(A)dS solution.

The presence of a horizon is indicated by numerical calculations shown in Fig. 8 (however, note that the numerical calculation breaks down at the horizon). Recall that the condition $\mathcal{H} = 0$ implies $f(\bar{r}) = 0 = h(\bar{r})$, see (11). This suggests that, at least for some values of b_L, these solutions represent a *new family of black holes with* Λ. Such solutions can naturally be considered as a generalization of the Schwarzschild–(A)dS family since, in addition to mass encoded by the parameter c_3 and a cosmological constant Λ, they contain further physical/geometrical parameter b_L. With this parameter, the Bach

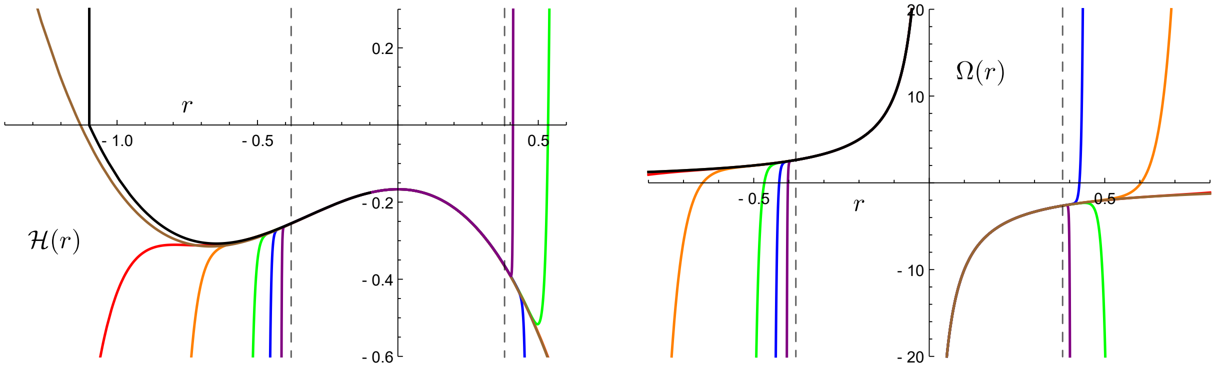

Figure 8: The metric functions $\mathcal{H}(r)$ (left) and $\Omega(r)$ (right) for the non-Schwarzschild (Bachian) solution $[-1,0]$ with the parameters $L=0$, $k=1/2$, $\Lambda=-1/2$, $r_0=0$, $a_0=-1$, $c_3=-1$ and $b_0\equiv c_4=1/20$. First 10 (red), 20 (orange), 50 (green), 100 (blue), and 300 (purple) terms in the expansions have been used. Furthermore, we plot the corresponding Schwarzschild–AdS solution obtained by setting $b_0\equiv c_4=0$ (brown). A numerical solution with initial values taken from the first 100 terms of the metric functions at $r=-0.2$ is also plotted (black). Note that the numerics breaks down on the horizon given by $\mathcal{H}=0$. Vertical dashed lines indicate the radius of the convergence. From the behaviour of the metric function Ω we can read off that, in the standard spherical coordinates $\bar{r}$, the metric functions converge from $\bar{r}\gtrapprox 2.63$. (In contrast with the $[0,1]$ case, the series in the $[-1,0]$ class do not approach geometric series asymptotically. Thus we have estimated the radius of convergence by searching for which values of r these series can be bounded by appropriately chosen geometric series.) All the plotted functions visually overlap within the radius of convergence, validating the results.

tensor B_{ab} becomes non-zero, see (216) with (23), and due to (2), the Ricci tensor R_{ab} is also non-trivial. Since they are vacuum solutions, the Birkhoff theorem is clearly violated in QG. Such solutions are not possible in the Einstein theory since $k=0$ formally corresponds to infinite value of Λ in (208).

For a general value of integer L ($L\in\mathbb{N}_0$), the metric functions read

$$\Omega(r) = \frac{a_0}{\Delta} + a_{L+4}\,\Delta^{L+3} + \cdots\,, \tag{218}$$

$$\mathcal{H}(r) = \frac{\Lambda}{3}\,a_0^2 - \Delta^2 + c_3\,\Delta^3 + c_{L+4}\,\Delta^{L+4} + \cdots\,. \tag{219}$$

Let us present additional terms in the following special cases for the lowest orders of L:

- For $L=0$, Eq. (208) gives $\Lambda=-\dfrac{1}{4k}$. Then, the first coefficients in the series (29) and (30) are

$$a_1=a_2=a_3=0\,,\quad a_4=\frac{2k}{5a_0}\,b_0\,,\quad a_5=0\,,\quad a_6=-\frac{288k^2}{49a_0^3}\,b_0\,,\quad a_7=\frac{4k^2}{a_0^3}\,c_3b_0\,,\dots\,, \tag{220}$$

$$c_0=\frac{\Lambda}{3}\,a_0^2\,,\quad c_1=0\,,\quad c_2=-1\,,\quad c_5=0\,,\quad c_6=-\frac{72k}{35a_0^2}\,b_0\,,\quad c_7=\frac{4k}{5a_0^2}\,c_3b_0\,,\dots\,, \tag{221}$$

  with the free parameters a_0, c_3, and $b_0\equiv c_4$. Choosing $a_0=-1$, $r_0=0$ and denoting $c_3=-2m$, the metric functions become

$$\Omega=-\frac{1}{r}-\frac{2}{5}k\,b_0\,r^3+\cdots\,,\qquad \mathcal{H}=\frac{\Lambda}{3}-r^2-2m\,r^3+b_0\,r^4+\cdots\,, \tag{222}$$

  which reduces to the Schwarzschild–(A)dS solution (15) when $b_0=0$.

- For $L=1$, equation (208) gives $\Lambda = -\dfrac{1}{8k}$. The coefficients are

$$a_1 = a_2 = a_3 = a_4 = 0\,, \quad a_5 = \frac{4k}{3a_0}\,b_1\,, \quad a_6 = 0\,, \quad a_7 = -\frac{400k^2}{9a_0^3}\,b_1\,, \ldots\,, \tag{223}$$

$$c_0 = \frac{\Lambda}{3}\,a_0^2\,, \quad c_1 = 0\,, \quad c_2 = -1\,, \quad c_4 = 0\,, \quad c_6 = 0\,, \quad c_7 = -\frac{80k}{9a_0^2}\,b_1\,, \ldots\,, \tag{224}$$

with free parameters a_0, c_3, and $b_1 \equiv c_5$. Choosing $a_0 = -1$, $r_0 = 0$ and denoting $c_3 = -2m$,

$$\Omega = -\frac{1}{r} - \frac{4}{3}k\,b_1\,r^4 + \cdots\,, \qquad \mathcal{H} = \frac{\Lambda}{3} - r^2 - 2m\,r^3 + b_1\,r^5 + \cdots\,, \tag{225}$$

which also reduces to the Schwarzschild–(A)dS solution when $b_1 = 0$.

- Explicit expressions for $L \geq 2$ can be obtained analogously. Notice that as $L \to \infty$, the cosmological constant $\Lambda \to 0$, and the solution approaches the Schwarzschild black hole metric.

To conclude: The class $[-1,0]$ with the discrete values of Λ given by (208), with the metric functions expressed in terms of the series (218) and (219) around an arbitrary point r_0 corresponding to the asymptotic physical region $\bar{r} \to \infty$, represents the family of spherically symmetric higher-order discrete Schwarzschild–Bach–(A)dS black holes with an additional Bach parameter b_L (where $L = 0, 1, 2, \ldots$).

5.7 Bachian singularity in the class $[n,p] = [1,0]$

Now let us investigate the case $[1,0]$. Since the Bach tensor is always nonvanishing (see below), this class does not contain the Schwarzschild–(A)dS as a special case. Moreover, it possesses a curvature singularity in both the Bach and Weyl tensors at $r = r_0$ corresponding to $\bar{r} = 0$. Thus it can be nicknamed as *Bachian singularity*.

After relabeling $l \to l-3$, Eq. (36) for the case $[n,p] = [1,0]$ yields

$$c_{l+1} = \frac{3}{k\,(l+1)l(l-1)(l-2)} \sum_{i=0}^{l-3} a_i\,a_{l-3-i}(l-2-i)(l-5-3i)\,, \qquad \forall\, l \geq 3\,. \tag{226}$$

The lowest order $l=0$ of (38) gives

$$a_1 = -\frac{a_0 c_1}{2c_0}\,, \tag{227}$$

while higher orders imply for $l \geq 1$ (the first sum is empty for $l = 1, 2$)

$$a_{l+1} = \frac{1}{(l+1)(l+2)\,c_0}\Bigg[\tfrac{2}{3}\Lambda \sum_{j=0}^{l-3}\sum_{i=0}^{j} a_i a_{j-i} a_{l-j-3} - \tfrac{1}{3}a_{l-1} - \sum_{i=1}^{l+1} c_i a_{l-i+1}\big[(l-i+2)(l+1) + \tfrac{1}{6}i(i-1)\big]\Bigg]. \tag{228}$$

Finally, the lowest order $l=0$ of (37) leads to

$$c_3 = \frac{1}{6kc_1}\left[9a_0^2 c_0 + 2k(c_2^2 - 1)\right]. \tag{229}$$

All the coefficients a_{l+1}, c_{l+1} can be determined from the recurrent relations (226), (228), which give

$$\Omega(r) = (r - r_0)\Big[a_0 + \sum_{i=1}^{\infty} a_i\,(r - r_0)^i\Big]\,, \qquad \mathcal{H}(r) = c_0 + \sum_{i=1}^{\infty} c_i\,(r - r_0)^i\,, \tag{230}$$

where the coefficients a_1, c_3 are given by (229) and (227), respectively, while

$$\begin{aligned} a_2 &= -\frac{a_0}{18c_0^2}\left[c_0(1 + 7c_2) - 6c_1^2\right]\,, \\ a_3 &= -\frac{a_0}{36kc_0^3c_1}\left[18a_0^2c_0^3 + k[4c_0^2(c_2^2 - 1) - 2c_0c_1^2(1 + 10c_2) + 9c_1^4]\right]\,, \dots\,, \\ c_4 &= -\frac{a_0^2}{4k}\,, \qquad c_5 = \frac{3a_0^2c_1}{40kc_0}\,, \dots\,. \end{aligned} \tag{231}$$

Here a_0, c_0, c_1, c_2 are four free parameters (using the gauge freedom (13), one can fix some of these parameters, e.g., $a_0 = 1$ and $r_0 = 0$). The coefficients (231) coincide with those of [15] since the cosmological constant Λ enters only the coefficients a_i with $i \geq 4$ and c_i with $i \geq 9$.

From the scalar invariants (19), (20),

$$B_{ab}\,B^{ab}(r) = \frac{3c_0^2}{4a_0^4k^2}\frac{1}{(r - r_0)^8} + \dots\,, \qquad C_{abcd}\,C^{abcd}(r) = \frac{4}{3a_0^4}\frac{(1 + c_2)^2}{(r - r_0)^4} + \dots\,, \tag{232}$$

it follows that the Bach tensor B_{ab} cannot be set to zero since, by definition, $c_0 \neq 0$. Consequently, the Schwarzschild–(A)dS solution does not belong to this class and the Bach invariant always diverges at $r = r_0$. Notice that there is also a Weyl curvature singularity at $r = r_0$ (in the special case $c_2 = -1$, $C_{abcd}\,C^{abcd} \propto (r - r_0)^{-2}$).

For (230), the limit $r \to r_0$ implies

$$\bar{r} = \Omega(r) \to 0\,, \tag{233}$$

and thus the Bach/Weyl curvature *singularity is located at the origin*, which is also indicated by the behaviour of the metric functions (10), (11) in terms of the physical radial coordinate $\bar{r}$,

$$h \sim -c_0\,\bar{r}^2 \to 0\,, \qquad f \sim -a_0^2c_0\,(\bar{r})^{-2} \to \infty\,. \tag{234}$$

Notice also that in the $\Lambda = 0$ case, this $[n, p] = [1, 0]$ class of solutions corresponds to the $(s, t) = (2, 2)$ family [8, 10, 15, 26, 27].

To conclude: The class $[1, 0]$, with the metric functions expressed in terms of the series (230) around an arbitrary point r_0 corresponding to the physical/geometrical origin $\bar{r} = 0$, represents a spherically symmetric spacetime with Bachian singularity.

5.8 Empty class $[n, p] = [0, > 2]$

In what follows we show that, in fact, this class is empty.

For $p \notin \mathbb{N}$, Eq. (36) implies $c_0 = 0$, which must be nonzero by definition. For all integers $p > 2$ at the order Δ^0, Eqs. (37) and (38) give the conditions $3a_0^2(1 - \Lambda a_0^2) = 2k$ and $2\Lambda a_0^2 = 1$, respectively, which together imply

$$\Lambda = \frac{3}{8k}\,, \qquad a_0 = \pm\sqrt{\frac{4k}{3}}\,. \tag{235}$$

First, let us discuss the case $[0, 3]$. At the order Δ^0, Eq. (36) gives $a_2 = (a_1^2 + 4kc_1)/a_0$. At the order Δ^1, Eq. (38) gives $a_1 = \frac{3}{2}a_0c_0$, while Eq. (36) gives $a_3 = 27a_0c_0^3/8 + 6kc_0c_1/a_0 + 20kc_2/(3a_0)$. Finally, at the order Δ^2, Eq. (38) implies $c_0 = 0$, and thus this case cannot occur.

Let us show that this is also the case for all integers $p > 3$. For $p > 3$ at the order Δ^1, Eq. (38) implies $a_1 = 0$. Using the mathematical induction, we show that all a_i for $1 \leq i \leq p-1$ vanish. So let us assume that all a_i for $1 \leq i \leq j-1$, $j < p-2$, vanish. Then Eq. (36) at the order Δ^{j-2} gives $a_0a_j(j-1) = 0$ which implies that also $a_j = 0$. Thus the second nonvanishing coefficient (after a_0) is a_{p-2} which is determined by the order Δ^{p-4} of Eq. (36) as

$$a_{p-2} = kp(p-1)\frac{c_0}{3a_0}\,. \tag{236}$$

Then using (235), Eqs. (36) and (38) at the orders Δ^{2p-6} and Δ^{2p-4} yield

$$2(2p-5)a_0a_{2p-4} - (p-1)a_{p-2}^2 - \tfrac{4}{3}kc_{p-2}(p-1)(2p-3)(2p-5) = 0\,, \tag{237}$$

$$-2a_{2p-4} + c_0a_{p-2}\left[3(p-2)(2p-3) + \tfrac{1}{2}p(p-1)\right] - 6\Lambda a_0a_{p-2}^2 + a_0c_{p-2}(p-1)(2p-3) = 0\,, \tag{238}$$

respectively. A linear combination of these two equations, using also (235) and (236), leads to

$$p(p-1)(p-2)(5p^2 - 21p + 20)c_0^2 = 0\,. \tag{239}$$

This, for integers $p > 3$, implies $c_0 = 0$ which is a contradiction.

To conclude: The class $[0, > 2]$ is empty and thus, under our assumptions, there are no black hole solutions to QG field equations with a higher-than-doubly degenerate horizon.

5.9 Solutions with regular Bachian infinity in the class $[n, p] = [< 0, 2n + 2]$

Finally, we will show that there are *infinitely many vacuum solutions* in the class $[n, p] = [< 0, 2n + 2]$, namely $[n, p] = [-J/2, 2 - J]$, where $J \in \mathbb{N}$, $J > 2$. Each of these solutions contains an asymptotic region $\bar{r} \to \infty$ where both the Bach and Weyl invariants approach a nontrivial constant. Therefore, these solutions *do not admit the classical Schwarzschild–(A)dS limit.*

In Eq. (38), the first and last terms start with the power Δ^{3n}, while the second one with $\frac{1}{3}a_0\Delta^n$. Since in all sums there are integer steps, to allow $a_0 \neq 0$, the expression $3n - n$ has to be a (negative) integer. Thus,

$$n = -J/2\,, \quad \text{where} \quad J \in \mathbb{N}\,, \quad J \geq 3\,, \tag{240}$$

and the coefficient p is $p = 2n + 2 = -J + 2$. Then from Eq. (64) we obtain a *unique value of the cosmological constant*,

$$\Lambda = \frac{3}{32k}\frac{11J^2 - 12J + 4}{1 - J^2}\,, \quad |\Lambda| \geq \frac{201}{256|k|}\,, \qquad \text{and} \quad c_0 = \frac{3a_0^2}{4k(1 - J^2)}\,. \tag{241}$$

The subleading terms in Eqs. (36) and (38) give

$$\begin{aligned} c_1 &= -\frac{3a_0a_1(n+2)}{2k(n+1)(2n+1)(2n+3)}\,,\\ 0 &= c_0a_1(11n^2 + 18n + 7) + a_0c_1(11n^2 + 11n + 3) - 6\Lambda a_0^2a_1\,, \end{aligned} \tag{242}$$

respectively. Their combination yields

$$a_1\,(55J^4 - 335J^3 + 554J^2 - 412J + 120) = 0\,, \tag{243}$$

which cannot be fulfilled for $J \in \mathbb{N}$, $J > 2$ unless

$$a_1 = 0 = c_1 \,. \tag{244}$$

As an illustration of such solutions, let us present two explicit examples. The highest possible value of n is obtained for $J = 3$, namely $n = -3/2$, $p = -1$. For $J = 4$, we obtain $n = -2$, $p = -2$:

- In the case $[-3/2, -1]$ we get

$$\Lambda = -\frac{201}{256k} \,, \tag{245}$$

$$a_1 = a_2 = a_3 = a_4 = 0 \,, \quad a_5 = \frac{16kc_5}{67a_0} \,, \quad a_6 = \frac{43520k^2}{117117a_0^3} \,, \quad a_7 = 0 \,, \dots \,, \tag{246}$$

$$c_0 = -\frac{3a_0^2}{32k} \,, \quad c_1 = c_2 = 0 \,, \quad c_3 = -\frac{4}{13} \,, \quad c_4 = 0 \,, \quad c_6 = \frac{16864k}{39039a_0^2} \,, \quad c_7 = 0 \,, \dots \,, \tag{247}$$

  and thus

$$\Omega(r) = a_0(r-r_0)^{-3/2} + \frac{16kc_5}{67a_0}(r-r_0)^{7/2} + \frac{43520k^2}{117117a_0^3}(r-r_0)^{9/2} + \cdots \,, \tag{248}$$

$$\mathcal{H}(r) = -\frac{3a_0^2}{32k}(r-r_0)^{-1} - \frac{4}{13}(r-r_0)^2 + c_5(r-r_0)^4 + \frac{16864k}{39039a_0^2}(r-r_0)^5 + \cdots \,, \tag{249}$$

  with two free parameters a_0, c_5.

- In the case $[-2, -2]$ we get

$$\Lambda = -\frac{33}{40k} \,, \tag{250}$$

$$a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0 \,, \quad a_7 = \frac{20kc_7}{33a_0} \,, \quad a_8 = \frac{400k^2}{441a_0^3} \,, \quad a_9 = a_{10} = 0 \,, \dots \,, \tag{251}$$

$$c_0 = -\frac{a_0^2}{20k} \,, \quad c_1 = c_2 = c_3 = 0 \,, \quad c_4 = -\frac{1}{7} \,, \quad c_5 = c_6 = 0 \,, \quad c_8 = \frac{40k}{63a_0^2} \,, \quad c_9 = c_{10} = 0 \,, \dots \,, \tag{252}$$

  with two free parameters a_0, c_7.

For all admitted values of n, the invariants (19), (20) approach a constant as $r \to r_0$, namely

$$B_{ab}\,B^{ab} \;=\; p^2(p-1)^2(p-3)^2(11p^2-32p+24)\frac{c_0^4}{144a_0^8} + \cdots \,, \tag{253}$$

$$C_{abcd}\,C^{abcd} \;=\; p^2(p-1)^2\frac{c_0^2}{3a_0^4} + \cdots \,. \tag{254}$$

Note that for the permitted values of p, the invariant (253) *cannot vanish*, and therefore this class does not admit the Schwazschild–(A)dS solution as a limit.

Moreover, since $n < 0$, it follows from (10) that

$$\bar{r} = \Omega = a_0\,\Delta^n + \cdots \;\to\; \infty \tag{255}$$

as $r \to r_0$. Thus this limit corresponds to an *asymptotic region*, where the standard metric functions (11) behave as

$$h = -a_0^2 c_0 \, \Delta^{4n+2} + \cdots \;\sim\; \bar{r}^{\,4+2/n} + \cdots \to \;\infty\,, \tag{256}$$

$$f = -n^2 c_0 \, \Delta^{2n} + \cdots \qquad \sim\; \bar{r}^{\,2} + \cdots \to \;\infty\,. \tag{257}$$

Note that, in the notation of [8,10], these solutions would be described as families $(s,t) = \big(-2, 4 - \frac{4}{J}\big)_\infty$ with $J \in \mathbb{N}$, $J \geq 3$, i.e., with the parameter $t \in [\frac{8}{3}, 4)$.

To conclude: The class $[-J/2, 2-J]$ with Λ uniquely determined by (241) is, in fact, an infinite discrete family of metrics parametrized by an integer $J \geq 3$. They all have a regular Bachian infinity because both the Bach and Weyl invariants approach a finite nonzero value (253), (254) in the asymptotic physical region $\bar{r} \to \infty$. In particular, for $J = 3$ the metric functions are expressed in terms of the series (248) and (249).

6 Discussion of solutions using the expansion in powers of r^{-1}

Now let us study and classify all possible solutions to the field equations of QG in the case of an asymptotic expansion as $r \to \infty$. Here we assume that the metric functions $\Omega(r)$, $\mathcal{H}(r)$ of (9) can be expanded in (negative) powers of r as (33), (34), that is

$$\Omega(r) = r^N \sum_{i=0}^{\infty} A_i\, r^{-i}\,, \qquad \mathcal{H}(r) = r^P \sum_{i=0}^{\infty} C_i\, r^{-i}\,. \tag{258}$$

Employing these expansions in the field equation (25), we obtain

$$\begin{aligned}\sum_{l=-2N+2}^{\infty} r^{-l} \sum_{i=0}^{l+2N-2} A_i\, A_{l-i+2N-2}\,(l-i+N-2)(l-3i+3N-1) \\ = \tfrac{1}{3}k \sum_{l=-P+4}^{\infty} r^{-l}\, C_{l+P-4}\,(l-4)(l-3)(l-2)(l-1)\,.\end{aligned} \tag{259}$$

The second field equation (26) gives

$$\begin{aligned}&\sum_{l=-2N-P+2}^{\infty} r^{-l} \sum_{j=0}^{l+2N+P-2} \sum_{i=0}^{j} A_i\, A_{j-i}\, C_{l-j+2N+P-2}\,(j-i-N)(l-j+3i-N-2) \\ &\quad + \sum_{l=-2N}^{\infty} r^{-l} \sum_{i=0}^{l+2N} A_i\, A_{l-i+2N} - \Lambda \sum_{l=-4N}^{\infty} r^{-l} \sum_{m=0}^{l+4N} \Big(\sum_{i=0}^{m} A_i\, A_{m-i}\Big)\Big(\sum_{j=0}^{l-m+4N} A_j\, A_{l-m-j+4N}\Big) \\ &= \tfrac{1}{3}k\Big[2 + \sum_{l=-2P+4}^{\infty} r^{-l} \sum_{i=0}^{l+2P-4} C_i\, C_{l-i+2P-4}\,(i-P)(l-i+P-4)(l-i+P-3)(l-\tfrac{3}{2}i+\tfrac{3}{2}P-\tfrac{5}{2})\Big],\end{aligned} \tag{260}$$

and finally, the trace equation (27) leads to

$$\begin{aligned}\sum_{l=-N-P+2}^{\infty} r^{-l} \sum_{i=0}^{l+N+P-2} C_i\, A_{l-i+N+P-2}\,\big[(l-i+P-2)(l-1) + \tfrac{1}{6}(i-P)(i-P+1)\big] \\ + \tfrac{1}{3} \sum_{l=-N}^{\infty} r^{-l}\, A_{l+N} = \tfrac{2}{3}\Lambda \sum_{l=-3N}^{\infty} r^{-l} \sum_{j=0}^{l+3N} \sum_{i=0}^{j} A_i\, A_{j-i}\, A_{l-j+3N}\,.\end{aligned} \tag{261}$$

Expressions for the coefficients C_j in terms of A_js can be obtained by comparing coefficients appearing at the same powers of r^{-l} on both sides of (259). The terms with the lowest order imply that there are three distinct cases:

- Case I^∞: $-2N+2 < -P+4\,,$ i.e., $P < 2N+2\,,$
- Case II^∞: $-2N+2 > -P+4\,,$ i.e., $P > 2N+2\,,$
- Case III^∞: $-2N+2 = -P+4\,,$ i.e., $P = 2N+2\,.$

In what follows, we derive all possible solutions in these cases.

6.1 Case I^∞

For $-2N+2 < -P+4$, the highest order (namely r^{-l}, $-l = 2N-2$) in (259) gives

$$N(N+1) = 0\,, \tag{262}$$

and thus only the cases $N = 0$ and $N = -1$ are allowed.

Since the leading orders appearing in (261),

$$\big[6N(N+P-1) + P(P-1)\big]C_0\, r^{N+P-2} + \cdots = -2\, r^N + \cdots + 4\Lambda A_0^2 r^{3N}\,, \tag{263}$$

for $N = 0$ are r^{P-2}, r^0, and r^0, respectively, the condition $P-2 < 2N = 0$ implies that the highest order is $0 = 2(-1+\Lambda A_0^2)r^0$, leading to $A_0^2 = 1/\Lambda$. Then Eq. (261) implies $\Lambda = 3/(8k)$.

In contrast, the case $N = -1$ cannot occur since the leading powers are r^{P-3}, r^{-1}, and r^{-3}, respectively, however, the condition $P-3 < 2N-1 = -3 < -1$ implies that the highest order in (263) is $0 = -2r^{-1}$, which is a contradiction.

To summarize: The only possible solutions in Case I^∞ are

$$[N, P] = [0, <2]^\infty \quad \text{with} \quad A_0^2 = \frac{1}{\Lambda} \quad \text{and} \quad \Lambda = \frac{3}{8k} \quad \Big(\text{so that } A_0^2 = \frac{8k}{3}\Big)\,. \tag{264}$$

6.2 Case II^∞

The condition $-2N+2 > -P+4$ implies that the highest order (namely r^{-l}, $l = -P+4$) in (259) is on the right hand side, which gives

$$P(P-1)(P-2)(P-3) = 0\,, \tag{265}$$

leading to four possible cases $P = 0$, $P = 1$, $P = 2$, and $P = 3$. For these four cases, the leading orders of (261), i.e., (263), read

$$\text{for} \quad P=0,\ N<-1: \qquad [6N(N-1)]C_0\, r^{N-2} + \cdots = -2\, r^N + \cdots + 4\Lambda r^{3N} + \cdots\,, \tag{266}$$

$$\text{for} \quad P=1,\ N<-1/2: \qquad [6N^2]C_0\, r^{N-1} + \cdots = -2\, r^N + \cdots + 4\Lambda r^{3N} + \cdots\,, \tag{267}$$

$$\begin{aligned} \text{for} \quad P=2,\ N<0: \qquad & \big[6N(N+1)+2\big]C_0\, r^N + \cdots = -2\, r^N + \cdots + 4\Lambda r^{3N} + \cdots \\ & \Rightarrow \ \text{necessarily} \quad (3N^2+3N+1)C_0 = -1\,, \end{aligned} \tag{268}$$

$$\begin{aligned} \text{for} \quad P=3,\ N<1/2: \qquad & \big[6N(N+2)+6\big]C_0\, r^{N+1} + \cdots = -2\, r^N + \cdots + 4\Lambda r^{3N} + \cdots \\ & \Rightarrow \ \text{necessarily} \quad N = -1\,. \end{aligned} \tag{269}$$

Eqs. (266), (267) do not admit solutions for $N < -1$ and $N < -1/2$, respectively. For the case $P = 2$, implying $N < 0$, we employ the leading orders of Eq. (260),

$$3A_0^2\,[N(3N+2)C_0 + 1]\, r^{2N} + 2k(C_0^2 - 1) - 3\Lambda A_0^4 r^{4N} + \cdots = 0\,, \tag{270}$$

which requires $(3N^2+2N)C_0 = -1$. In combination with (268), we obtain $N = -1$, $C_0 = -1$.

To summarize: The only possible two classes of solutions in Case II^∞ are

$$[N, P] \,=\, [-1, 3]^\infty\,, \tag{271}$$

$$[N, P] \,=\, [-1, 2]^\infty\,. \tag{272}$$

6.3 Case III$^\infty$

For $P = 2N + 2$, the highest order (which is r^{-l}, $l = 2 - 2N$) in (259) appears on both sides and implies

$$P(P-2)\big[3A_0^2 + 4kC_0(P-1)(P-3)\big] = 0\,, \tag{273}$$

leading to three possible subcases $P = 0$, $P = 2$, and $3A_0^2 = -4kC_0(P-1)(P-3)$ with $P \neq 0,1,2,3$, corresponding to $N = -1$, $N = 0$, and $3A_0^2 = -4kC_0(4N^2 - 1)$ with $N \neq -1, -1/2, 0, 1/2$, respectively.

For these three cases, the leading orders of (261),

$$\big[(11N^2 + 6N + 1)C_0 - 2\Lambda A_0^2\big]\, r^{3N} + \cdots = -r^N + \cdots\,, \tag{274}$$

read (note that necessarily $N \geq 0$):

$$\text{for}\quad N = -1 \Leftrightarrow P = 0: \qquad (6C_0 - 2\Lambda A_0^2)\, r^{-3} + \cdots = -\, r^{-1} + \cdots \quad \text{not compatible}\,, \tag{275}$$

$$\text{for}\quad N = 0 \Leftrightarrow P = 2: \qquad C_0 - 2\Lambda A_0^2 + \cdots = -1 + \cdots \quad \Rightarrow C_0 = -1 + 2\Lambda A_0^2\,, \tag{276}$$

$$\text{for}\quad 3A_0^2 = 4kC_0(1 - 4N^2): \qquad (11N^2 + 6N + 1)C_0 - 2\Lambda A_0^2 + \cdots = 0\,. \tag{277}$$

Eq. (260) for the case (276) requires $3A_0^2(1 - \Lambda A_0^2) + 2k(C_0^2 - 1) = 0$, which after substituting for C_0 from (276) gives $(1 - \Lambda A_0^2)(8k\Lambda - 3) = 0$. Thus, there are two possibilities:

$$\Lambda A_0^2 = 1 \quad \Rightarrow \quad C_0 = 1\,, \tag{278}$$

and

$$\Lambda = \frac{3}{8k} \quad \Rightarrow \quad C_0 = \frac{3}{4k} A_0^2 - 1\,. \tag{279}$$

In the case (277), the two conditions, namely $3A_0^2 = 4kC_0(1 - 4N^2) = -4kC_0(P-3)(P-1)$, with $P \neq 0,1,2,3$ (and thus $N = P/2 - 1 \neq -1, -1/2, 0, 1/2$), and $(11N^2 + 6N + 1)C_0 = 2\Lambda A_0^2$ (for $N > 0$, $N \neq 0, 1/2$) imply

$$\Lambda = \frac{3}{8k}\frac{11N^2 + 6N + 1}{1 - 4N^2} \quad \Rightarrow \quad C_0 = \frac{3}{4k}\frac{A_0^2}{1 - 4N^2}\,. \tag{280}$$

The highest order of Eq. (270) is then identically satisfied.

To summarize: In Case III$^\infty$, there are three possible classes of solutions

$$[N,P] = [0,2]^\infty \quad \text{with} \quad \Lambda A_0^2 = 1\,, \quad C_0 = 1\,, \tag{281}$$

$$[N,P] = [0,2]^\infty \quad \text{with} \quad \Lambda = \frac{3}{8k}\,, \quad C_0 = \frac{3}{4k}A_0^2 - 1\,, \tag{282}$$

$$[N,P] = [>0, 2N+2]^\infty \quad \text{with} \quad \Lambda = \frac{3}{8k}\frac{11N^2 + 6N + 1}{1 - 4N^2}\,,\ C_0 = \frac{3}{4k}\frac{A_0^2}{1 - 4N^2}\,,\ N \neq \frac{1}{2}\,. \tag{283}$$

7 Description and study of all possible solutions in powers of r^{-1}

In this section, by solving Eqs. (259), (260), and the trace (261), we will study all spherically symmetric solutions contained in Cases I$^\infty$, II$^\infty$, III$^\infty$ in the asymptotic region in the coordinate r, i.e., as $r \to \infty$. As follows from the previous section, there are six classes of solutions to be discussed, namely (264), (271), (272), (281), (282), and (283).

7.1 Schwarzschild–Bach–(A)dS black hole in the class $[N, P] = [-1, 3]^\infty$: near the singularity

The expansions (33), (34) in negative powers of r for $N = -1$, $P = 3$, which is the class (271), gives

$$\Omega(r) \;=\; -\frac{1}{r} + \frac{B}{r}\left(\frac{2}{9}\,\frac{1}{C_0^3 r^3} + \frac{1}{6}\,\frac{1}{C_0^4 r^4} + \frac{2}{15}\,\frac{1}{C_0^5 r^5} + \dots\right), \tag{284}$$

$$\mathcal{H}(r) \;=\; \frac{\Lambda}{3} - r^2 - \left(\frac{\Lambda}{3} - r_h^2\right)\frac{r^3}{r_h^3} + B\left(\frac{1}{C_0^2} - \frac{1}{90k}\,\frac{1}{C_0^3 r^3} - \frac{1}{140k}\,\frac{1}{C_0^4 r^4} - \frac{1}{210k}\,\frac{1}{C_0^5 r^5} + \dots\right), \tag{285}$$

which represents the Schwarzschild–Bach–(A)dS black hole in QG, and for $B = 0$ reduces to the Schwarzschild–(A)dS solution (87) with the horizon located at r_h.

Since in the limit $r \to \infty$ we obtain $\bar{r} = \Omega(r) \sim -1/r \to 0$ and $\mathcal{H} \to \infty$, the physical origin $\bar{r} = 0$ represents the curvature singularity, cf. (304). Note also that $h(\bar{r}) \sim 1/(r_h\,\bar{r}) \to \infty$ and $f(\bar{r}) \sim h(\bar{r})$, see (11). In what follows, we derive this class of solutions.

Eq. (259) gives (relabeling $l \to l + 2$) all C_{l+1} in terms of $A_0, \dots, A_{l-2}$, starting from $C_4 = 0$:

$$C_{l+1} = \frac{3}{k\,(l-2)(l-1)l(l+1)} \sum_{i=0}^{l-2} A_i\, A_{l-2-i}(l-1-i)(l-2-3i)\,, \qquad \forall\, l \geq 3\,. \tag{286}$$

Eq. (261) determines all A_l in terms of $A_0, \dots, A_{l-1}$ and $C_1, \dots, C_l$, starting from A_1:

$$l^2 C_0 A_l = \tfrac{2}{3}\Lambda \sum_{j=0}^{l-3}\sum_{i=0}^{j} A_i A_{j-i} A_{l-j-3} - \tfrac{1}{3}\,A_{l-1} - \sum_{i=1}^{l} C_i A_{l-i}\left[\,l(l-i) + \tfrac{1}{6} i(i+1)\right], \qquad \forall\, l \geq 1\,. \tag{287}$$

Finally, the additional constraint, namely

$$C_2 = \frac{C_1^2 - 1}{3C_0}\,, \tag{288}$$

follows from the lowest nontrivial order $l = 0$ of the field equation (260).

Thus, this class has four initial parameters A_0, C_0, C_1, C_3. The constant C_2 is given by (288), and A_l, C_{l+3} for all $l \geq 1$ by the recurrent relations (287), (286), namely

$$C_4 = 0\,, \quad C_5 = 0\,, \quad C_6 = A_0^2\,\frac{(C_1+1)^2(C_1-2) + 9C_0^2(A_0^2\Lambda - 3C_3)}{2430 C_0^3 k}\,, \dots\,, \tag{289}$$

$$A_1 = -A_0\frac{(C_1+1)}{3C_0}\,, \quad A_2 = A_0\frac{(C_1+1)^2}{9C_0^2}\,, \tag{290}$$

$$A_3 = -A_0\frac{(C_1+1)^2(13 + 7C_1) + 54C_0^2 C_3 - 18\Lambda A_0^2 C_0^2}{243 C_0^3}\,, \dots\,. \tag{291}$$

7.1.1 Identification of the Schwarzschild–(A)dS black hole

For this class of solutions of the form (258), the curvature invariants (19), (20) read

$$B_{ab}\, B^{ab}(r \to \infty) = \left(45\frac{C_0}{A_0^4}\, C_6\right)^2, \qquad C_{abcd}\, C^{abcd}(r \to \infty) \sim 12\frac{C_0^2}{A_0^4}\, r^6\,. \tag{292}$$

To identify the Schwarzschild–(A)dS solution, we have to set the Bach tensor to zero, which is achieved by setting

$$C_6 = 0\,. \tag{293}$$

This is equivalent to $27C_0^2C_3 = (C_1+1)^2(C_1-2)+9\Lambda A_0^2C_0^2$. Then the sequence A_i reduces to a geometrical series, while the sequence C_i truncates to a 3rd-order polynomial

$$A_i = A_0\Big(-\frac{C_1+1}{3C_0}\Big)^i \quad \text{for all } i\geq 0\,, \tag{294}$$

$$C_2 = \frac{C_1^2-1}{3C_0}\,,\quad C_3 = \frac{(C_1+1)^2(C_1-2)+9\Lambda A_0^2C_0^2}{27C_0^2}\,,\quad C_i = 0 \quad \text{for all } i\geq 4\,, \tag{295}$$

and the metric functions can be expressed in the closed form

$$\Omega(r) \,=\, \frac{A_0}{r}\sum_{i=0}^{\infty}\Big(-\frac{C_1+1}{3C_0\,r}\Big)^i = \frac{A_0}{r+(C_1+1)/(3C_0)}\,, \tag{296}$$

$$\mathcal{H}(r) \,=\, C_0\,r^3 + C_1\,r^2 + \frac{C_1^2-1}{3C_0}\,r + \frac{(C_1+1)^2(C_1-2)+9\Lambda A_0^2C_0^2}{27C_0^2}\,. \tag{297}$$

Choosing a gauge (13) such that

$$A_0 = -1\,,\qquad C_1 = -1\,, \tag{298}$$

the metric functions simplify to

$$\bar r = \Omega(r) = -\frac{1}{r}\,,\qquad \mathcal{H}(r) = \frac{\Lambda}{3} - r^2 + C_0\,r^3\,, \tag{299}$$

which is the *Schwarzschild–(A)dS black hole metric* (15), where C_0 is given by the horizon position r_h and the cosmological constant Λ as

$$C_0 = \Big(r_h^2 - \frac{\Lambda}{3}\Big)\frac{1}{r_h^3}\,, \tag{300}$$

see (87).

7.1.2 More general Schwarzschild–Bach–(A)dS black hole

For a more general solution with a nonvanishing Bach tensor, see (292), it is convenient to introduce a *dimensionless Bach parameter* B proportional to C_6. Employing (298), $C_6 = -(C_3 - \frac{\Lambda}{3})/(90kC_0)$, and thus we choose

$$B \equiv C_0^2\left(C_3 - \frac{\Lambda}{3}\right)\,. \tag{301}$$

Using (301) and the gauge (298), the recurrent relations (287), (286) simplify to

$$\begin{aligned} &A_0 = -1\,,\qquad A_1 = 0\,,\qquad A_2 = 0\,,\qquad A_3 = \frac{2B}{9C_0^3}\,,\qquad A_4 = \frac{B}{6C_0^4}\,,\\ &A_5 = \frac{2B}{15C_0^5}\,,\qquad A_6 = -\frac{B(10B-9)}{81C_0^6} - \frac{7B(3+40k\Lambda)}{9720kC_0^4}\,,\ldots\,, \end{aligned} \tag{302}$$

$$\begin{aligned} &C_0 = \Big(r_h^2 - \frac{\Lambda}{3}\Big)\frac{1}{r_h^3}\,,\qquad C_1 = -1\,,\qquad C_2 = 0\,,\qquad C_3 = \frac{B}{C_0^2} + \frac{\Lambda}{3}\,,\qquad C_4 = 0\,,\\ &C_5 = 0\,,\qquad C_6 = -\frac{B}{90kC_0^3}\,,\qquad C_7 = -\frac{B}{140kC_0^4}\,,\qquad C_8 = -\frac{B}{210kC_0^5}\,,\ldots\,, \end{aligned} \tag{303}$$

giving the explicit expansions (284), (285).

Finally, the corresponding curvature invariants (292) at $\bar r = 0$ read

$$B_{ab}\,B^{ab}(r\to\infty) = \frac{1}{4k^2C_0^4}\,B^2\,,\qquad C_{abcd}\,C^{abcd}(r\to\infty) \sim 12C_0^2\,r^6 \to \infty\,, \tag{304}$$

To conclude: The class $[-1,3]^\infty$, with the metric functions expressed in terms of the series (284) and (285) around the physical origin $\bar{r}=0$, represents the Schwarzschild–Bach–(A)dS black hole. As was already pointed out in [15] for the $\Lambda=0$ case, this metric may, in fact, represent a distinct Schwarzschild–Bach–(A)dS black hole from the one in the $[0,1]$ class discussed in Sec. 5.2.

7.2 Bachian–(A)dS vacuum in the class $[N,P]=[-1,2]^\infty$

Now we will analyze the second possibility (272) in Case II$^\infty$. After relabeling $l\to l+2$, Eq. (259) for $N=-1$, $P=2$ yields

$$C_l = \frac{3}{k\,(l-2)(l-1)l(l+1)}\sum_{i=0}^{l-2} A_i\,A_{l-2-i}(l-1-i)(l-2-3i) \qquad \forall\, l\geq 3\,. \tag{305}$$

Eq. (261) in its lowest orders r^{-1}, r^{-2} determines A_1 and C_0 as

$$A_1 = \tfrac{1}{2}A_0C_1\,, \qquad C_0=-1\,, \tag{306}$$

while for higher orders gives

$$l(l-1)C_0A_{l-1} = \tfrac{2}{3}\Lambda\sum_{j=0}^{l-3}\sum_{i=0}^{j} A_iA_{j-i}A_{l-j-3} - \sum_{i=1}^{l-1} C_iA_{l-i-1}\left[(l-i)(l-1)+\tfrac{1}{6}(i-2)(i-1)\right] \quad \forall\, l\geq 3\,. \tag{307}$$

There are no additional constraints following from Eq. (260). There are thus three free parameters, A_0, C_1, C_2, and all other coefficients are determined by the recurrent relations (305), (307), e.g.,

$$\begin{aligned}
A_2 &= \frac{A_0}{3}(C_1^2+C_2-\tfrac{1}{3}\Lambda A_0^2)\,, \qquad A_3=\frac{A_0}{4}C_1(C_1^2+2C_2-\tfrac{2}{3}\Lambda A_0^2)\,,\\
A_4 &= \frac{A_0}{5}\left[C_1^4+3C_1^2C_2+C_2^2+\frac{A_0^2}{192k}\left[C_1^2+4C_2-8(23C_1^2+12C_2)k\Lambda\right]+\frac{A_0^4}{144k}\Lambda(8k\Lambda-1)\right], \ldots\,,
\end{aligned} \tag{308}$$

$$\begin{aligned}
C_3 &= 0\,, \qquad C_4=\frac{A_0^2}{240k}(C_1^2+4C_2-\tfrac{4}{3}\Lambda A_0^2)\,, \qquad C_5=\frac{A_0^2}{240k}C_1(C_1^2+4C_2-\tfrac{4}{3}\Lambda A_0^2)\,,\\
C_6 &= \frac{A_0^2}{67200k^2}\left[A_0^2(3-\tfrac{32}{3}k\Lambda)+4k(59C_1^2+26C_2)\right](C_1^2+4C_2-\tfrac{4}{3}\Lambda A_0^2)\,,\ldots\,.
\end{aligned} \tag{309}$$

7.2.1 Identification of Minkowski and (A)dS spaces

First, let us observe that in the limit $r\to\infty$, the scalar invariants (19), (20) remain finite,

$$B_{ab}\,B^{ab}(r\to\infty)=\frac{300}{A_0^8}\,C_4^2\,, \qquad C_{abcd}\,C^{abcd}(r\to\infty)\sim\frac{12}{A_0^4\,r^4}\,C_4^2\,, \tag{310}$$

suggesting that there is no physical singularity there. Both the Bach and Weyl tensor invariants vanish iff $C_4=0$, i.e., $C_2=-\frac{1}{4}C_1^2+\frac{1}{3}\Lambda A_0^2$. Then all the coefficients (308), (309) simplify to $A_i=A_0\,(\frac{1}{2}C_1)^i$ for all i, and $C_i=0$ for all $i\geq 3$, respectively, so that the metric functions reduce to

$$\Omega(r)=\frac{A_0}{r}\sum_{i=0}^{\infty}\left(\frac{C_1}{2\,r}\right)^i=\frac{A_0}{r-\frac{1}{2}C_1}\,, \qquad \mathcal{H}(r)=\frac{1}{3}\Lambda A_0^2-\left(r-\frac{1}{2}C_1\right)^2\,. \tag{311}$$

Employing the gauge freedom (13), we may set

$$A_0 = -1\,, \qquad C_1 = 0\,. \tag{312}$$

Then the metric functions simplify to

$$\bar{r} = \Omega(r) = -\frac{1}{r}\,, \qquad \mathcal{H}(r) = \frac{\Lambda}{3} - r^2\,. \tag{313}$$

Comparing with (15), this case $C_4 = 0$ corresponds to *Minkowski* or *(anti-)de Sitter* space with vanishing Bach and Weyl tensors. For $\Lambda > 0$, there is a cosmological horizon at $r_{cosm} = \sqrt{\frac{\Lambda}{3}}$.

7.2.2 Bachian–(A)dS vacuum

Let us return to the class $[N, P] = [-1, 2]^\infty$ with a general Bach tensor (310). Using the gauge (312), we introduce a dimensionless Bach parameter

$$B_v \equiv \Big(C_2 - \frac{\Lambda}{3}\Big)k \tag{314}$$

so that $C_4 = (C_2 - \frac{1}{3}\Lambda)/(60k) = B_v/(60k^2)$, cf. (309).

The coefficients (308), (309) then take the form

$$\begin{aligned}
&A_0 = -1\,, \quad A_1 = 0\,, \quad A_2 = -\frac{B_v}{3k}\,, \quad A_3 = 0\,,\\
&A_4 = -\frac{B_v}{240k^2}\,(1 + 8k\Lambda + 48B_v)\,, \quad A_5 = 0\,, \dots\,,
\end{aligned} \tag{315}$$

$$\begin{aligned}
&C_0 = -1\,, \quad C_1 = 0\,, \quad C_2 = \frac{B_v}{k} + \frac{\Lambda}{3}\,, \quad C_3 = 0\,,\\
&C_4 = \frac{B_v}{60k^2}\,, \quad C_5 = 0\,, \quad C_6 = \frac{B_v}{16800k^3}\,(3 + 24k\Lambda + 104B_v)\,, \dots\,,
\end{aligned} \tag{316}$$

leading to the metric functions

$$\Omega(r) = -\frac{1}{r} - \frac{B_v}{k\,r^3}\left(\frac{1}{3} + \frac{1}{5kr^2}\Big(\frac{1}{48} + \frac{1}{6}k\Lambda + B_v\Big) + \dots\right), \tag{317}$$

$$\mathcal{H}(r) = \frac{\Lambda}{3} - r^2 + \frac{B_v}{k}\left(1 + \frac{1}{60kr^2} + \frac{1}{700k^2r^4}\Big(\frac{1}{8} + k\Lambda + \frac{13}{3}B_v\Big) + \dots\right), \tag{318}$$

and the curvature invariants (310)

$$B_{ab}\,B^{ab}(r \to \infty) = \frac{B_v^2}{12k^4}\,, \qquad C_{abcd}\,C^{abcd}(r \to \infty) \sim \frac{B_v^2}{300k^4r^4} \to 0\,. \tag{319}$$

In the limit $r \to \infty$, the metric functions behave as $\Omega \sim -1/r$ and $\mathcal{H} \sim \frac{\Lambda}{3} - r^2 + \frac{B_v}{k}$. From (10), (11) we thus obtain

$$\bar{r} = \Omega(r) \to 0\,, \qquad h \sim 1\,, \qquad f \sim 1\,, \tag{320}$$

i.e., at the physical origin $\bar{r} = 0$, both metric functions h and f remain nonzero and finite, and there is no horizon, nor singularity therein.

To conclude: The class $[-1, 2]^\infty$, with the metric functions expressed in terms of the series (317) and (318) around the physical origin $\bar{r} = 0$, describes a one-parameter Bachian generalization of Minkowski or (A)dS spacetime with a nonzero Bach tensor whose magnitude is determined by the parameter B_v. This Bachian–(A)dS vacuum is a "massless limit" of the class $[-1, 3]^\infty$, corresponding to $C_0 = 0$ in (299), cf. Sec. 7.2.3.

7.2.3 The class $[-1,2]^\infty$ as a limit of the $[-1,3]^\infty$ class

A limiting procedure between the class of solutions $[-1,2]^\infty$, described by (305)–(307) with the coefficients denoted here by hats, and the class $[-1,3]^\infty$, described by (286)–(288), requires

$$C_0 \to 0\,, \qquad C_i \to \hat{C}_{i-1}, \quad i\geq 1\,, \qquad A_i \to \hat{A}_i, \quad i \geq 0\,. \tag{321}$$

Then the relation (287) for $l=1$, i.e., $3C_0A_1 = -A_0(1+C_1)$, gives

$$C_1 \to -1, \quad \text{i.e.,} \quad \hat{C}_0 = -1\,. \tag{322}$$

The relations (286) for C_{l+1} and (305) for $\hat{C}_l$ are the same, and the relation (287) for A_l

$$l^2C_0A_l = \tfrac{2}{3}\Lambda\sum_{j=0}^{l-3}\sum_{i=0}^{j}A_iA_{j-i}A_{l-j-3} - \tfrac{1}{3}\,A_{l-1} - \sum_{i=1}^{l}C_iA_{l-i}\left[\,l(l-i)+\tfrac{1}{6}i(i+1)\right], \quad \forall\, l\geq 1\,, \tag{323}$$

for $C_0=0$ leads to

$$\hat{A}_{l-1} = \frac{1}{l(l-1)}\bigg\{\tfrac{2}{3}\Lambda\sum_{j=0}^{l-3}\sum_{i=0}^{j}\hat{A}_i\hat{A}_{j-i}\hat{A}_{l-j-3} + \sum_{i=1}^{l-1}\hat{C}_i\,\hat{A}_{l-1-i}\left[(l-1)(l-i)+\tfrac{1}{6}(i-2)(i-1)\right]\bigg\}, \quad \forall\, l\geq 2\,, \tag{324}$$

which is exactly (307).

Note that the four free parameters A_0, C_0, C_1, C_3 of the $[-1,3]^\infty$ family reduce to three free parameters $\hat{A}_0, \hat{C}_1, \hat{C}_2$ of the $[-1,2]^\infty$ family since two parameters become fixed ($C_0 \to 0$, $C_1 \to \hat{C}_0 = -1$) and one parameter $C_2 \to \hat{C}_1$ becomes free ($3C_0C_2 = C_1^2 - 1 \to 0$).

7.3 Nariai–Bach solutions in the class $[N,P] = [0,2]^\infty$

In this section, we will show that the class $[0,2]^\infty$ contains *static spherically symmetric Nariai spacetime and its Bachian generalizations*. Only the non-Kundt solutions with *discrete values* of Λ given by Eq. (343) can be transformed into the standard static spherically symmetric form (8).

For $l\geq 3$, Eq. (259) gives

$$A_0A_{l-2}(l-2)(l-1) = \tfrac{1}{3}k\,C_{l-2}(l-4)(l-3)(l-2)(l-1) - \sum_{i=1}^{l-3}A_iA_{l-i-2}(l-i-2)(l-3i-1)\,, \tag{325}$$

which for $l=3,4$ yields

$$A_1 = 0\,, \qquad A_2 = 0\,, \tag{326}$$

respectively. For $l\geq 1$, the trace equation (261) implies

$$\begin{aligned} &A_l\left[2\Lambda A_0^2\left(l(l-1)-\tfrac{2}{3}\right) - l(l-1)\right] + \tfrac{1}{6}C_lA_0(l-2)(l-1) \qquad (327)\\ &= -\sum_{i=1}^{l-1}C_iA_{l-i}\left[(l-i)(l-1)+\tfrac{1}{6}(i-2)(i-1)\right] + \tfrac{2}{3}\Lambda\left[\sum_{i=1}^{l-1}A_0A_iA_{l-i} + \sum_{j=1}^{l-1}\sum_{i=0}^{j}A_iA_{j-i}A_{l-j}\right], \end{aligned}$$

where we employed the relation

$$C_0 = 2\Lambda A_0^2 - 1\,, \tag{328}$$

which follows from the leading order of Eq. (261). Finally, the leading term of Eq. (260) gives

$$(\Lambda A_0^2 - 1)(8k\Lambda - 3) = 0\,. \tag{329}$$

Similarly as in the class [0, 2] investigated in Section 5.5, we have two cases to consider, namely

$$\text{a)}\ \ \Lambda A_0^2 = 1\,, \quad \Lambda > 0\,, \tag{330}$$

$$\text{b)}\ \ \Lambda = \frac{3}{8k}\,, \tag{331}$$

see Eqs. (281) and (282). Let us discuss these two distinct cases separately.

7.3.1 Case a) $\Lambda A_0^2 = 1$: Nariai(–Bach) spacetime

Recall that $A_1 = 0 = A_2$, see (326), and from (328) with (330) we obtain

$$C_0 = 1\,. \tag{332}$$

Assuming $A_i = 0$ for all $1 \le i \le l-1$, and $C_i = 0$ for $3 \le i \le l-1$, $l \ge 4$, Eqs. (325) and (327) give (also for $l = 3$) the relations

$$A_l\, A_0 - C_l\, \tfrac{1}{3}k\,(l-1)(l-2) \,=\, 0\,, \tag{333}$$

$$A_l\left[2\Lambda A_0^2\left(l(l-1) - \tfrac{2}{3}\right) - l(l-1)\right] + C_l\, A_0 \tfrac{1}{6}(l-1)(l-2) \,=\, 0\,. \tag{334}$$

This is a system of two linear equations for two unknowns A_l, C_l with the determinant

$$\tfrac{1}{6}(l-1)(l-2)\Big[A_0^2 + 2k\left[2\Lambda A_0^2\left(l(l-1) - \tfrac{2}{3}\right) - l(l-1)\right]\Big]\,, \tag{335}$$

which, after substituting from (330), reduces to

$$\tfrac{1}{6}(l-1)(l-2)\Big[\,\frac{1}{\Lambda} + 2k\left(l(l-1) - \tfrac{4}{3}\right)\Big]\,, \qquad l \ge 3\,. \tag{336}$$

Therefore, we have to distinguish two subcases:

- For a *generic value* of $\Lambda > 0$ this determinant is nonvanishing, and necessarily

$$C_l = 0 = A_l \qquad \text{for all} \quad l \ge 3\,. \tag{337}$$

The only such solution is thus

$$\Omega = A_0 = \frac{1}{\sqrt{\Lambda}}\,, \qquad \mathcal{H}(r) = r^2 + C_1\, r + C_2\,, \tag{338}$$

so that the metric (9) reads

$$\mathrm{d}s^2 = A_0^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big) + A_0^2\big[(r^2 + C_1\, r + C_2)\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r\big]\,. \tag{339}$$

After performing the transformation

$$r \,=\, \tfrac{1}{4}(4C_2 - C_1^2 + \Lambda^2\tilde{u}^2)\,\tilde{r} + \tfrac{1}{2}(\Lambda\tilde{u} - C_1)\,, \tag{340}$$

$$u \,=\, \frac{4}{\sqrt{4C_2 - C_1^2}}\arctan\left(\frac{\Lambda\tilde{u}}{\sqrt{4C_2 - C_1^2}}\right), \tag{341}$$

(and dropping tilde symbols), or for $C_1^2 = 4C_2$ using the gauge freedom (13) and the scaling $A_0^2\, u \to u$, the solution reduces to

$$\mathrm{d}s^2 = \frac{1}{\Lambda}\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big) - 2\,\mathrm{d}u\,\mathrm{d}r + \Lambda\, r^2 \mathrm{d}u^2\,. \tag{342}$$

This is the metric of the *Nariai spacetime* [28] which in Einstein's theory is a vacuum solution of algebraic type D. In fact, it belongs to the class of direct-product geometries, see Chapter 7 of [17]. In particular, it has the Kundt form (18.49) for $\alpha = \Lambda/2$ therein (see also [29,30]). It is a nonsingular, homogeneous and spherically symmetric Einstein space. Interestingly, since the conformal factor $\Omega = A_0$ is constant, and thus $\bar{r} = \Omega = \text{const.}$, this solution *cannot be transformed into the standard static spherically symmetric coordinates* (8).

To conclude: The class $[0,2]^\infty$ with $\Lambda A_0^2 = 1$ and arbitrary $\Lambda > 0$, with the metric functions expressed in terms of (338), represents the direct-product ($S^2 \times \mathrm{d}S_2$) Nariai solution (342).

• The second branch of the $[0,2]^\infty$ class of solutions obeying (330) exists only for special *discrete values* of the cosmological constant $\Lambda > 0$, such that

$$\Lambda = -\frac{3}{2k\,[3L(L-1)-4]}\,, \qquad \text{where} \quad L \in \mathbb{N}\,, \quad L \geq 3\,, \quad k < 0\,. \tag{343}$$

The determinant (336) vanishes, in which case C_1, C_2, together with a new *Bach parameter*

$$\tilde{B}_L \equiv C_L\,, \tag{344}$$

are three free parameters. Recall that

$$A_0 = \frac{1}{\sqrt{\Lambda}}\,, \qquad C_0 = 1\,, \tag{345}$$

but $A_i = 0$ for $1 \leq i \leq L-1$, and also $C_i = 0$ for $3 \leq i \leq L-1$. The first nontrivial coefficient A_L is given by (333),

$$A_L \,=\, \tfrac{1}{3}k\,\sqrt{\Lambda}\,(L-1)(L-2)\,\tilde{B}_L\,, \tag{346}$$

and all the subsequent coefficients C_i, A_i for $i > L$ are determined by (325), (327) as

$$\begin{aligned} C_l \,&=\, 6\frac{[3l(l-1)-4]\,U_{l-1} - 3A_0 l(l+1)V_{l-1}}{(l+1)l(l-1)(l-2)[3A_0^2 - 8k + 6kl(l-1)]}\,, \\ A_l \,&=\, -3\frac{A_0 U_{l-1} + 2kl(l+1)V_{l-1}}{(l+1)l[3A_0^2 - 8k + 6kl(l-1)]}\,, \end{aligned} \tag{347}$$

where

$$\begin{aligned} U_{l-1} \,&=\, \sum_{i=1}^{l-1} A_i A_{l-i}(l-i)(l-3i+1)\,, \qquad (348)\\ V_{l-1} \,&=\, \sum_{i=1}^{l-1} C_i A_{l-i}\,\big[(l-i)(l-1) + \tfrac{1}{6}(i-1)(i-2)\big] - \tfrac{2}{3}\Lambda\Big(\sum_{i=1}^{l-1} A_0 A_i A_{l-i} + \sum_{j=1}^{l-1}\sum_{i=0}^{j} A_i A_{j-i} A_{l-j}\Big)\,. \end{aligned}$$

Thus, the metric functions take the explicit form

$$\Omega(r) \,=\, \frac{1}{\sqrt{\Lambda}} + A_L\, r^{-L} + \sum_{i=L+1}^{\infty} A_i\, r^{-i}\,, \tag{349}$$

$$\mathcal{H}(r) \,=\, r^2 + C_1\, r + C_2 + \tilde{B}_L\, r^{2-L} + \sum_{i=L+1}^{\infty} C_i\, r^{2-i}\,, \tag{350}$$

where, A_L and A_i, C_i are determined by (346) and (347).

This is clearly a *Bachian generalization of the Nariai metric* (339), to which it reduces when $\tilde{B}_L = 0$. Since now $\bar{r} = \Omega(r)$ is *not a constant*, it is possible to transform these Nariai–Bach spacetimes to standard static spherically symmetric coordinates (8).

For $r \to \infty$, the leading terms of the curvature invariants (19) and (20) read

$$B_{ab}\, B^{ab}(r \to \infty) \,=\, \frac{1}{72}\,(3L^2 - 8L + 8)(L-2)^2(L-1)^2(L+1)^2\, \Lambda^4\, \tilde{B}_L^2\, r^{-2L} + \cdots\,, \tag{351}$$

$$C_{abcd}\, C^{abcd}(r \to \infty) \,=\, \frac{16}{3}\, \Lambda^2 + \cdots\,. \tag{352}$$

Also, from (10), (11) we obtain

$$\bar{r} = \Omega(r) \to A_0 = \frac{1}{\sqrt{\Lambda}}\,, \tag{353}$$

$$h \sim -\frac{(A_L)^{2/L}}{\Lambda(\bar{r} - A_0)^{2/L}} \;\to\; \infty\,, \qquad f \sim -L^2 \Lambda(\bar{r} - A_0)^2 \;\to\; 0\,. \tag{354}$$

Although the discrete-Λ spectrum of possible solutions (343) in this case $[0,2]^\infty$ is the same as the spectrum (171) in the case $[0,2]$, these two metrics do not describe the same solution since only the $[0,2]$ case (the extreme higher-order (discrete) Schwarzschild–Bach–dS black holes) contains the Schwarzschild–de Sitter limit for vanishing Bach parameter. The Robinson–Trautman-type metric $[0,2]^\infty$ given by (349), (350) is a generalization of the Nariai metric (of the Kundt type) to a nonvanishing Bach tensor.

To conclude: The class $[0,2]^\infty$ with $\Lambda A_0^2 = 1$ and the discrete values of Λ given by (343), for which the metric functions are expressed in terms of the series (349), (350) around a finite point $\bar{r} = \frac{1}{\sqrt{\Lambda}}$, represents a spherically symmetric generalization of the Nariai metric to a nonzero Bach tensor, which can be called Nariai–Bach spacetimes.

7.3.2 Case b) $\Lambda = \frac{3}{8k} \equiv \frac{3\gamma}{8(\alpha - 3\beta)}$: another Bachian generalization of the Nariai solution

Let us now investigate the second distinct case of possible solutions in the class $[0,2]^\infty$, namely (331).

Equations (325) and (327) lead, as in previous Case a), to the system (333), (334) with the determinant (335), which for (331) is proportional to

$$3A_0^2 - 4k\,. \tag{355}$$

Therefore, either the determinant vanishes for $A_0^2 = \frac{4}{3}k \Leftrightarrow 2\Lambda A_0^2 = 1$, which using (328) implies $C_0 = 0$, i.e., a contradiction with our assumptions, or the determinant is nonvanishing, leading to trivial solutions $C_i = 0$ for all $i \geq 3$, and $A_i = 0$ for all $i \geq 1$. This solution thus has only three free parameters A_0, C_1, C_2, while C_0 is determined as

$$C_0 = 2\Lambda A_0^2 - 1 = \frac{3}{4k} A_0^2 - 1\,, \tag{356}$$

see (328) and (282). The explicit metric functions read

$$\Omega = A_0\,, \qquad \mathcal{H}(r) = C_0\, r^2 + C_1\, r + C_2\,, \tag{357}$$

and lead to a direct-product metric

$$\mathrm{d}s^2 = A_0^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big) + A_0^2\Big[\big((2\Lambda A_0^2 - 1)\, r^2 + C_1\, r + C_2\big)\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r\Big]\,. \tag{358}$$

whose invariants (19), (20) are *constants*

$$B_{ab}\, B^{ab} = \left(\frac{4\Lambda}{3A_0^2}(1 - \Lambda A_0^2)\right)^2 = \left(\frac{8k - 3A_0^2}{16k^2 A_0^2}\right)^2\,, \tag{359}$$

$$C_{abcd}\, C^{abcd} = \frac{16}{3}\Lambda^2 = \frac{3}{4k^2}\,. \tag{360}$$

Clearly, the Bach tensor vanishes and the solution (358) becomes the Nariai spacetime (342), which is an Einstein space (with $B_{ab} = 0$), if and only if $A_0^2 = 1/\Lambda \Leftrightarrow A_0^2 = \frac{8}{3}k$.

The direct-product solution (358) is of the Kundt type, and cannot be transformed into the standard static spherically symmetric coordinates (8) since $\bar{r} = \Omega = \text{const.}$

To conclude: The class $[0,2]^\infty$ with $\Lambda = 3/(8k)$ necessarily leads to the metric (358). It represents another Bachian generalization of the Nariai solution (342) which belongs to the Kundt family.

7.4 Plebański–Hacyan spacetime in the class $[N,P] = [0,<2]^\infty$

In what follows we will show that, in fact, this class admits only the Kundt-type spacetimes $[0,1]^\infty$ and $[0,0]^\infty$.

First, for $P \notin \mathbb{N}$, Eq. (259) implies $C_0 = 0$ which must be nonzero by definition. Therefore, P is necessarily an integer.

For all $P < 2$, Eqs. (260) and (261) at the order r^0 give the conditions $3A_0^2(1 - \Lambda A_0^2) - 2k = 0$ and $2\Lambda A_0^2 - 1 = 0$, respectively, which imply

$$\Lambda = \frac{3}{8k} > 0\,, \qquad \text{and} \qquad A_0 = \pm\sqrt{\frac{4k}{3}} = \pm\frac{1}{\sqrt{2\Lambda}}\,. \tag{361}$$

Let us now investigate the following three distinct possibilities, namely $P = 1$, $P = 0$, and $P < 0$.

• In the case $[0,1]^\infty$, Eq. (259) at the orders r^{-3} and r^{-4} gives $A_1 = A_2 = 0$, respectively, while at the orders r^{-5}, r^{-6} and r^{-7} it yields $A_3 = \frac{2kC_2}{3A_0}$, $A_4 = \frac{2kC_3}{A_0}$, and $A_5 = \frac{4kC_4}{A_0}$, respectively. At the order r^{-4}, Eq. (261) implies $C_2 = 0$. Let us use the mathematical induction to prove that all C_ℓ, $A_{\ell-1}$ vanish for $\ell \geq 2$. So, let us assume that $A_1 = A_2 = \cdots = A_j = 0$, $C_2 = C_3 = \cdots = C_{j-1} = 0$, and $A_{l+1} = \frac{k}{3A_0}(l-1)l\, C_l$ for $l \in \{j,\, j+1\}$. We prove that then $C_j = 0 = A_{j+1}$ and also that $A_{j+3} = \frac{k}{3A_0}(j+1)(j+2)\, C_{j+2}$. Indeed, Eq. (259) at the order $r^{-(j+5)}$ gives

$$A_{j+3} = \frac{k}{3A_0}(j+1)(j+2)\, C_{j+2}\,, \tag{362}$$

while (261) at the order $r^{-(j+2)}$, using (361) and also the expressions for A_{j+1}, A_{j+2}, yields

$$C_0\,C_j = 0\,, \tag{363}$$

which implies $C_j = 0 = A_{j+1}$, and thus C_ℓ, $A_{\ell-1}$ vanish for all $\ell \geq 2$. Therefore,

$$\bar{r} = \Omega = A_0\,, \qquad \mathcal{H}(r) = C_0\,r + C_1\,. \tag{364}$$

The metric (9) with such functions can further be simplified using the coordinate transformation

$$u = \frac{2}{C_0}\,\ln\left(\frac{\tilde{u}}{A_0^2}\right), \qquad r = \frac{\tilde{u}}{A_0^2}\left(\frac{C_0}{2}\,\tilde{r} + \frac{1}{C_0}\right) - \frac{C_1}{C_0}\,, \tag{365}$$

to the form

$$\mathrm{d}s^2 = A_0^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big) - 2\,\mathrm{d}u\,\mathrm{d}r\,. \tag{366}$$

This is a simple direct-product $S^2 \times M_2$ geometry. Actually, this is the *Plebański–Hacyan spacetime* with $\Lambda > 0$, see the metric (7.19) in [17]. Interestingly, it is a type D electrovacuum spacetime in Einstein's theory, while here it is a spherically symmetric *vacuum solution to QG*, with the radius A_0 of the sphere S^2 given by

$$A_0^2 = \frac{4k}{3} = \frac{1}{2\Lambda}\,. \tag{367}$$

The curvature invariants are

$$R_{ab}\,R^{ab} = 8\Lambda^2\,, \qquad C_{abcd}\,C^{abcd} = \frac{16}{3}\Lambda^2\,, \qquad B_{ab}\,B^{ab} = \frac{16}{9}\Lambda^4\,, \tag{368}$$

i.e., the curvature invariants are constant, and the nonvanishing Bach tensor is uniform. Notice again that, due to $\bar{r} = A_0$, this Kundt-type solution cannot be put into the standard static spherically symmetric coordinates.

• In the case $[0,0]^\infty$, Eq. (259) at the orders r^{-3} and r^{-4} again gives $A_1 = A_2 = 0$, respectively, while at the orders r^{-5}, r^{-6} and r^{-7}, $A_3 = \frac{2kC_1}{3A_0}$, $A_4 = \frac{2kC_2}{A_0}$, and $A_5 = \frac{4kC_3}{A_0}$, respectively. At the orders r^{-5}, r^{-6}, r^{-7}, Eq. (261) implies $C_1 = C_2 = C_3 = 0$, respectively. Using the mathematical induction we again prove that all $C_\ell = 0 = A_\ell$ for $\ell \geq 1$. We assume that $A_1 = A_2 = \cdots = A_{j+1} = 0$, $C_1 = C_2 = \cdots = C_{j-1} = 0$, and $A_{l+2} = \frac{k}{3A_0}(l+1)l\,C_l$ for $l \in \{j,\ j+1,\ j+2\}$ and prove that then $C_j = 0 = A_{j+2}$ and $A_{j+5} = \frac{k}{3A_0}(j+4)(j+3)C_{j+3}$. Indeed, Eq. (259) at the order $r^{-(j+7)}$ gives

$$A_{j+5} = \frac{k}{3A_0}(j+3)(j+4)C_{j+3}\,, \tag{369}$$

while (261) at the order $r^{-(j+4)}$, using (361) and the expressions for A_{j+2}, A_{j+4}, yields

$$C_0\,C_j = 0\,, \tag{370}$$

which implies $C_j = 0 = A_{j+2}$ and thus C_ℓ, A_ℓ vanish for all $\ell \geq 1$. Therefore,

$$\bar{r} = \Omega = A_0\,, \qquad \mathcal{H}(r) = C_0\,, \tag{371}$$

which is the special case of (364). With the coordinate transformation

$$r = \frac{\tilde{r}}{A_0^2} + \frac{C_0}{2}\,u\,, \tag{372}$$

we again obtain the metric (366) of the Plebański–Hacyan direct-product spacetime.

• Finally, the case $[0,<0]^\infty$ with $P<0$ is, in fact, *empty*. Indeed, using the mathematical induction we show that A_i for $1\le i\le 1-P$ vanish. For all $P<0$, Eq. (259) at the order r^{-3} implies $A_1=0$. Let us assume that all A_i for $1\le i\le j-1$, $j<2-P$, vanish. Then Eq. (259) at the order r^{-j-2} gives $A_0A_j(j+1)=0$, which implies that also $A_j=0$. Thus the second nonvanishing coefficient (after A_0) is A_{2-P} which is determined by the order r^{P-4} of Eq. (259),

$$A_{2-P}=kP(P-1)\frac{C_0}{3A_0}\,. \tag{373}$$

Then, using (361), Eqs. (259) and (261) at the orders r^{2P-6} and r^{2P-4} yield

$$\begin{aligned} 2(5-2P)A_0A_{4-2P}+(P-1)A_{2-p}^2-\tfrac{4}{3}kC_{2-P}(1-P)(3-2P)(5-2P)\;&=\;0\,,\\ -2A_{4-2P}+C_0A_{2-P}\left[3(2-P)(3-2P)+\tfrac{1}{2}P(P-1)\right]-6\Lambda A_0A_{2-P}^2+A_0C_{2-P}(1-P)(3-2P)\;&=\;0\,, \end{aligned}$$

respectively. A linear combination of these two equations, using also (361) and (373), leads to

$$P(P-1)(P-2)(5P^2-21P+20)C_0^2=0\,, \tag{374}$$

which for integers $P<0$ would imply $C_0=0$, a contradiction. Therefore, this class is indeed empty.

To conclude: The class $[0,<2]^\infty$ includes only the $S^2\times M_2$ direct-product Plebański–Hacyan spacetime (366), (367) which is the spherically symmetric vacuum solution to QG with $\Lambda=3/(8k)>0$ and a nonvanishing Bach tensor. As in the previous case, this Kundt metric cannot be put into the standard static spherically symmetric coordinates.

7.5 Solutions with regular Bachian infinity in the class $[N,P]=[>0,2N+2]^\infty$

Let us investigate the last possible class (283). Similarly as in Section 5.9, in Eq. (261) the first and last terms start with the power r^{3N}, while the second one with $\frac{1}{3}A_0r^N$. Since in all sums there are integer steps, to allow $A_0\neq 0$, the expression $3N-N$ has to be a positive integer. Thus, N must have the form

$$N=J/2\,,\qquad \text{where}\quad J\in\mathbb{N}\,,\quad J\ge 2\,, \tag{375}$$

(recall that $N=1/2$ is not admitted, see (283)). Consequently, $N\ge 1$ and $P=2N+2=J+2\ge 4$. Then from Eq. (280) we obtain *discrete values of the cosmological constant* Λ for $J=2,3,\dots$ as

$$\Lambda\;=\;\frac{3}{32k}\frac{11J^2+12J+4}{1-J^2}\,, \tag{376}$$

$$\text{so that}\quad \Lambda\in\left[-\tfrac{9}{4k},-\tfrac{33}{32k}\right)\ \text{for}\ k>0\,,\ \text{and}\quad \Lambda\in\left(\tfrac{33}{32|k|},\tfrac{9}{4|k|}\right]\ \text{for}\ k<0\,,$$

$$C_0\;=\;\frac{3}{4k}\frac{A_0^2}{1-J^2}\,. \tag{377}$$

Notice that the product $k\Lambda$ is *always negative*. From the subleading term of (261) we get

$$A_1=-k\frac{2N(2N+1)(2N-1)}{3(N+1)}\frac{C_1}{A_0}\,, \tag{378}$$

and similarly we can express further coefficients for any $N = J/2$. Let us present two examples for $J = 2$ and $J = 3$.

• In the case $[1, 4]^\infty$, there are three free parameters A_0, C_1, C_5 with the remaining coefficients determined by Eqs. (259)–(261) as

$$\begin{aligned}
\Lambda &= -\frac{9}{4k}\,, \\
A_1 &= -k\frac{C_1}{A_0}\,, \quad A_2 = A_3 = A_4 = 0\,, \quad A_5 = k\frac{2C_5}{9A_0}\,, \quad \dots\,, \\
C_0 &= -\frac{A_0^2}{4k}\,, \quad C_2 = -\frac{1}{7} - \frac{3kC_1^2}{2A_0^2}\,, \quad C_3 = \frac{2kC_1}{7A_0^2} + \frac{k^2C_1^3}{A_0^4}\,, \quad C_4 = \frac{32k}{147A_0^2} - \frac{k^2C_1^2}{7A_0^4} - \frac{k^3C_1^4}{4A_0^6}\,, \dots
\end{aligned} \tag{379}$$

and thus the metric functions take the form

$$\begin{aligned}
\Omega &= A_0\,r - k\frac{C_1}{A_0} + k\frac{2C_5}{9A_0}\,r^{-4} + \cdots\,, \\
\mathcal{H} &= -\frac{A_0^2}{4k}\,r^4 + C_1\,r^3 - \frac{1}{7}\left(1 + \frac{21kC_1^2}{2A_0^2}\right) r^2 + \frac{2kC_1}{7A_0^2}\left(1 + \frac{7kC_1^2}{2A_0^2}\right) r + \cdots\,.
\end{aligned} \tag{380}$$

• In the case $\left[\frac{3}{2}, 5\right]^\infty$, there are three free parameters A_0, C_1, C_7, and

$$\begin{aligned}
\Lambda &= -\frac{417}{256k}\,, \\
A_1 &= -\frac{16kC_1}{5A_0}\,, \quad A_2 = \frac{128k^2C_1^2}{75A_0^3}\,, \quad A_3 = \frac{2048k^3C_1^3}{3375A_0^5}\,, \quad A_4 = \frac{8192k^4C_1^4}{16875A_0^7}\,, \quad A_5 = \frac{131072k^5C_1^5}{253125A_0^9}\,, \dots \\
C_0 &= -\frac{3A_0^2}{32k}\,, \quad C_2 = -\frac{64kC_1^2}{15A_0^2}\,, \quad C_3 = -\frac{4}{11025A_0^4}(225A_0^4 - 25088k^2C_1^3)\,, \\
C_4 &= \frac{256kC_1}{165375A_0^6}(225A_0^4 - 6272k^2C_1^3)\,, \quad C_5 = -\frac{4096k^2C_1^2}{12403125A_0^8}(1125A_0^4 - 12544k^2C_1^3)\,, \\
C_6 &= \frac{8480k}{38073A_0^2}\,, \quad \dots\,,
\end{aligned} \tag{381}$$

with C_7 appearing in $A_7, A_8, \dots$ and $C_8, C_9, \dots$.

For all allowed values of N, the curvature invariants (19), (20) *approach a constant* as $r \to \infty$,

$$B_{ab}\,B^{ab}(r \to \infty) \,=\, P^2(P-1)^2(P-3)^2(11P^2 - 32P + 24)\frac{C_0^4}{144A_0^8} + \cdots\,, \tag{382}$$

$$C_{abcd}\,C^{abcd}(r \to \infty) \,=\, P^2(P-1)^2\frac{C_0^2}{3A_0^4} + \cdots\,. \tag{383}$$

For all the permitted values of $P = 4, 5, \dots$, the *Bach invariant* (382) *cannot vanish*, and therefore this class does not include the Schwazschild–(A)dS solution as a special subcase or a limit.

Since

$$\bar{r} = \Omega(r) = A_0\,r^N + \cdots \;\to\; \infty\,, \tag{384}$$

cf. (10), the limit $r \to \infty$ corresponds to an *asymptotic region* far away from the origin at $\bar{r} = 0$. In this asymptotic region, the metric functions (11) behave as

$$h \,=\, -A_0^2C_0\,r^{4N+2} + \cdots \;\sim\; \bar{r}^{\,4+2/N} + \cdots \to \infty\,, \tag{385}$$

$$f \,=\, -N^2C_0\,r^{2N} + \cdots \;\sim\; \bar{r}^{\,2} + \cdots \to \infty\,. \tag{386}$$

Finally, let us observe that in the notation of [8–10], these solutions correspond to the families $(s,t) = (-2, 4+4/J)_\infty$ with $J \in \mathbb{N}$, $J \geq 2$, i.e., with the parameter $t \in (4,6]$. Interestingly, together with the case of Section 5.9 with the same spectrum of Λ (cf. (241) and (376)), they both describe solutions with regular Bachian infinity and together correspond to families $(s,t) = (-2, 4+4/J)_{\infty, J\geq 2} \cup (-2, 4-4/J)_{\infty, J\geq 3}$, i.e., $(s,t) = (-2,t)_\infty$, where $t \in [\frac{8}{3}, 4) \cup (4,6]$.

To conclude: The class $[J/2, J+2]^\infty$ with Λ uniquely determined by (376) is, in fact, an infinite discrete family of metrics parameterized by an integer $J \geq 2$. They all have a regular Bachian infinity since both the Bach and Weyl invariants approach finite nonzero values (382), (383) in the asymptotic physical region $\bar{r} \to \infty$. In particular, for $J = 2$, the metric functions are expressed in terms of the series (380).

8 Identifying the classes in the spherically symmetric coordinates

Up to now, we mostly used the Kundt coordinates to derive and analyze the vacuum spherically symmetric solutions to QG with a nonvanishing cosmological constant, found in this paper as power series in powers of $\Delta \equiv r - r_0$ and r^{-1}. In this section, we identify them in terms of the standard notation used in the literature in the spherically symmetric coordinates. This allows us to establish the connection with the previously known classes for $\Lambda = 0$.

In Table 1, solutions with $\Lambda = 0$ found in [8,10] near the origin, and in [10,27] near a finite point $\bar{r} \to \bar{r}_0 \neq 0$ are listed with (s,t) and $(w = -s, t)$ denoting the powers of the leading terms of a Laurent expansion of the two metric functions in the standard spherically symmetric form (8), respectively,[9]

$$f^{-1}(\bar{r}) = A(\bar{r}) \sim \bar{r}^{\,s} \sim \bar{r}^{\,-w}\,, \tag{387}$$

$$h(\bar{r}) = B(\bar{r}) \sim \bar{r}^{\,t}\,. \tag{388}$$

Family $(s,t)_0$ near the origin	Family $(w,t)_{\bar{r}_0}$ near $\bar{r}_0$
$(0,0)_0$	$(1,1)_{\bar{r}_0}$
$(1,-1)_0$	$(0,0)_{\bar{r}_0}$
$(2,2)_0$	$(1,0)_{\bar{r}_0}$

Table 1: Families of spherically symmetric solutions to QG with $\Lambda = 0$ near the origin found in [8,10], and near a finite point $\bar{r} \to \bar{r}_0 \neq 0$ found in [10,27]. The subscripts "$_0$" and "$_{\bar{r}_0}$" indicate an expansion around $\bar{r} = 0$ and $\bar{r} = \bar{r}_0$, respectively.

Since our calculations are performed in the "nonphysical" Kundt coordinates, for the physical interpretation, it is important to identify the corresponding points in the spherically symmetric coordinates, around which expansions are taken. This can be obtained using (10), (29), and (33), and is summarized in Table 2.

[9] Here, the arguments of the metric functions $A(r), B(r)$ of [10] are relabeled to $\bar{r}$.

$\bar{r} \to$	Corresponding n for $r \to r_0$	Corresponding N for $r \to \infty$
0	> 0	< 0
$\bar{r}_0$	$= 0$	$= 0$
∞	< 0	> 0

Table 2: The correspondence between the points around which expansions are being performed in the Kundt coordinate r and the spherically symmetric radial coordinate $\bar{r} = \Omega(r)$.

Now, let us study the relation between the families (s,t) or (w,t), and $[n,p]$ and $[N,P]^\infty$. The cases with $n = 0$ in (32) or $N = 0$ in (35) have to be treated separately. Moreover, they contain special subcases that in the notation (w,t) represent *new classes with noninteger steps* in $\bar{\Delta} = \bar{r} - \bar{r}_0$.

8.1 Classes $[n \neq 0, p]$ and $[N \neq 0, P]^\infty$

Using the expressions (11) with $\bar{r} = \Omega(r)$ and (29), (30) for $r \to r_0$ when $n \neq 0$, and (33), (34) for $r \to \infty$ when $N \neq 0$, we obtain the relations between (s,t), introduced by (387), (388), and $[n,p]$, $[N,P]^\infty$ as

$$(s,t) = \left(\tfrac{2-p}{n}, 2 + \tfrac{p}{n}\right), \tag{389}$$

$$(s,t) = \left(\tfrac{2-P}{N}, 2 + \tfrac{P}{N}\right), \tag{390}$$

respectively. For the solutions found in this paper, these are summarized in Table 3.

Class $[n,p]$	Class $[N,P]^\infty$	Corresponding family (s,t)
$[n,p]$	–	$\left(\frac{2-p}{n}, 2 + \frac{p}{n}\right)$
$[1,0]$	–	$(2,2)_0$
$[-1,2]$	–	$(0,0)_\infty$
$[-1,0]$	–	$(-2,2)_\infty$
$\left[-\frac{J}{2}, 2-J\right]$	–	$\left(-2, 4\frac{J-1}{J}\right)_\infty$
–	$[N,P]^\infty$	$\left(\frac{2-P}{N}, 2 + \frac{P}{N}\right)$
–	$[-1,3]^\infty$	$(1,-1)_0$
–	$[-1,2]^\infty$	$(0,0)_0$
–	$\left[\frac{J}{2}, J+2\right]^\infty$	$\left(-2, 4\frac{J+1}{J}\right)_\infty$

Table 3: Values of (s,t) for all solutions $[n,p]$ with $n \neq 0$, and $[N,P]^\infty$ with $N \neq 0$ found in this paper. The subscripts "$_0$" and "$_\infty$" indicate the expansion for $\bar{r} \to 0$ and $\bar{r} \to \infty$, respectively.

Note that power series expansions with integer steps in the powers of Δ in $[n,p]$ with $n \neq 0$, and in the powers of r^{-1} in $[N,P]^\infty$ with $N \neq 0$ correspond to integer steps in $\bar{\Delta}$ since $\Delta \propto \bar{\Delta}$ and $r \propto \bar{\Delta}^{-1}$, respectively.

Additional information about the cases discussed in this section is summarized in subsequent Tables 8, 9, and 10.

8.2 Classes $[0, p]$ and $[0, P]^\infty$

When $n = 0$ or $N = 0$, there are the following subcases (recall that in the class $[0, 2]^\infty$, necessarily $A_1 = A_2 = 0$, see Section 7.3):

• **Classes $[0, p]$ with $a_1 \neq 0$**
In the generic case with $a_1 \neq 0$, using (11), (29), (30), the relations are

$$(w, t) = (p, p)_{\bar{r}_0} \,. \tag{391}$$

These solutions are summarized in Table 4.

Class $[n = 0, p]$	Corresponding family (w, t) for $\bar{r} \to \bar{r}_0$
$[0, p]$	$(p, p)_{\bar{r}_0}$
$[0, 0]$	$(0, 0)_{\bar{r}_0}$
$[0, 1]$	$(1, 1)_{\bar{r}_0}$
$[0, 2]$	$(2, 2)_{\bar{r}_0}$

Table 4: Values of (w, t) for all solutions $[0, p]$ with $a_1 \neq 0$.

Similarly as in the previous case, integer steps in Δ correspond to integer steps in $\bar{\Delta}$ since $\bar{\Delta} \equiv \bar{r} - \bar{r}_0 \sim a_1 \Delta$.

• **Classes $[0, p]$ with $a_1 = 0 \neq a_2$**
In this case,

$$(w, t) = \left(\tfrac{p+2}{2}\, , \tfrac{p}{2}\right)_{\bar{r}_0, 1/2} \tag{392}$$

and these solutions are summarized in Table 5.

Class $[n = 0, p]$	Corresponding family (w, t) for $\bar{r} \to \bar{r}_0$
$[0, p]_{a_1=0}$	$\left(\frac{p+2}{2}\, , \frac{p}{2}\right)_{\bar{r}_0, 1/2}$
$[0, 0]_{a_1=0}$	$(1, 0)_{\bar{r}_0, 1/2}$
$[0, 1]_{a_1=0}$	$\left(\frac{3}{2}, \frac{1}{2}\right)_{\bar{r}_0, 1/2}$

Table 5: Values of (w, t) for all solutions $[0, p]$ with $a_1 = 0 \neq a_2$.

Integer steps in Δ lead to half integer steps in $\bar{\Delta}$ since $\bar{\Delta} \sim a_2 \Delta^2$.

• **Classes $[0, p]$ with $a_1 = a_2 = 0 \neq a_3$, and $[0, P]^\infty$ with $A_1 = A_2 = 0 \neq A_3$**
In this case, using also (33), (34),

$$(w, t) = \left(\tfrac{p+4}{3}, \tfrac{p}{3}\right) \,, \tag{393}$$

$$(w, t) = \left(\tfrac{8-P}{3}, -\tfrac{P}{3}\right)_{\bar{r}_0, 1/3} \tag{394}$$

and these solutions are summarized in Table 6.

Class $[n=0,p]$	Class $[N=0,P]^\infty$	Corresponding family (w,t) for $\bar r \to \bar r_0$
$[0,p]_{a_1=a_2=0}$	–	$\left(\frac{p+4}{3}, \frac{p}{3}\right)_{\bar r_0, 1/3}$
$[0,0]_{a_1=a_2=0}$	–	$\left(\frac{4}{3}, 0\right)_{\bar r_0, 1/3}$
$[0,1]_{a_1=a_2=0}$	–	$\left(\frac{5}{3}, \frac{1}{3}\right)_{\bar r_0, 1/3}$
–	$[0,P]^\infty_{A_1=A_2=0}$	$\left(\frac{8-P}{3}, -\frac{P}{3}\right)_{\bar r_0, 1/3}$
–	$[0,2]^\infty_{A_1=A_2=0}$	$\left(2, -\frac{2}{3}\right)_{\bar r_0, 1/3}$

Table 6: Values of (w,t) for all solutions $[0,p]$ with $a_1 = a_2 = 0 \neq a_3$ and $[0,P]^\infty$ with $A_1 = A_2 = 0 \neq A_3$.

Integer steps in Δ and r^{-1} lead to steps in $\bar\Delta^{1/3}$ since $\bar\Delta \sim a_3\Delta^3$ and $\bar\Delta \sim A_3 r^{-3}$, respectively.

• **Classes** $[0,P]^\infty$ **with** $A_1 = A_2 = \cdots = A_{L-1} = 0 \neq A_L\,,\ L \geq 4$
In this case,

$$(w,t) = \left(\tfrac{2L+2-P}{L}, -\tfrac{P}{L}\right)_{\bar r_0, 1/L}\,, \tag{395}$$

see Table 7.

Class $[N=0,P]^\infty_{A_1=\cdots=A_{L-1}=0}$	Corresponding family (w,t) for $\bar r \to \bar r_0$
$[0,2]^\infty_{A_1=\cdots=A_{L-1}=0}$	$\left(2, -\frac{2}{L}\right)_{\bar r_0, 1/L}$

Table 7: Values of (w,t) for the solutions $[0,2]^\infty$ with $A_1 = A_2 = \cdots = A_{L-1} = 0 \neq A_L$.

Integer steps in r^{-1} lead to steps in $\bar\Delta^{-1/L}$ since $\bar\Delta \sim A_L r^{-L}$.

Apart from the generic cases $[0,0]$, $[0,1]$, $[0,2]$, $[0,2]^\infty$, discussed in Sections 5.4, 5.2, 5.5, 7.3, respectively, only the special case $[0,2]_{a_1=0}$ has been studied so far, see Sec. 5.5.2. For completeness, let us briefly discuss both generic and also special cases in the $[0,p]$ and $[0,P]^\infty$ classes. Recall that the classes with $[n=0, p>0]$ contain horizons at $r = r_0 = r_h$.

• Generic family $[n,p] = [0,0]$ has the highest number of free parameters, and it seems that it can be connected to all the other solutions. It represents an expansion around a generic point in these spacetimes;

• family $[0,0]_{a_1=0}$, for which the Bach invariant (19) is always nonvanishing, represents a generalization of the family $(1,0)_{\bar r_0, 1/2}$ of [10, 27] with $\Lambda \neq 0$. It is the only family that describes a wormhole since $f = 0$ and $h \neq 0$ at $\bar r_0$ ($\mathcal{H} \neq 0$, $\Omega' = 0$ implies $n = 0 = p$, $a_1 = 0$), see [10]. In particular, it corresponds to a wormhole with two different patches (half-integer wormhole), see [27];

• family $[0,0]_{a_1=0,E}$ (only even powers in Δ are considered, this is indicated by the subscript "$_E$"), for which the Bach invariant (19) is always nonvanishing, is a generalization of the family $(1,0)_{\bar r_0}$ of [10, 27] for $\Lambda \neq 0$ and describes a wormhole with two same patches (integer wormhole), see [27];

• family $[0,0]_{a_1=0=a_2}$, which can be denoted as $\left(\frac{4}{3}, 0\right)_{\bar r_0, 1/3}$ in the notation of [10], is a generalization of the family found in [15] in the case $\Lambda = 0$;

• family $[0,1] = (1,1)_{\bar r_0}$ is the Schwarzschild–Bach–(A)dS black hole, see Sec. 5.2;

• family $[0,1]_{a_1=0} = \left(\frac{3}{2}, \frac{1}{2}\right)_{\bar r_0, 1/2}$ is a generalization of the black hole found in [15] in the case $\Lambda = 0$;

• family $[0,2] = (2,2)_{\bar{r}_0}$ represents extreme black holes (for generic Λ the extreme Schwarzschild–dS black hole, for discrete values of Λ the extreme higher-order discrete Schwarzschild–Bach–dS black holes, for $\Lambda = \frac{3}{8k}$ the extreme Bachian-dS black hole, see Sec. 5.5;

• family $[0,2]_{a_1=0}$ with $\Lambda = \frac{3}{8k}$ represents the Bachian generalization of the Nariai spacetime with an extreme horizon, belonging to the Kundt class, see Sec. 5.5.2;

• family $[0,2]^\infty$ with a generic Λ (necessarily $A_i = 0$ for all $i \geq 1$) represents the Nariai spacetime belonging to the Kundt family, see Sec. 7.3.1;

• families $[0,2]^\infty_{A_1=\cdots=A_{L-1}=0} = \left(2, -\frac{2}{L}\right)_{\bar{r}_0, 1/L}$, starting with $L=3$, and Λ given by (343) represent the higher-order discrete Nariai-Bach spacetimes with steps in $\bar{r}^{-1/L}$, see Sec. 7.3.1;

• family $[0,2]^\infty$ with $\Lambda = \frac{3}{8k}$ (necessarily $A_i = 0$ for all $i \geq 1$) represents another Bachian generalization of the Nariai spacetime belonging to the Kundt family, see Sec. 7.3.2;

• family $[0, <2]^\infty$: only the solutions $[0,0]^\infty$, $[0,1]^\infty$ exist, and they represent the Plebański–Hacyan solution belonging to the Kundt family, see Sec. 7.4.

9 Summary

To conclude, let us summarize the solutions discussed in this paper.

First, all families compatible with the field equations (25)–(26) in the Kundt coordinates as $r \to r_0$ and $r \to \infty$, in terms of the series expansions (29)–(31) and (33)–(34), are summarized in Tables 8 and 9, respectively. Their physical interpretation and the reference to the corresponding section, in which these solutions are studied, are also indicated. Note that the special subcases of the cases with $n = 0$ are not included here, and can be found in Section 8.2 and in Table 10.

In Table 10, all the classes and subclasses found and identified in this paper, both in the standard and Kundt coordinates, are summarized. They are arranged according to the regions, in which the expansions are taken in the "physical" radial coordinate $\bar{r}$.

Class $[n,p]$	Family (s,t)	Λ	Interpretation	Section
$[-1,2]$	$(0,0)_\infty$	0	Schwarzschild black hole (S)	5.1
$[0,1]$	$(-1,1)_{\bar{r}_0}$	any	Schwarzschild–Bach–(A)dS black hole (near the horizon) (S)	5.2, 5.3
$[0,0]$	$(0,0)_{\bar{r}_0}$	any	generic solution, including the Schwa–Bach–(A)dS black hole (S)	5.4
$[0,2]$	$(-2,2)_{\bar{r}_0}$	any	extreme Schwarzschild–dS black hole (near the horizon) (S)	5.5.1
$[0,2]$	$(-2,2)_{\bar{r}_0}$	disc.	extreme higher-order discrete Schwa–Bach–dS black holes (S)	5.5.1
$[0,2]$	$(-2,2)_{\bar{r}_0}$	$\frac{3}{8k}$	extreme Bachian–dS black hole (S)	5.5.2
$[-1,0]$	$(-2,2)_\infty$	any	Schwarzschild–(A)dS black hole (S)	5.6.1
$[-1,0]$	$(-2,2)_\infty$	disc.	higher-order discrete Schwa–Bach–(A)dS black holes (S)	5.6.2
$[1,0]$	$(2,2)_0$	any	Bachian singularity (near the singularity) (nS)	5.7
$[0,>2]$			empty	5.8
$\left[-\frac{J}{2},2-J\right]$, $J\in\mathbb{N}$, $J\geq 3$	$\left(-2,\left[\frac{8}{3},4\right)\right)_\infty$	disc.	solutions with regular Bachian infinity (nS)	5.9

Table 8: All solutions to QG that can be written as the power series (29)–(30), expanded in the Kundt coordinates around any constant value r_0. For some solutions, only specific discrete values of Λ are allowed (indicated by "disc."). The symbols "(S)" and "(nS)" indicate that a class contains or does *not* contain the Schwarzschild–(A)dS black hole as a special case. Note that the second and fourth columns apply only to the generic cases with $a_1\neq 0$ (see Section 8.2 for the special subcases of the cases with $n=0$).

Class $[N,P]^\infty$	Family (s,t)	Λ	Interpretation	Section
$[-1,3]^\infty$	$(1,-1)_0$	any	Schwa–Bach–(A)dS black hole (near the singularity) (S)	7.1
$[-1,2]^\infty$	$(0,0)_0$	any	Bachian (A)dS vacuum (near the origin) (nS)	7.2
$[0,2]^\infty$	–	any	Nariai spacetime (K, nS)	7.3.1
$[0,2]^\infty$	$(-2,-2/L)_{\bar{r}_0}$	disc.	higher-order discrete Nariai–Bach solutions (nS)	7.3.1
$[0,2]^\infty$	–	$\frac{3}{8k}$	Bachian generalization of the Nariai spacetime (K, nS)	7.3.2
$[0,1]^\infty$	–	$\frac{3}{8k}$	Plebański–Hacyan spacetime (K, nS)	7.4
$[0,0]^\infty$	–	$\frac{3}{8k}$	Plebański–Hacyan spacetime (K, nS)	7.4
$\left[\frac{J}{2},J+2\right]^\infty$ $J\in\mathbb{N}$, $J\geq 2$	$(-2,(4,6])_\infty$	disc.	solutions with regular Bachian infinity (nS)	7.5

Table 9: All solutions to QG that can be written as the power series (33)–(34), expanded in the Kundt coordinates as $r\to\infty$. For some solutions only discrete values of Λ are allowed (indicated by "disc."). The symbols "(S)" and "(nS)" indicate that the class contains or does *not* contain the Schwarzschild–(A)dS black hole as a special case. Note that some of these solutions are written only in the Kundt form (indicated by"(K)") and cannot be transformed to the standard spherically symmetric coordinates.

Family	$[n,p]$ or $[N,P]^\infty$	Parameters	Free param.	Interpretation
(s,t)	$\bar{r} \to 0$			
$(2,2)_0$	$[1,0]$	a_0, c_0, c_1, c_2, r_0 $\Lambda \in \mathbb{R}$	$6 \to 4$	Bachian singularity (nS)
$(2,2)_{0,E}$	$[1,0]_E$ $c_1 = 0 = c_3$	a_0, c_0, r_0 $\Lambda \in \mathbb{R}$	$4 \to 2$	Bachian singularity (nS)
$(1,-1)_0$	$[-1,3]^\infty$	A_0, C_0, C_1, C_3 $\Lambda \in \mathbb{R}$	$5 \to 3$	Schwa–Bach–(A)dS black hole (S)
$(0,0)_0$	$[-1,2]^\infty$	A_0, C_1, C_2 $\Lambda \in \mathbb{R}$	$4 \to 2$	Bachian–(A)dS vacuum (nS)
(w,t)	$\bar{r} \to \bar{r}_0$			
$(1,1)_{\bar{r}_0}$	$[0,1]$	a_0, c_0, c_1, r_h $\Lambda \in \mathbb{R}$	$5 \to 3$	Schwa–Bach–(A)dS black hole (S)
$(\frac{3}{2},\frac{1}{2})_{\bar{r}_0,1/2}$	$[0,1]$ $a_1 = 0$	a_0, c_0, r_0 $\Lambda \in \mathbb{R}$	$4 \to 2$	“unusual” horizon (nS)
$(0,0)_{\bar{r}_0}$	$[0,0]$	$a_0, a_1, c_0, c_1, c_2, r_0$ $\Lambda \in \mathbb{R}$	$7 \to 5$	generic solution (S)
$(1,0)_{\bar{r}_0,1/2}$	$[0,0]$ $a_1 = 0$	a_0, c_0, c_1, c_2, r_0 $\Lambda \in \mathbb{R}$	$6 \to 4$	half-integer wormhole (nS)
$(1,0)_{\bar{r}_0,E}$	$[0,0]$ $a_1 = 0 = c_1 = c_3$	a_0, c_0, r_0 $\Lambda \in \mathbb{R}$	$4 \to 2$	symmetric wormhole (nS)
$(\frac{4}{3},0)_{\bar{r}_0,1/3}$	$[0,0]$ $a_1 = 0 = a_2$	a_0, c_0, c_1, r_0 $\Lambda \in \mathbb{R}$	$5 \to 3$	not known (nS)
$(2,2)_{\bar{r}_0}$	$[0,2]$	c_1, r_h $\Lambda \geq 0$	$3 \to 1$	extreme Schwarzschild–dS black hole (S)
$(2,2)_{\bar{r}_0}$	$[0,2]$	c_1, c_L, r_h $\Lambda \geq 0$ disc.	$3 \to 1$	h-o discrete extreme Schwa–Bach–dS (S)
$(2,2)_{\bar{r}_0}$	$[0,2]$	a_0, c_1, r_h $\Lambda = 3/(8k)$	$3 \to 1$	extreme Bachian–dS black hole (nS/S)
$(2,-\frac{2}{L})_{\bar{r}_0,1/L}$	$[0,2]^\infty$ $A_1 = \ldots A_{L-1} = 0$	C_1, C_L, r_0 $\Lambda \geq 0$ disc.	$3 \to 1$	h-o discrete Nariai–Bach solutions (nS)
(s,t)	$\bar{r} \to \infty$			
$(0,0)_\infty$	$[-1,2]$	a_0, c_1, r_0 $\Lambda = 0$	$3 \to 1$	Schwarzschild black hole (S)
$(-2,2)_\infty$	$[-1,0]$	a_0, c_3, r_0 $\Lambda \in \mathbb{R}$	$4 \to 2$	Schwarzschild–(A)dS black hole (S)
$(-2,2)_\infty$	$[-1,0]$	a_0, c_3, c_{L+3}, r_0 Λ disc.	$4 \to 2$	h-o discrete Schwa–Bach–(A)dS (S)
$\big(-2, \big(4\frac{J-1}{J}\big)\big)_\infty$ $t \in \big[\frac{8}{3}, 4\big)$	$\big[-\frac{J}{2}, 2-J\big]$ $J \geq 3$	a_0, c_{2J-1}, r_0 Λ disc.	$3 \to 1$	solutions with regular Bachian infinity (nS)
$\big(-2, \big(4\frac{J+1}{J}\big)\big)_\infty$ $t \in (4, 6]$	$\big[\frac{J}{2}, J+2\big]^\infty$ $J \geq 2$	A_0, C_1, C_{2J+1} Λ disc.	$3 \to 1$	solutions with regular Bachian infinity (nS)

Table 10: All non-Kundt solutions, found and analyzed in this paper, sorted according to the physical regions in which the expansions are taken in the standard spherically symmetric coordinates. Subscripts “$_0$”, “$_{\bar{r}_0}$”, and “$_\infty$” denote solutions (s,t) or (w,t) (cf. (387), (388)) near $\bar{r} = 0$, $\bar{r} = \bar{r}_0$, and $\bar{r} \to \infty$, respectively. Subscript “$_E$” indicates that only even powers are present in the expansion, while “$_{1/2}$”, and “$_{1/3}$”, and “$_{1/L}$” indicate that fractional powers are present. The number of free parameters is given before and after removing two parameters by the gauge freedom (13) in the Kundt coordinates. In usual coordinates, only one parameter can be removed by rescaling (17). The symbols “(S)” or “(nS)” indicate that a class of solutions contains or does *not* contain the Schwarzschild–(A)dS black hole as a special case, respectively. “h-o” stands for the abbreviation of “higher-order”.

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A Derivation and simplification of the field equations

To derive the QG field equations (25) and (26) (with the condition (27)) for the static spherically symmetric metric in the Kundt coordinates (9), we employ Appendices A–C of [15]. After substituting (A13)–(A16), (B5), and (B7)–(B10) of [15] into the field equations (2),

$$R_{ab} - \Lambda\, g_{ab} = 4k\, B_{ab}\,, \tag{396}$$

we obtain

$$\Omega\Omega'' - 2\Omega'^2 = \tfrac{1}{3}k\,\mathcal{H}'''' \,, \tag{397}$$

$$\big(\Omega^2\mathcal{H}\big)'' - 2\Lambda\Omega^4 = -\tfrac{2}{3}k\big(2\,\mathcal{H}\mathcal{H}'''' + \mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2\big)\,, \tag{398}$$

$$\big(\mathcal{H}\Omega\Omega'\big)' + \Omega^2 - \Lambda\Omega^4 = \tfrac{1}{3}k\,\big(\mathcal{H}\mathcal{H}'''' + \mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2\big)\,, \tag{399}$$

which are the nontrivial components rr, ru, and xx, respectively. The yy component is identical to the xx one, and uu is a multiple of the ru component.

Using (B11) of [15], the trace (7) of the field equations (396), $R = 4\Lambda$, takes the form

$$\mathcal{T} \equiv \mathcal{H}\Omega'' + \mathcal{H}'\Omega' + \tfrac{1}{6}(\mathcal{H}'' + 2)\Omega = \tfrac{2}{3}\Lambda\,\Omega^3\,, \tag{400}$$

which indeed follows from (397)–(399).

In what follows, our goal is to show that the three nontrivial field equations (397)–(399) for the two functions $\Omega(r)$ and $\mathcal{H}(r)$ *can be reduced just to two equations.*

Introducing a symmetric tensor J_{ab} as

$$J_{ab} \equiv R_{ab} - \tfrac{1}{2}R\, g_{ab} + \Lambda\, g_{ab} - 4k\, B_{ab}\,, \tag{401}$$

the vacuum QG field equations (3), assuming R =const., take the form

$$J_{ab} = 0\,. \tag{402}$$

For the metric (9), the non-trivial components of (401) are

$$J_{rr}\,, \qquad J_{uu} = -\mathcal{H}\, J_{ru}\,, \qquad J_{xx} = \mathcal{J}(r)\, g_{xx} = J_{yy}\,, \tag{403}$$

where the function $\mathcal{J}(r)$ is defined as

$$\mathcal{J} \equiv \Omega^{-2}\Big[\big(\mathcal{H}\Omega\Omega'\big)' + \Omega^2 + \Lambda\Omega^4 - 3\,\mathcal{T}\,\Omega - \tfrac{1}{3}k\,\big(\mathcal{H}\mathcal{H}'''' + \mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2\big)\Big]\,, \tag{404}$$

and

$$J_{rr} \,=\, 2\Omega^{-2}\Big[-\Omega\Omega'' + 2\Omega'^2 + \tfrac{1}{3}k\,\mathcal{H}''''\Big]\,, \tag{405}$$

$$J_{ru} \,=\, \Omega^{-2}\Big[-\tfrac{1}{2}\big(\Omega^2\mathcal{H}\big)'' - \Lambda\Omega^4 + 3\,\mathcal{T}\,\Omega - \tfrac{1}{3}k\big(2\,\mathcal{H}\mathcal{H}'''' + \mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2\big)\Big]\,. \tag{406}$$

Since the Bach tensor is conserved, $\nabla^b B_{ab} = 0$, see (6), the contracted Bianchi identities $\nabla^b R_{ab} = \frac{1}{2} R_{,a}$ then yield

$$\nabla^b J_{ab} \equiv 0\,, \tag{407}$$

which is valid regardless of the form of the field equations.

For the metric (9), this leads to identities

$$\nabla^b J_{rb} = -\Omega^{-3}\Omega'\big(J_{ij}\, g^{ij} + \mathcal{H} J_{rr}\big) - \Omega^{-2}\big(\mathcal{H} J_{rr,r} + J_{ru,r} + \tfrac{3}{2}\mathcal{H}' J_{rr}\big) \quad \equiv 0\,, \tag{408}$$

$$\nabla^b J_{ub} = -2\Omega^{-3}\Omega'\big(J_{uu} + \mathcal{H} J_{ru}\big) - \Omega^{-2}\big(J_{uu} + \mathcal{H} J_{ru}\big)_{,r} \quad \equiv 0\,, \tag{409}$$

$$\nabla^b J_{ib} = \Omega^{-2} J_{ik||l}\, g^{kl} \quad \equiv 0\,, \tag{410}$$

where the spatial covariant derivative $_{||}$ is calculated with respect to the spatial (2-sphere) part g_{ij} of the Kundt seed metric (9).

Using (403), equations (409) and (410) are identically satisfied, while (408) is the only nontrivial one. If the field equations $J_{rr}=0$ and $J_{ru}=0$ hold, then from (408) it necessarily follows that $J_{ij}\, g^{ij} = J_{xx}\, g^{xx} + J_{yy}\, g^{yy} = 2\mathcal{J}(r) \equiv 0$ and thus $J_{xx} = 0 = J_{yy}$. Therefore, due to the Bianchi identities, there are only two equations that have to be satisfied, namely $J_{rr} = 0$ yielding (397), and $J_{ru} = 0$ implying (398), where $\mathcal{T} = \frac{2}{3}\Lambda\,\Omega^3$ given by (400) has been used. They completely determine all vacuum QG solutions of the form (9). The remaining equations $J_{xx} = 0 = J_{yy}$ are then automatically satisfied since necessarily $\mathcal{J} = 0$, i.e., using (404)

$$\big(\mathcal{H}\Omega\Omega'\big)' + \Omega^2 + \Lambda\Omega^4 - 3\,\mathcal{T}\,\Omega = \tfrac{1}{3}k\,\big(\mathcal{H}\mathcal{H}'''' + \mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2\big)\,. \tag{411}$$

By substituting $\mathcal{T} = \frac{2}{3}\Lambda\,\Omega^3$, see (400), into (411) we immediately obtain equation (399).

Thus solving the QG field equations (2) for the metric (9) is equivalent to solving (397) and

$$\big(\Omega^2\mathcal{H}\big)'' + 2\Lambda\Omega^4 - 6\mathcal{T}\Omega = -\tfrac{2}{3}k\big(2\,\mathcal{H}\mathcal{H}'''' + \mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2\big)\,. \tag{412}$$

Substituting for $\mathcal{H}''''$ from (397), these two equations (397) and (412) can be simplified to the final set of the field equations (25) and (26) for the metric functions $\Omega(r)$ and $\mathcal{H}(r)$, namely

$$\Omega\Omega'' - 2\Omega'^2 = \tfrac{1}{3}k\,\mathcal{H}''''\,, \tag{413}$$

$$\Omega\Omega'\mathcal{H}' + 3\Omega'^2\mathcal{H} + \Omega^2 - \Lambda\Omega^4 = \tfrac{1}{3}k\big(\mathcal{H}'\mathcal{H}''' - \tfrac{1}{2}\mathcal{H}''^2 + 2\big)\,. \tag{414}$$

Let us also note that, instead of the system (413) and (414), one can alternatively solve equation (413) and any two equations from the set (398), (399), (400).

References

[1] Abbott B P et al. 2016 Tests of General Relativity with GW150914 *Phys. Rev. Lett.* **116** 221101

[2] Akiyama K et al. 2019 First M87 Event Horizon Telescope Results. V. Physical Origin of the Asymmetric Ring *Astrophys J. Lett.* **875** L5

[3] Abuter R et al. 2020 Detection of the Schwarzschild precession in the orbit of the star S2 near the Galactic centre massive black hole *Astron. Astrophys.* **636** L5

[4] Capozziello S and De Laurentis M 2011 Extended Theories of Gravity *Phys. Rep.* **509** 167

[5] Stelle K S 1977 Renormalization of higher derivative quantum gravity *Phys. Rev. D* **16** 953

[6] Salvio A 2018 Quadratic gravity *Front. Phys.* **6** 77

[7] Smilga A V 2014 Supersymmetric field theory with benign ghosts *J. Phys. A* **47** 052001

[8] Stelle K S 1978 Classical gravity with higher derivatives *Gen. Relativ. Gravit.* **9** 353

[9] Lü H, Perkins A, Pope C N and Stelle K S 2015 Black holes in higher derivative gravity *Phys. Rev. Lett.* **114** 171601

[10] Lü H, Perkins A, Pope C N and Stelle K S 2015 Spherically symmetric solutions in higher derivative gravity *Phys. Rev. D* **92** 124019

[11] Nelson W 2010 Static solutions for fourth order gravity *Phys. Rev. D* **82** 104026

[12] Pravda V, Pravdová A, Podolský J and Švarc R 2017 Exact solutions to quadratic gravity *Phys. Rev. D* **95** 084025

[13] Podolský J, Švarc R, Pravda V and Pravdová A 2018 Explicit black hole solutions in higher-derivative gravity *Phys. Rev. D* **98** 021502(R)

[14] Švarc R, Podolský J, Pravda V, and Pravdová A 2018 Exact black holes in quadratic gravity with any cosmological constant *Phys. Rev. Lett* **121** 231104

[15] Podolský J, Švarc R, Pravda V and Pravdová A 2020 Black holes and other exact spherical solutions in quadratic gravity *Phys. Rev. D* **101** 024027

[16] Glavan D and Lin Ch 2020 Einstein-Gauss-Bonnet gravity in four-dimensional spacetime *Phys. Rev. Lett.* **124** 081301

[17] Griffiths J and Podolský J 2009 *Exact Space-Times in Einstein's General Relativity* (Cambridge: Cambridge University Press)

[18] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press)

[19] Wald R M 1984 *General Relativity* (Chicago: University of Chicago Press)

[20] Zhong-Ying Fan and Lü H 2015 Thermodynamical first laws of black holes in quadratically-extended gravities *Phys. Rev. D* **91** 064009

[21] Wald R M 1993 Black hole entropy is the Noether charge *Phys. Rev. D* **48** R3427(R)

[22] Iyer V and Wald R M 1994 Some properties of Noether charge and a proposal for dynamical black hole entropy *Phys. Rev. D* **50** 846

[23] Lü H and Pope C N 2011 Critical gravity in four dimensions *Phys. Rev. Lett.* **106** 181302

[24] Bičák J and Podolský J 1999 Gravitational waves in vacuum spacetimes with cosmological constant. II. Deviation of geodesics and interpretation of nontwisting type N solutions *J. Math. Phys. (N.Y.)* **40** 4506

[25] Podolský J and Švarc R 2012 Interpreting spacetimes of any dimension using geodesic deviation *Phys. Rev. D* **85** 044057

[26] Holdom B and Ren J 2017 Not quite a black hole *Phys. Rev. D* **95** 084034

[27] Perkins A 2016 *Static spherically symmetric solutions in higher derivative gravity* (Ph.D. thesis, Imperial College London)

[28] Nariai H 1951 On a new cosmological solution of Einstein's field equations of gravitation *Sci. Rep. Tôhoku Univ.* **35** 62

[29] Podolský J and Ortaggio M 2003 Explicit Kundt type II and N solutions as gravitational waves in various type D and O universes *Class. Quantum Grav* **20** 1685

[30] Kadlecová H, Zelnikov A, Krtouš P and Podolský J 2009 Gyratons on direct-product spacetimes *Phys. Rev. D* **80** 024004