A novel classification method of 2+1 spacetimes based on the Cotton scalars

J. Podolský and M. Papajčík*

*Charles University, Faculty of Mathematics and Physics,
Institute of Theoretical Physics, V Holešovičkách 2, 180 00 Prague 8, Czechia.*
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A new effective approach to the algebraic classification of geometries in 2+1 gravity is presented. It uses five real Cotton scalars Ψ_A of distinct boost weights, which are $3D$ analogues of the Newman–Penrose scalars representing the Weyl tensor in $4D$. The classification into types I, II, D, III, N, O is directly related to the multiplicity of the four Cotton-aligned null directions (CANDs). We derive a synoptic algorithm based on the invariants constructed from Ψ_A, and we show its agreement with the Petrov scheme based on eigenvalues and canonical Jordan form of the Cotton–York tensor. Our method is simpler and also general because it can be used in any 2+1 theory, such as Einstein's gravity or topologically massive gravity. As an example we analyze the algebraic structure of Robinson–Trautman spacetimes which include charged black holes with a cosmological constant.

I. INTRODUCTION

Einstein's general relativity (GR), conceived in 1915, is a highly successful physical theory. Formulating gravity as specific curvature of $3 + 1$ spacetime, it plays a fundamental role in contemporary astrophysics and cosmology, in particular in the theory of black holes and gravitational waves. Its predictions are in excellent agreement with various tests and observations [1].

Despite this enormous success, GR has its own limits. It predicts physical singularities inside black holes or at an initial big bang, where quantum effects become crucial. Many attempts to construct a quantum gravity theory have been made [2], but it seems that none of them is yet conceptually clear, mathematically consistent, complete and (above all) experimentally tested.

In order to attack the difficult problem of quantizing gravity, various reformulations, modifications and generalizations of GR have been proposed. These include extensions to higher dimensions $D > 4$, as in the string theory approach, but also study of lower-dimensional $D = 3$ spacetimes. Such a *2+1 gravity*, proposed in the 1960s, is nowadays a very popular research arena of fundamental physics. It provides *models of quantum gravity* using various approaches [3], but also gives a great number of explicit *exact spacetimes on a classical level* [4].

To classify the vast number of 2+1 spacetimes and to understand their properties, classification schemes were proposed, analogous to those in $D = 4$ GR [5]. The most important is an *algebraic classification* into types I, II, D, III, N, O developed in 1954–60 by Petrov, Géhéniau, Pirani, Bell, Debever and Penrose. In its best formulation it is based on the multiplicity of 4 *principal null directions* (PNDs) of the curvature Weyl tensor C_{abcd} encoded in the Newman–Penrose scalars Ψ_A (A = 0, 1, 2, 3, 4) [5–7]. In 2004 this was extended to $D > 4$ by Coley, Milson, Pravda and Pravdová who introduced the concept of *Weyl-aligned null directions* (WANDs), see the reviews [8, 9] and [10] using the notation Ψ_A.

The algebraic classification in $D = 3$ was introduced by Barrow, Burd and Lancaster [11] and later refined by García, Hehl, Heinicke and Macías [12]. Instead of using the Weyl tensor C_{abcd} (which vanishes identically), it is necessary to employ the *Cotton tensor* C_{abc} [13]. In 2+1 gravity it has 5 components and can be Hodge-mapped onto the *Cotton–York tensor* Y_{ab} [14]. It is a symmetric traceless 3×3 matrix, and its algebraic type can be determined by the *eigenvalues* and related *Jordan forms* (for details see [4], which also reviews another important classification method to Segre–Plebański types based on the traceless Ricci tenor).

Here we propose a *novel, practical method* of algebraic classification in 2+1 gravity, consistent with [12]. Assuming no field equations, it is also more *general*. We directly use the Cotton scalars Ψ_A which are the null-triad projections of C_{abc}. These are the $D = 3$ counterparts of the Newman–Penrose Weyl scalars of GR, and determine the multiplicity of the CANDs — in analogy with PNDs (in $D = 4$) and WANDs (in $D > 4$).

II. COTTON SCALARS Ψ_A

In a general $3D$ spacetime with the metric g_{ab} of signature $(-, +, +)$, the curvature is given by the *Ricci tensor* R_{ab}, *Ricci scalar* $R \equiv R_a{}^a$ and *Cotton tensor* [4], Ch. 20,

$$C_{abc} \equiv 2 \left(\nabla_{[a} R_{b]c} - \tfrac{1}{4} \nabla_{[a} R \, g_{b]c} \right), \qquad (2.1)$$

where ∇ is the metric connection. It is antisymmetric ($C_{(ab)c} = 0 = C_{[abc]}$) and traceless ($C_{ab}{}^a = 0$), so that it has *5 independent components*. The Cotton tensor C_{abc} plays the role similar to the Weyl tensor C_{abcd} representing a free gravitational field in Einstein's $4D$ gravity. In the Newman–Penrose formalism [5–7] the Weyl tensor is encoded in 5 complex scalars Ψ_A, defined as projections of C_{abcd} onto specific combinations of some *null tetrad*

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* jiri.podolsky@mff.cuni.cz; matus.papajcik@matfyz.cuni.cz

vectors. Since $C_{abcd} \equiv 0$ in any 2+1 geometry, it cannot be used for classification in $3D$. However, one can introduce the *null triad* basis $\{e_a\} \equiv \{k, l, m\}$ normalized as

$$k_a \, l^a = -1, \qquad m_a \, m^a = 1, \qquad (2.2)$$

$k_a \, k^a = 0 = l_a \, l^a$ and $k_a \, m^a = 0 = l_a \, m^a$. It means that k and l are *null vectors*, while m is the *spatial unit vector* orthogonal to both k and l. Next we define the Newman–Penrose-type real *Cotton scalars* Ψ_A as the projections

$$\begin{aligned}
\Psi_0 &\equiv C_{abc} \, k^a \, m^b \, k^c, \\
\Psi_1 &\equiv C_{abc} \, k^a \, l^b \, k^c, \\
\Psi_2 &\equiv C_{abc} \, k^a \, m^b \, l^c, \\
\Psi_3 &\equiv C_{abc} \, l^a \, k^b \, l^c, \\
\Psi_4 &\equiv C_{abc} \, l^a \, m^b \, l^c.
\end{aligned} \qquad (2.3)$$

Equivalently, $\Psi_1 = C_{abc} \, k^a \, m^b \, m^c$, $\Psi_3 = C_{abc} \, l^a \, m^b \, m^c$, $\Psi_2 = C_{abc} \, m^a \, l^b \, k^c = \frac{1}{2} C_{abc} \, k^a \, l^b \, m^c$.

III. THE CLASSIFICATION BASED ON Ψ_A

The specific *algebraic types* of 2+1 geometries can now be defined by very simple conditions, namely that *in a suitable null triad* $\{k, l, m\}$ some of the *Cotton scalars* Ψ_A *vanish*, as given in Tab. I.

TABLE I. A simple algebraic classification of 2+1 geometries.

type	the conditions	
I	$\Psi_0 = 0,$	$\Psi_1 \neq 0$
II	$\Psi_0 = \Psi_1 = 0,$	$\Psi_2 \neq 0$
III	$\Psi_0 = \Psi_1 = \Psi_2 = 0,$	$\Psi_3 \neq 0$
N	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0,$	$\Psi_4 \neq 0$
D	$\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4,$	$\Psi_2 \neq 0$
O	all $\Psi_A = 0$	

In fact, formally these are the same conditions as for the $D = 4$ classification related to the multiplicity of the PNDs of the Weyl tensor, see Sec. 4.3 of [5], or the multiplicity of the WANDs in $D > 4$ [9].

IV. CLASSIFICATION INVARIANTS

A *general choice* of the null triad $\{k, l, m\}$ does not provide vanishing Cotton scalars as prescribed in Tab. I. However, for each algebraic type the required canonical forms of Ψ_A can be achieved by a suitable Lorentz transformation. This also leads to the important concept of Cotton-aligned null direction (CAND), see Sec. V.

In practise it is not necessary to find the null triad with the canonical forms of Ψ_A. It can be proven [15] that the algebraic type of a given 2+1 geometry can be
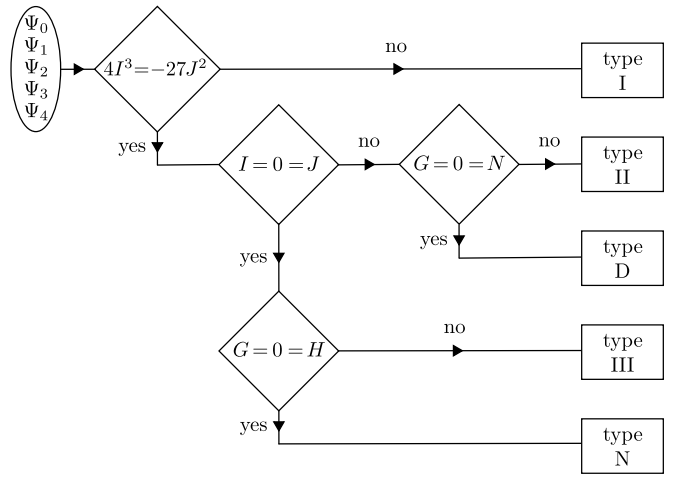

FIG. 1. Flow diagram for determining the algebraic type of a 2+1 geometry using the invariants (4.1) constructed from the Cotton scalars Ψ_A (if $\Psi_4 \neq 0$). Type O is a conformally flat spacetime with vanishing Cotton tensor. Application of the diagram on various explicit geometries can be found in [15].

determined using the Cotton scalars (2.3) evaluated *in any null triad* satisfying (2.2). Such a classification can be done using the *scalar polynomial invariants*

$$\begin{aligned}
I &\equiv \Psi_0 \Psi_4 - 2 \, \Psi_1 \Psi_3 - 3 \, \Psi_2^2, \\
J &\equiv 2 \, \Psi_0 \Psi_2 \Psi_4 + 2 \, \Psi_1 \Psi_2 \Psi_3 + 2 \, \Psi_2^3 + \Psi_0 \Psi_3^2 - \Psi_4 \Psi_1^2, \\
G &\equiv \Psi_1 \Psi_4^2 - 3 \, \Psi_2 \Psi_3 \Psi_4 - \Psi_3^3, \\
H &\equiv 2 \, \Psi_2 \Psi_4 + \Psi_3^2, \\
N &\equiv 3 \, H^2 + \Psi_4^2 \, I.
\end{aligned} \qquad (4.1)$$

In fact, $I = \frac{1}{4} C_{abc} \, C^{abc}$ and $J = \frac{1}{6} C_{abc} \, C^{abd} \, Y^c{}_d$, where Y_{ab} is the Cotton–York tensor (7.1). Useful *algorithm of algebraic classification* is given by the flow diagram in Fig. 1 (it is an analogue of the $D = 4$ diagram presented in Fig. 9.1 of [5]). The procedure is *not applicable if* $\Psi_4 = 0$. In this case, when $\Psi_0 \neq 0$ we can perform a swap $\Psi_0 \leftrightarrow -\Psi_4$, $\Psi_1 \leftrightarrow -\Psi_3$, after which Fig. 1 can be used. When $\Psi_4 = 0 = \Psi_0$, it is necessary to employ Tab. II.

V. COTTON-ALIGNED NULL DIRECTION (CAND)

The Cotton scalars Ψ_A depend on the choice of the null triad $\{k, l, m\}$. However, as in $D \geq 4$ this freedom is simply given by *Lorentz transformations* at a given point of the spacetime. There are 3 subgroups preserving (2.2):

$$k' = B \, k, \quad l' = B^{-1} l, \quad m' = m, \qquad (5.1)$$

$$k' = k, \quad l' = l + \sqrt{2} \, L \, m + L^2 \, k, \quad m' = m + \sqrt{2} \, L \, k, \qquad (5.2)$$

$$k' = k + \sqrt{2} \, K \, m + K^2 \, l, \quad l' = l, \quad m' = m + \sqrt{2} \, K \, l, \qquad (5.3)$$

TABLE II. Algebraic classification of 2+1 geometries for the special case $\Psi_4 = 0 = \Psi_0$.

$\Psi_1 = 0$	$\Psi_2 = 0$	$\Psi_3 = 0$	type O
		$\Psi_3 \neq 0$	type III
	$\Psi_2 \neq 0$	$\Psi_3 = 0$	type D
		$\Psi_3 \neq 0$	type II
$\Psi_1 \neq 0$	$\Psi_2 = 0$	$\Psi_3 = 0$	type III
		$\Psi_3 \neq 0$	type I
	$\Psi_2 \neq 0$	$\Psi_3 = 0$	type II
		$\Psi_3 \neq 0$: $9\Psi_2^2 = -8\Psi_1\Psi_3$	type II
		$9\Psi_2^2 \neq -8\Psi_1\Psi_3$	type I

where B, K, L are *real parameters*. Under the *boost* (5.1), $\Psi_A' = B^{2-A}\,\Psi_A$, i.e. Ψ_A in (2.3) are *ordered according to their specific boost weights*, which is the corresponding *power* $(2-A)$ of the boost parameter B. Under the *null rotation* (5.2) *with fixed* $\boldsymbol{k} = \boldsymbol{k}'$, the scalars transform as $\Psi_0' = \Psi_0$, $\Psi_1' = \Psi_1 + \sqrt{2}\,L\,\Psi_0$, etc. [15]. The classification in Tab. I is thus *invariant with respect to both* (5.1) *and* (5.2). The *null rotation* (5.3) *with fixed* $\boldsymbol{l} = \boldsymbol{l}'$ transforms the Cotton scalars (2.3) as

$$\Psi_0' = \Psi_0 + 2\sqrt{2}\,K\,\Psi_1 + 6K^2\,\Psi_2 - 2\sqrt{2}\,K^3\,\Psi_3 - K^4\,\Psi_4\,, \tag{5.4}$$

etc. Now, a crucial observation is that (5.4) *always* allows us to *achieve* $\Psi_0' = 0$ by a *suitable choice* of the parameter K, so that in the new triad $\{\boldsymbol{k}', \boldsymbol{l}', \boldsymbol{m}'\}$ the condition for type I given in Tab. I is satisfied. Such a frame can be called the *principal null triad*, and we name its null vector $\boldsymbol{k}'$ the *Cotton-aligned null direction*, abbreviated as CAND. A CAND is the 2+1 analogue of a PND (principal null direction) of the Weyl tensor in $D = 4$ GR, and of a WAND (Weyl-aligned null direction) in $D \geq 4$ gravity [5, 9].

This also proves that *all 2+1 geometries with $C_{abc} \neq 0$ are of algebraic type I, or more special*. This is also true for $3 + 1$ geometries, considering the Weyl tensor instead of the Cotton tensor.

Moreover, the CAND can be explicitly found. In view of (5.4), the condition $\Psi_0' = 0$ reads

$$\Psi_4\,K^4 + 2\sqrt{2}\,\Psi_3\,K^3 - 6\,\Psi_2\,K^2 - 2\sqrt{2}\,\Psi_1\,K - \Psi_0 = 0\,. \tag{5.5}$$

It is an algebraic equation of the 4th order for K, which admits *four complex solutions* (not necessarily distinct). Therefore, at any event of the 2+1 spacetime *there exist four CANDs* determined by the local algebraic structure of the Cotton tensor. Each of these four CANDs $\boldsymbol{k}'$ is obtained by (5.3) where the parameter K is the root of (5.5), although it is generally difficult to find them explicitly. Multiplicity of the roots K thus implies the same *multiplicity of the CANDs*. These are uniquely related to the algebraic types, as we discuss next.

VI. CANDS MULTIPLICITY, COMPLEXITY, AND THE ALGEBRAIC (SUB)TYPES

A 2+1 spacetime is said to be *algebraically general* if its CANDs, i.e. the four roots of (5.5), are *all distinct*. Such a spacetime is of algebraic *type I*. A spacetime is *algebraically special* if at least two of its CANDs *coincide*. If *just two* CANDs $\boldsymbol{k}$ coincide, it is of *type II*. Higher multiplicity defines *type III* (triple CAND/root) and the most special *type N* (quadruple CAND/root) geometries. In addition, there is a degenerate case *type D* — a subtype of type II with two distinct CANDs $\boldsymbol{k}$ and $\boldsymbol{l}$, both of multiplicity 2 (*two pairs* of coinciding roots), see Tab. III.

TABLE III. Algebraic types of 2+1 geometries are uniquely related to the multiplicity of the Cotton-aligned null directions (CANDs), i.e. to the multiplicity of the four roots of the key equation (5.5). The last column contains the roots K and L for the canonical Cotton scalars Ψ_A given in Tab. I.

type	CANDs	multiplicity	canonical roots
I	↖↑↗↗	$1+1+1+1$	$K = 0$, 3 other distinct roots
II	↖↑↗	$1+1+2$	$K^2 = 0$, 2 other distinct roots
D	↘↗	$2+2$	$K^2 = 0$, $L^2 = 0$
III	↖↗↗	$1+3$	$K^3 = 0$, $L = 0$ or $K \neq 0$
N	↗↗↗↗	4	$K^4 = 0$

If $\boldsymbol{k}$ is a CAND then $\Psi_0 = 0$, corresponding to the root $K = 0$ of (5.5). Type II arises when $\Psi_1 = 0$, type III when $\Psi_1 = \Psi_2 = 0$, and type N when $\Psi_1 = \Psi_2 = \Psi_3 = 0$, in which case $K^4 = 0$ gives the quadruple CAND $\boldsymbol{k}$. For type D spacetimes with $\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4$, Eq. (5.5) reduces to $\Psi_2 K^2 = 0$, so that $\boldsymbol{k}$ is the double CAND. Moreover (swapping $\boldsymbol{k} \leftrightarrow \boldsymbol{l}$, so that $\Psi_0 \leftrightarrow \Psi_4$, $\Psi_1 \leftrightarrow \Psi_3$, $\Psi_2 \leftrightarrow -\Psi_2$), the vector $\boldsymbol{l}$ is another double CAND, and the condition $\Psi_4 = \Psi_3 = 0$ implies $L^2 = 0$.

Such a classification in 2+1 gravity is actually more subtle because the key *real* equation (5.5) can have some *complex roots* K. Some of the vectors $\boldsymbol{k}$ representing CANDs can thus *formally be complex*. This cannot happen in 3+1 gravity or in $D > 4$. It is thus natural to suggest a *subclassification*:

- subtypes I_r, II_r and D_r: *all four (possibly multiple) CANDs are real,*

- subtypes I_c, II_c and D_c: *some of the CANDs are complex.*

It can be shown that type $III \equiv III_r$ and type $N \equiv N_r$.

Explicit conditions for these subtypes, the relation to Class I' geometries with complex eigenvalues of the Cotton–York tensor, and the Petrov–Segre types $I_{\mathbb{R}}$ and $I_{\mathbb{C}}$ in topologically massive gravity will be presented in [15], together with the Bel–Debever criteria for all types.

TABLE IV. Algebraic classification based on the Jordan normal forms and eigenvalues λ_i of the Cotton–York tensor $Y_a{}^b$, the corresponding special values of the Cotton scalars Ψ_A, and the scalar invariants. It shows the equivalence of these classifications.

algebraic type	Jordan normal form of $Y_a{}^b$	special Cotton scalars	invariants
I	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 \end{pmatrix}$	$\Psi_1 = 0 = \Psi_3$ $\Psi_0 = \frac{1}{2}(\lambda_1 - \lambda_2) = -\Psi_4$ $\Psi_2 = \frac{1}{2}(\lambda_1 + \lambda_2)$	$I = \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2)^2$ $J = \lambda_1 \lambda_2 (\lambda_1 + \lambda_2)$
II	$\begin{pmatrix} \lambda_1 - 1 & -1 & 0 \\ 1 & \lambda_1 + 1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix}$	$\Psi_0 = 0, \ \Psi_1 = 0 = \Psi_3$ $\Psi_2 = \lambda_1$ $\Psi_4 = 2$	$I = -3\lambda_1^2$ $J = 2\lambda_1^3$ $G = 0, \ N = 36\lambda_1^2$
D	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix}$	$\Psi_0 = 0 = \Psi_4$ $\Psi_1 = 0 = \Psi_3$ $\Psi_2 = \lambda_1$	$I = -3\lambda_1^2$ $J = 2\lambda_1^3$ $G = 0 = N$
III	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$	$\Psi_0 = 0 = \Psi_4$ $\Psi_1 = 0 = \Psi_2$ $\Psi_3 = \sqrt{2}$	$I = 0 = J$ $G = -2\sqrt{2}, \ H = 2$
N	$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\Psi_0 = 0 = \Psi_2$ $\Psi_1 = 0 = \Psi_3$ $\Psi_4 = 2$	$I = 0 = J$ $G = 0 = H$

VII. EQUIVALENCE WITH CLASSIFICATION BASED ON THE COTTON–YORK TENSOR

In previous studies, the Hodge dual of the Cotton tensor C_{abc} was employed. The *Cotton–York tensor* [14] is defined as $Y_{ab} \equiv e_a \rfloor {}^*\mathbf{C}_b$, where $\mathbf{C}_b$ is the Cotton 2-form $\mathbf{C}_b \equiv \frac{1}{2} C_{mnb} \, \boldsymbol{\omega}^m \wedge \boldsymbol{\omega}^n$, see Eq. (20.111) in [4]. Explicitly,

$$Y_{ab} = \tfrac{1}{2} \, g_{ak} \, \omega^{kmn} \, C_{mnb} \, , \qquad (7.1)$$

where the Levi-Civita tensor reads $\omega_{abc} = -\sqrt{-g} \, \varepsilon_{abc}$. It has 5 components because it is symmetric and traceless ($Y_{ab} = Y_{ba}$ and $Y_a{}^a = 0$). It can be shown [15] that the *general Cotton–York tensor* in the triad (2.2) is

$$\begin{aligned} Y_{ab} = & - \Psi_0 \, l_a \, l_b + \Psi_1 (l_a \, m_b + m_a \, l_b) \\ & - \Psi_2 (l_a \, k_b + k_a \, l_b + 2 \, m_a \, m_b) \\ & - \Psi_3 (k_a \, m_b + m_a \, k_b) + \Psi_4 \, k_a \, k_b \, . \end{aligned} \qquad (7.2)$$

This can be used to show the equivalence of our new method of classification, based on the Cotton scalars Ψ_A and the multiplicity of CANDs, with the "Petrov" scheme based on the Jordan form and eigenvalues of the Cotton–York tensor [4, 11, 12]. To this end, we express Y_{ab} in an *orthonormal basis* $\{\boldsymbol{E}_0, \boldsymbol{E}_1, \boldsymbol{E}_2\}$ associated with $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}\}$ via $\boldsymbol{E}_0 \equiv \frac{1}{\sqrt{2}}(\boldsymbol{k} + \boldsymbol{l})$, $\boldsymbol{E}_1 \equiv \frac{1}{\sqrt{2}}(\boldsymbol{k} - \boldsymbol{l})$, $\boldsymbol{E}_2 \equiv \boldsymbol{m}$, so that in this basis the metric reads $g_{ab} = \mathrm{diag}(-1, 1, 1)$. Using (7.2) we calculate all orthonormal projections (such as $Y_{00} \equiv E_0^a E_0^b Y_{ab}$ etc.),

and evaluate $Y_a{}^b \equiv Y_{ac} \, g^{cb}$ as

$$Y_a{}^b = \begin{pmatrix} \Psi_2 + \dfrac{\Psi_0 - \Psi_4}{2} & -\dfrac{\Psi_0 + \Psi_4}{2} & -\dfrac{\Psi_1 - \Psi_3}{\sqrt{2}} \\[2mm] \dfrac{\Psi_0 + \Psi_4}{2} & \Psi_2 - \dfrac{\Psi_0 - \Psi_4}{2} & -\dfrac{\Psi_1 + \Psi_3}{\sqrt{2}} \\[2mm] \dfrac{\Psi_1 - \Psi_3}{\sqrt{2}} & -\dfrac{\Psi_1 + \Psi_3}{\sqrt{2}} & -2\Psi_2 \end{pmatrix} . \qquad (7.3)$$

This matrix is traceless but *not symmetric*, so the roots of the characteristic polynomial $\det (Y_a{}^b - \lambda \, \delta_a{}^b) = 0$ may be *complex*. According to the *eigenvalues* λ_1, λ_2 and $\lambda_3 = -\lambda_1 - \lambda_2$, the *Petrov types* are defined such that the case $\lambda_1 \neq \lambda_2$ gives type I, $\lambda_1 = \lambda_2 \neq 0$ gives type II or type D, while $\lambda_1 = \lambda_2 = 0 = \lambda_3$ gives type III or type N.

We can find the *Jordan normal forms* defining the algebraic types of 2+1 geometries, as presented in Tab. IV. For types I and D the first two columns are a copy of Tab. 1.2.1 of [4]. For types II and N we performed a *similarity transformation* between the Jordan form J presented in [4] and the *normal form* $A \, J \, A^{-1}$ with $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. For type III we employed $A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$. Such forms of $Y_a{}^b$ can be uniquely identified with the special values of the Cotton scalars Ψ_A using (7.3), see the third column of Tab. IV. It confirms that the "Petrov types" defined by the Jordan forms of $Y_a{}^b$ are equivalent to our approach. Indeed, evaluating the invariants (4.1)

for the special values of Ψ_A (the last column in Tab. IV) the algorithm in Fig. 1 *gives the same algebraic types* (for D and III we must employ Tab. II because $\Psi_0 = 0 = \Psi_4$). Our method also shows the unique relation to multiplicity of CANDs, in full analogy with PNDs in $D = 4$ gravity, see Sec. 4.3 of [5], and WANDs in $D > 4$ theories [9].

VIII. ROBINSON–TRAUTMAN SPACETIMES

The usefulness of the new classification method can be demonstrated on treating a class of 2+1 Robinson–Trautman spacetimes with a *cosmological constant* Λ and an *aligned electromagnetic field*, which include black holes. In [16] we derived their general form

$$\mathrm{d}s^2 = \frac{r^2}{P^2}\big(\mathrm{d}x + e\,P^2\mathrm{d}u\big)^2 - 2\,\mathrm{d}u\,\mathrm{d}r - 2H\,\mathrm{d}u^2\,, \quad (8.1)$$

$$2H = -m + \kappa_0\,Q^2\ln\left|\frac{Q}{r}\right| - 2\,(\ln Q)_{,u}\,r - \Lambda\,r^2\,,$$

with the Maxwell field potential $\mathbf{A} = Q\,\ln(r/r_0)\,\mathrm{d}u$, see Eqs. (180), (182) of [16]. Here $m \equiv \mu\,Q^2$, μ is a constant and $Q(u)$ is *any* function of u, while the functions $P(u,x)$, $e(u,x)$ satisfy the Einstein field equation $(Q/P)_{,u} = Q\,(e\,P)_{,x}$. The Cotton tensor (2.1) of the solution (8.1) has components $C_{urr} = \frac{1}{2}\kappa_0\,Q^2\,r^{-3}$, $C_{xrx} = \frac{1}{2}\kappa_0\,(Q^2/P^2)\,r^{-1}$, $C_{xru} = C_{urx} = \frac{1}{2}\kappa_0\,e\,Q^2\,r^{-1}$, and much more complicated expressions for C_{ruu}, C_{uxu} and C_{uxx}. However, using the null triad $\boldsymbol{k} = \partial_r$, $\boldsymbol{l} = \partial_u - H\,\partial_r - e\,P^2\,\partial_x$, $\boldsymbol{m} = (P/r)\,\partial_x$, definition (2.3) yields *very simple Cotton scalars*

$$\Psi_0 = 0 = \Psi_4\,, \quad \Psi_2 = 0\,, \quad (8.2)$$

$$\Psi_1 = -\frac{\kappa_0 Q^2}{2r^3}\,, \quad \Psi_3 = \frac{1}{2}\Big(m - \kappa_0\,Q^2\ln\left|\frac{Q}{r}\right| + \Lambda\,r^2\Big)\,\Psi_1\,.$$

Because $\Psi_0 = 0$ and generally $\Psi_1 \neq 0$, Tab. I indicates that such spacetimes are *of algebraic type I*. Moreover, it follows from Secs. V, VI that $\boldsymbol{k} = \partial_r$ is the CAND. It coincides with the null direction of the aligned electromagnetic field. Because $\Psi_4 = 0$, the second distinct CAND is $\boldsymbol{l}$.

All scalars (8.2) vanish when $Q = 0$, corresponding to *vacuum* solutions with Λ, and thus (anti-)de Sitter or Minkowski spaces which are *conformally flat* (type O). In the case $Q \neq 0$ with an aligned electromagnetic field

the main invariants (4.1) are $I = -2\,\Psi_1\Psi_3$ and $J = 0$. Because $\Psi_4 = 0 = \Psi_0$, we must employ Tab. II instead of Fig. 1. Then $\Psi_1 \neq 0$, $\Psi_2 = 0$, $\Psi_3 \neq 0$ determine *type I*.

In [16] we identified the class of (cyclic symmetric) *charged black hole electrostatic solutions* [17], the 2+1 analogue to the Reissner–Nordström–(anti-)de Sitter solution (Eq. (192) of [16] and Sec. 11.2 of [4]). This arises as the special subcase $Q = \mathrm{const.}$, $e = 0$ of (8.1). Such spacetimes are also of type I, in agreement with Sec. 11.1.5 of [4]. Interestingly, *on the horizons*, localized by the condition $H = 0$, the scalar Ψ_3 in (8.2) vanishes. So according to Tab. II these horizons are of *type III*. Eq. (5.5) determining the CANDs for (8.2) becomes

$$\left(1 + H\,K^2\right)K = 0\,. \quad (8.3)$$

For $H > 0$ (above the horizon) there are thus *two complex CANDs*, so that this region is of algebraic subtype $\mathrm{I_c}$. In contrast, for $H < 0$ (below the horizon) there are four real CANDs, and thus the region is of subtype $\mathrm{I_r}$.

SUMMARY

We have presented a new convenient method of algebraic classification for 2+1 geometries. It is independent on any field equations. The procedure has 5 simple steps:

1. Calculate the Cotton tensor C_{abc} using (2.1).

2. Choose any null triad $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}\}$ satisfying (2.2).

3. Evaluate the Cotton scalars Ψ_A by (2.3).

4. Calculate the invariants I, J, G, H, N by (4.1).

5. Use the algorithm presented in Fig. 1 (or Tab. II).

The types I, II, III, N, D, O correspond to canonical forms of the Cotton scalars (Tab. I), with specific multiplicity of the 4 CANDs (Tab. III). Our method also suggests the refinement into the subtypes $\mathrm{I_r}$, $\mathrm{II_r}$, $\mathrm{D_r}$ (all CANDs are real) and subtypes $\mathrm{I_c}$, $\mathrm{II_c}$, $\mathrm{D_c}$ (complex CANDs). We proved that this agrees with previous classification scheme based on eigenvalues and Jordan forms of the Cotton–York tensor $Y_a{}^b$ (Tab. IV). As demonstrated in an explicit example (Sec. VIII), it provides additional insight into the structure of exact spacetimes in 2+1 gravity.

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