INSTITUTE OF PHYSICS PUBLISHING EUROPEAN JOURNAL OF PHYSICS

Eur. J. Phys. **26** (2005) 1037–1045 doi:10.1088/0143-0807/26/6/011

Buquoy's problem

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Received 17 June 2005, in final form 19 June 2005
Published 8 August 2005
Online at stacks.iop.org/EJP/26/1037

Abstract
We analyse the one-dimensional motion of a uniform thin fibre which is pulled upwards from a horizontal plane by a constant vertical force exerted against the homogeneous gravitational field. The solution of the equation of motion which describes this variable mass problem is discussed and the character of the resulting damped oscillations is described.

1. Historical background and formulation of the problem

The problem we will discuss here is associated with the name of count Jiří František August Buquoy (in German transcription Graf Georg von Buquoy), Czech aristocrat, mathematician and gifted inventor (1781–1851). After his studies in Prague and Vienna where he was educated in mathematics, natural science, philosophy, law and economy, he devoted his time from 1803 to taking care of the large family possessions and to his investigations. In 1810 he constructed a steam engine and did his best to apply it in practice. Above all, he was engaged in glass works. On the basis of many experiments he succeeded in inventing an original process technology (now forgotten) of a black opaque glass called hyalite (1817).

Buquoy was the first who investigated mechanical systems with a varying mass. In 1812 he explicitly formulated the correct dynamical equation of motion for the case when the mass of a moving object is changing (see [1], p 66). He subsequently suggested several nice concrete examples of such a motion [2] to which he applied his new generalized dynamical equation, attempting also to solve the corresponding differential equations. In August 1815 he presented his results at the Paris Academy of Sciences (Institut National des Sciences et des Arts, Première Classe) to Laplace, Poisson, Ampère, Delambre, Arago, Cauchy, Fourier and others [3, 4]. Nevertheless, apart from a single short article [5] by Poisson, his ideas did not attract attention, and they gradually became forgotten. Buquoy's general equation of motion and other explicit examples were later formulated independently by various authors [6–8]. The pioneering work of Buquoy on systems of non-constant mass was rediscovered and his

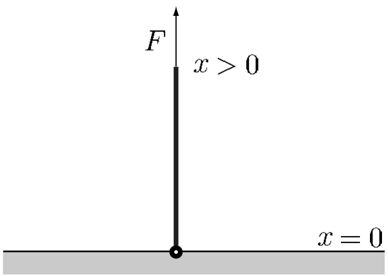

Figure 1. A schematic picture of the vertically moving fibre. The actual position of its end above the horizontal plane is represented by x.

role in the history of physics was recognized only recently by Mikhailov [9–11]. Since then it has begun to appear in some new textbooks on mechanics [12, 13].

The first explicit example of a system with a varying mass suggested by Buquoy in 1814 (see [2] on p 34) is the following:

> Consider an ideally flexible fibre lying reeled on a horizontal plane. Determine its motion when a constant vertical force (directed upward) is exerted on the end of the fibre.

We wish to analyse this problem here and to demonstrate that it exhibits some surprising properties which may be of pedagogical interest for undergraduate students. (Let us note that the original solution proposed in [2] was not correct. The particular case of the problem was correctly solved in [12].) Denoting the position of the end of the fibre above the horizontal plane by $x > 0$ (see figure 1), we make the following natural simplifying assumptions:

- the vertical gravitational field is homogeneous,
- the fibre is thin and its linear density η is constant,
- the fibre reels off without friction at the origin of the x-axis during the upward motion,
- the fibre 'smoothly disappears' at the origin of the x-axis during the motion downward, i.e., the part that has already landed on the plane does not move.

The problem could be modelled experimentally as the vertical motion of a balloon with a heavy rope hanging down. The constant upward force would be the buoyant force exerted on the balloon in air. The mass of the balloon and the friction forces would have to be neglected.

2. The equation of motion

The motion of the fibre is, of course, determined by Newton's law—the rate of change of the momentum p of the moving part of the fibre is due to the resultant force,

$$\dot{p} = F - mg, \tag{1}$$

where $F > 0$ is the vertically upward oriented constant force, and mg is the oppositely exerting weight of the reeled fibre. Supposing x to be the height of the end of the fibre above the plane then $p = m\dot{x}$, and the mass is given by $m = \eta x$ where η is a constant linear density of the fibre (we emphasize that only $x > 0$ has a physical meaning). Therefore, the equation of motion has the form

$$(x\dot{x})^{\cdot} = \frac{F}{\eta} - gx, \tag{2}$$

i.e.,

$$\ddot{x} = \frac{F}{\eta x} - g - \frac{\dot{x}^2}{x}. \tag{3}$$

A stationary solution when x is independent of time represents the *equilibrium* situation such that the end of the fibre is situated at a constant height

$$x_c = \frac{F}{\eta g}. \tag{4}$$

The problem thus seems to be reduced to finding the solution of equation (3). However, this equation only describes the *upward motion* of the fibre—it is only correct for a *growing* x, i.e. when $\dot{x} > 0$. The reason is that only in such a case is the resultant force $F - mg$ responsible for the rate of change of the total momentum of the fibre

$$\dot{p} = m\ddot{x} + \dot{m}\dot{x}. \tag{5}$$

The first term in equation (5) corresponds to a velocity change of the already reeled-off part of the fibre which has the mass $m = \eta x$. The second term expresses the part of the momentum which in the time dt has to be delivered by the resultant force in order to give velocity $\dot{x}$ to a piece of the fibre of mass dm, lying at rest on the plane—in this way the piece 'joins' the moving part of the fibre.

In the case of a *downward oriented motion* ($\dot{x} < 0$, i.e. decreasing x) the situation is *different*. The appropriate rate of change of the momentum, related to $F - mg$, only corresponds to the velocity change of the already reeled-off fibre. The second term $\dot{m}\dot{x}$ in equation (5) represents the rate of change of the momentum of a piece of fibre with mass dm which in time dt lands on the plane. Its velocity has to suddenly decrease from $\dot{x}$ to zero. However, the forces considered above are *not* responsible for this change. Such a change is due to the *interaction between the fibre and the plane* (for example, an ideally inelastic impact of the fibre in which case the appropriate momentum is absorbed by the 'infinitely massive' plane). Under these circumstances the term $\dot{m}\dot{x}$ *must be ignored* when $\dot{x} < 0$. Formally it means that we have to omit the velocity term from the right-hand side of equation (3). For a downward motion of the fibre the equation of motion thus has the form

$$\ddot{x} = \frac{F}{\eta x} - g. \tag{6}$$

Another possible view on the downward motion is the following. One can take into account the interaction between the fibre and the plane by introducing the reaction force of the form $F_r = \eta\dot{x}^2$. This is the third force exerted on the fibre during the downward motion, and it is oriented upward, against the motion. Replacing F by $F + F_r$ in equation (3) we thus naturally obtain equation (6). The work done by the reaction force during the downward motion is responsible for the dissipation of the total energy of the fibre. Let us also note that equation (6), contrary to equation (3), correctly describes the free-fall of the end of the fibre when we switch off the force F (i.e. for $F = 0$), as required by the principle of equivalence.

The solutions of both these equations merge together at the turning points where $\dot{x} = 0$, i.e. at the instants when the fibre is at rest—the 'switching' between the two equations of motion takes place as a result of the change of sign of the velocity. Naturally, both the equations have the same stationary solution x_c given by equation (4). The equations of motion (3) and (6) can thus be rewritten in a *unified form* as

$$\ddot{x} = g\left(\frac{x_c}{x} - 1\right) - \frac{1}{2}(1 + \operatorname{sgn}\dot{x})\frac{\dot{x}^2}{x}. \tag{7}$$

Interestingly, the Buquoy problem is thus described by an equation of motion which depends in a very specific way on the velocity.

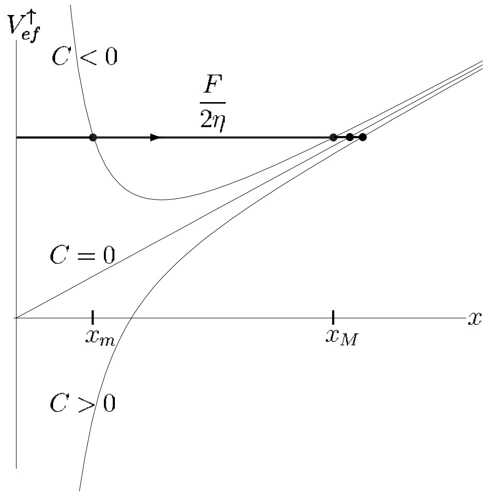

Figure 2. Three possible forms of the potential $V_{\mathrm{ef}}^{\uparrow}(x)$ for the upward motion.

3. Upward motion

To look for solutions for $\dot{x} > 0$ we may employ equation (2). For a non-trivial situation the term $x\dot{x}$ is non-zero, so that we can multiply both sides by $x\dot{x}$ and integrate to obtain

$$\frac{1}{2}(x\dot{x})^2 = \frac{F}{2\eta}x^2 - \frac{g}{3}x^3 + C, \tag{8}$$

where C is an integration constant. Its value is determined by the initial conditions x_0, $\dot{x}_0$ as

$$C = \frac{1}{2}(x_0\dot{x}_0)^2 - \frac{F}{2\eta}x_0^2 + \frac{g}{3}x_0^3. \tag{9}$$

Equation (8) can be rewritten in the form

$$\frac{1}{2}\dot{x}^2 + V_{\mathrm{ef}}^{\uparrow}(x) = \frac{F}{2\eta}, \qquad V_{\mathrm{ef}}^{\uparrow}(x) \equiv \frac{g}{3}x - \frac{C}{x^2}, \tag{10}$$

which may be formally interpreted as a conservation law of total mechanical energy (having the value $F/2\eta$), corresponding to the motion of a fictitious particle with unit mass in the force field with the effective potential $V_{\mathrm{ef}}^{\uparrow}(x)$. In fact, by substituting from (10) into (3) we obtain the equation

$$\ddot{x} = -\frac{g}{3} - \frac{2C}{x^3} \equiv F_{\mathrm{ef}}(x). \tag{11}$$

Possible forms of the potential $V_{\mathrm{ef}}^{\uparrow}(x)$ depending on the constant C are illustrated in figure 2. The value and the sign of C are determined by the initial conditions according to (9).

For $C \geqslant 0$ we observe from (11) that $F_{\mathrm{ef}} < 0$ and the initial condition must necessarily be $\dot{x}_0 > 0$. The initial condition $\dot{x}_0 = 0$ is only possible when $F_{\mathrm{ef}} \geqslant 0$, in which case $C \leqslant -\frac{g}{6}x_0^3 < 0$. For any value of the exerted force F during the upward motion there exists a *maximal height* x_{M} which is determined by the condition $V_{\mathrm{ef}}^{\uparrow}(x_{\mathrm{M}}) = \frac{F}{2\eta}$, i.e. given by the solution of cubic equation

$$x^3 - \frac{3}{2}x_c x^2 - \frac{3}{g}C = 0, \tag{12}$$

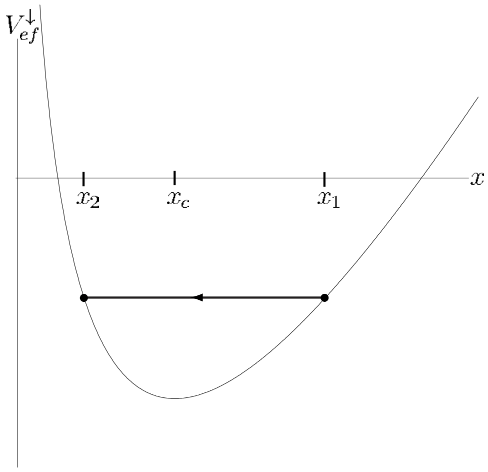

Figure 3. The potential $V_{\mathrm{ef}}^{\downarrow}(x)$ for the downward motion.

see equation (10). After reaching the turning point x_{M} the fibre necessarily starts moving downward. For $C < 0$ the potential has a form of a potential well. Thus, there generically exist *two* solutions of equation (12)—besides the maximal height x_{M} there is also the 'initial point' x_m (the starting point for the upward motion with zero initial velocity). In such a case it is possible to express the maximal height x_{M} explicitly in terms of the value x_m. By dividing (12) by the factor $x - x_m$ we obtain a quadratic equation for $x_{\mathrm{M}} > x_c > x_m$ which has a unique physical solution

$$x_{\mathrm{M}} = \tfrac{1}{2}\left(\tfrac{3}{2}x_c - x_m\right) + \tfrac{1}{2}\sqrt{\left(\tfrac{3}{2}x_c + x_m\right)^2 - 4x_m^2}. \tag{13}$$

Note also that for large values of x such that $x^3 \gg \frac{3}{g}|C|$ the effective potential (10) is approximately $V_{\mathrm{ef}}^{\uparrow}(x) \approx \frac{g}{3}x$, so that *independently of* C the maximal height is $x_{\mathrm{M}} \approx \frac{3}{2}x_c$.

4. Downward motion

By integrating equation (6) we obtain the effective potential for the downward motion,

$$\frac{1}{2}\dot{x}^2 + V_{\mathrm{ef}}^{\downarrow}(x) = V_{\mathrm{ef}}^{\downarrow}(x_1), \qquad V_{\mathrm{ef}}^{\downarrow}(x) = gx - \frac{F}{\eta}\ln x, \tag{14}$$

which is shown in figure 3. It has the form of a potential well with minimum at x_c corresponding to the stationary solution. As indicated in figure 3, for every potential level higher than $V_{\mathrm{ef}}^{\downarrow}(x_c)$ there exist *two turning points* such that $0 < x_2 < x_c < x_1$. The coordinates of these points fulfil the condition

$$x_1 - x_2 = x_c \ln(x_1/x_2). \tag{15}$$

An important observation is that necessarily $x_2 > 0$, and thus the upper end of the *fibre can never fall down back on the plane* (see also the form of the effective reaction force F_{r} introduced above).

If the downward motion follows the upward motion, the turning point x_1 is identical to the maximal value x_{M} introduced in the previous section. And similarly, the turning point x_2 becomes a *new* minimal value x_m for a subsequent upward motion.

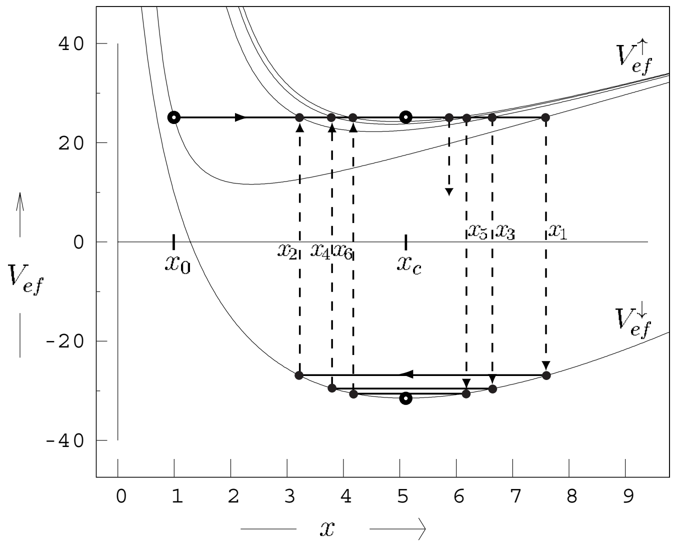

Figure 4. The sequence of the potentials $V_{\rm ef}^{\uparrow}(x)$ and $V_{\rm ef}^{\downarrow}(x)$. The sequence of the turning points $x_1, x_2, x_3, \ldots$ (see the text) approaches the stationary position x_c indicated by the circles at the very bottom of the potential wells. The values of the parameters are $\eta = 2 \times 10^{-3}$ kg m^{-1}, $F = 0.1$ N, $g = 9.8$ m s^{-2}, $x_0 = 1$ m, $\dot{x}_0 = 0$.

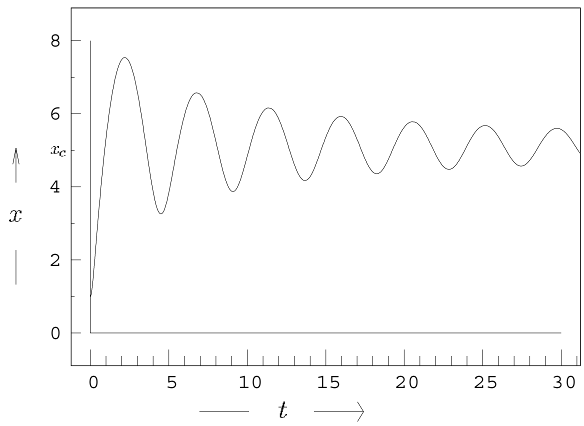

Figure 5. The complete motion of the Buquoy's fibre clearly exhibits damped oscillations around the stationary point x_c.

5. The complete motion

To describe the complete motion of the fibre we thus have to alternately join solutions of equation (10) valid for $\dot{x} > 0$ with solutions of (14) for $\dot{x} < 0$. The merging takes place at the corresponding turning points $\dot{x} = 0$, i.e. at the instants when the function $x(t)$ reaches its local maxima or minima, respectively. In figures 4 and 5 this procedure is explicitly demonstrated (supposing the initial conditions $x_0 > 0$, $\dot{x}_0 = 0$, such that $C < 0$ and $x_0 = x_m$) by calculating the sequence of the turning points $x_1, x_2, x_3, \ldots$ for subsequent upward and downward motions of the fibre.

Table 1. The times t of the heights x of the sequence of the turning points and passages through the (asymptotic) equilibrium point $x_c = 5.102$, and the corresponding amplitudes $\xi = x - x_c$. We also present the time intervals $T^{\uparrow}$ and $T^{\downarrow}$ between the two nearest passages through $x = x_c$ in the cases of the upward and downward motions separately (for example, $T^{\uparrow} = 4.573$ is given by the difference of $t = 5.514$ and $t = 0.941$). The periods $T^{\uparrow}$ decrease whereas $T^{\downarrow}$ increase to the value $T_s = 4.534$.

t	x	ξ	$T^{\uparrow}$	$T^{\downarrow}$
0	1	−4.102	–	–
0.941	x_c	0	–	–
2.186	7.536	2.434	–	–
3.528	x_c	0	4.573	–
4.470	3.259	−1.843	–	–
5.514	x_c	0	–	4.451
6.710	6.573	1.471	–	–
7.979	x_c	0	4.549	–
8.994	3.869	−1.233	–	–
10.063	x_c	0	–	4.496
11.245	6.161	1.059	–	–
12.475	x_c	0	4.543	–
13.518	4.172	−0.930	–	–
14.606	x_c	0	–	4.511
15.780	5.930	0.828	–	–
16.986	x_c	0	–	–

The procedure is the following. From the initial conditions we determine the potential $V_{\mathrm{ef}}^{\uparrow}(x)$ using (9) and (10). The first turning point $x_1 = x_{\mathrm{M}} > x_0$ is determined by $V_{\mathrm{ef}}^{\uparrow}(x_1) = \frac{F}{2\eta}$, or explicitly by (13). From (14) we then evaluate $V_{\mathrm{ef}}^{\downarrow}(x_1)$ and determine the second turning point for the downward motion $x_2 < x_1$ from the condition $V_{\mathrm{ef}}^{\downarrow}(x_2) = V_{\mathrm{ef}}^{\downarrow}(x_1)$, or using equation (15). The value x_2 plays the role of the new initial position x_m with zero velocity for the next upward motion. We thus determine the *new potential* $V_{\mathrm{ef}}^{\uparrow}(x)$ corresponding to a new value of C, and from $V_{\mathrm{ef}}^{\uparrow}(x_3) = \frac{F}{2\eta}$ we find the coordinate of the next upper turning point $x_3 = x_{\mathrm{M}} > x_2$. Then we find $V_{\mathrm{ef}}^{\downarrow}(x_3)$ and determine the following turning point for the downward motion $x_4 < x_3$ from the condition $V_{\mathrm{ef}}^{\downarrow}(x_4) = V_{\mathrm{ef}}^{\downarrow}(x_3)$, etc. As can be seen in figure 4, this sequence of turning points oscillates around the stationary position x_c, with the amplitude of such oscillations decreasing—the motion of the fibre thus exhibits *damped quasi-periodic oscillations*.

The harmonic nature of small oscillations around x_c can be demonstrated from the equations of motion (6) and (11) in the case of small amplitudes $\xi = x - x_c$ when $\xi \ll x_c = \frac{F}{\eta g}$. Neglecting higher terms, *both* equations take the form

$$\ddot{\xi} + \frac{\eta g^2}{F}\xi = 0. \tag{16}$$

This means that small oscillations are close to harmonic ones with period $T_s = \frac{2\pi}{g}\sqrt{\frac{F}{\eta}}$. It can also be shown that the time interval $T^{\uparrow}$ between two subsequent passages through $x = x_c$ in the case of the upward motion *decreases* with time, while the analogous time interval $T^{\downarrow}$ for the downward motion *increases* with time, see table 1. In fact, $\lim_{t\to\infty} T^{\uparrow} = \lim_{t\to\infty} T^{\downarrow} = T_s$.

The plot in figure 5 which displays the complete motion $x(t)$ was obtained numerically by solving equation (7) using the Runge–Kutta method (the appropriate upward and downward solutions being 'switched' at the turning points where the velocity changes its sign). The choice of the parameters is the same as in figure 4. In table 1 we also present the numerical values of the times t at which the turning points x_i, $i = 0, 1, 2, \ldots$, and x_c are reached, and the corresponding amplitudes $\xi = x_i - x_c$. The damping is obvious from the decreasing moduli of amplitude ξ. Moreover, the time intervals $T^{\uparrow}$ for the upward motion slowly decrease with time whereas the periods for the downward motion $T^{\downarrow}$ increase.

Except for very special initial conditions it is difficult to obtain analytic solutions to the equation of motion (7). An interesting particular upward motion that is explicitly solvable occurs when $C = 0$, in which case equation (11) admits a solution of the form

$$x = \frac{3}{2g}\left(\frac{F}{\eta} - \dot{x}_0^2\right) + \dot{x}_0\, t - \frac{g}{6}t^2, \tag{17}$$

with the initial conditions $x_0 = \frac{3}{2g}\left(\frac{F}{\eta} - \dot{x}_0^2\right)$ and $0 < \dot{x}_0 \leqslant \sqrt{\frac{F}{\eta}}$. The fibre moves upward with a constant deceleration $-\frac{g}{3}$. For the 'limiting' initial conditions $x_0 = 0$ and $\dot{x}_0 = \sqrt{\frac{F}{\eta}}$, an extraordinary situation happens—exerting of a finite constant force F on the 'zero' mass of the reeled fibre at $x = 0$ having a non-zero velocity results (in homogeneous gravitational field g) in a 'free-fall' motion with deceleration $-\frac{g}{3}$. The fibre reaches its maximal height $x_{\mathrm{M}} = \frac{3}{2}x_c$ at $t_M = \frac{3}{g}\sqrt{\frac{F}{\eta}}$. When calculating the time interval between t_M and the previous passage through $x = x_c$, we obtain just a quarter of a period of hypothetical 'parabolic' oscillations $T_p^{\uparrow} = \frac{4\sqrt{3}}{g}\sqrt{\frac{F}{\eta}}$, which is naturally higher than T_s.

6. Discussion

A physical interpretation of the complete solution to the Buquoy problem is obvious. It follows from the analysis presented above that the damped motion of the fibre results from the dissipative processes (of momentum and energy) during landing of the fibre on the plane. These processes have a one-way character since the mechanical energy is dissipated into the more inaccessible internal energy. From the point of view of effective potentials describing the motion of a virtual particle with a unit mass in a force field, the dissipation of energy is due to the changes of the potential for the upward motion after reaching the bottom turning point. In terms of the reaction force, it is the work done by this force during the downward motion of the fibre.

Various generalizations of the problem are clearly possible, for example by relaxing the assumed conditions or by replacing the model of a continuous fibre by a system of discrete elements (such as beads, chain links, people entering and leaving a paternoster elevator, etc). In all such cases it would, however, be necessary to carefully specify other presumptions, and the description of the system would go beyond the formalism of a one-dimensional motion of a fictitious particle which was sufficient in our analysis.

Another non-trivial situation would also appear in the case when the vector of the force F is not perpendicular to the horizontal plane (representing, e.g., a general motion of a rocket, pulling an optical fibre used for its remote control). In such a case the fibre sag necessarily arises, and a part of the fibre would also be dragged along the plane. The horizontal velocity component would thus decrease with time but it could not change its sign. On the other hand, concerning the height of the pulled fibre above the horizontal plane, similar oscillations as in Buquoy's problem discussed here can be expected due to the dissipation of the vertical momentum.

7. Conclusions

We presented the solution of a simple yet interesting problem of a one-dimensional oscillating motion of an open system with varying mass in the field of constant exerted force plus the gravitational force. It was shown that the energy dissipation in the system which is responsible for damped oscillations arises from a specific velocity dependence of the effective force. Small oscillations of the system around the equilibrium point are close to harmonic ones. A change of the effective potential corresponding to the exerted force arises due to the change of the velocity sign, so that in general the potential depends on the coordinate of the last turning point.

Acknowledgments

We wish to thank Jan Slavík for bringing our attention to Buquoy's problem, and Jerry Griffiths for reading the manuscript.

References

[1] von Buquoy G 1812 *Analytische Bestimmung des Gesetzes der virtuellen Geschwindigkeiten in mechanischer und statischer Hinsicht* (Leipzig: Breitkopf und Härtel)
[2] von Buquoy G 1814 *Weitere Entwickelung und Anwendung des Gesetzes der virtuellen Geschwindigkeiten in mechanischer und statischer Hinsicht* (Leipzig: Breitkopf und Härtel)
[3] von Buquoy G 1815 *Exposition d'un nouveau principe général de dynamique, dont le principe des vitesses virtuelles n'est qu'un cas particulier* (Paris: V Courcier)
[4] Hirsch E 1975 *Graf Georg Buquoy: ein vergessener Goetheanist* (Wien: Österreichische Nationalbibliothek) pp 19–34
[5] Poisson S D 1819 Sur le mouvement d'un système de corps, en supposant les masses variables *Bull. Sci. Soc. Philomat. Paris* (avril) 60–2
[6] Tait P G and Steele W L 1856 *A Treatise on Dynamics of a Particle* (London: Macmillan)
[7] Mescerskij I V 1897 *Dinamika tochki peremennoj massy* (St Petersburg: Akademia Nauk, Peterburskij Universitet)
[8] Levi-Civita T 1930 Ancora sul moto di un corpo di massa variabile *Rend. Accad. Lincei* **11** 626–32
[9] Mikhailov G K 1974 K istorii dinamiki sistem peremennogo sostava i teorii reaktivnogo dvizhenia (do nachala vtoroj mirovoj vojny) *Prepr. Instituta probl. mechaniki AN SSSR* (49) 106
[10] Mikhailov G K 1975 K istorii dinamiki sistem peremennogo sostava *Mechanika tverdogo tela, Izv. AN SSSR* (5) 41–51
[11] Mikhailov G K 1986 Georg Bukua i nachala dinamiki sistem s peremennymi massamy *Issledovanija po Istorii Fiziki i Mechaniki* (Moskva: Nauka) pp 191–238
[12] Panovko J G 1985 *Mechanika deformirujemogo tverdogo tela: sovremennyje koncepcii, oshibki i paradoksy* (Moskva: Fizmatgiz, Nauka) pp 173–8
[13] Slavík J 1994 *Teoretická mechanika* (Plzen: Západoceská Univerzita) pp 22–3