Pushing the Limits of General Relativity:

Curved Space-Time Quantum Field Theory, Analogues, and Exotica

Sebastian Schuster

together with **Ana Alonso Serrano**, **Pavel Krtouš**, **Christian Pfeifer**, **Jessica Santiago**, **Matt Visser**, [. . .]

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Matematicko-Fyzikální Fakulta
Univerzita Karlova

16$^{\text{th}}$ July 2021, Physikalisches Kolloquium, Bonn

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YOUTH AND SPORTS

Outline

Introduction: Notation & Context

My Troubles to Come

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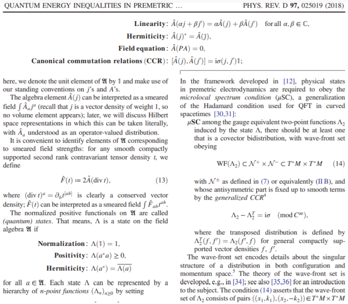
QUANTUM ENERGY INEQUALITIES IN PREMETRIC … PHYS. REV. D **97**, 025019 (2018)

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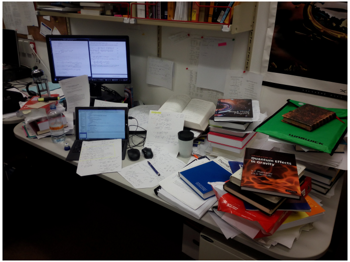

A 'Where's Waldo' for bibliophile physicists. . .

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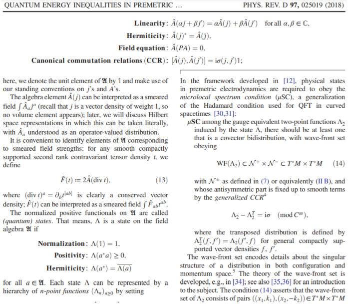

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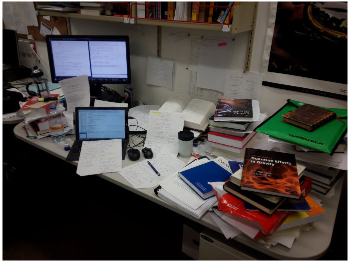

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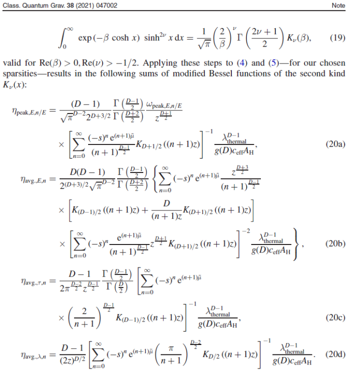

Class. Quantum Grav. **38** (2021) 047002 Note

$$\int_0^\infty \exp(-\beta\cosh x)\,\sinh^{2\nu} x\,\mathrm{d}x = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\beta}\right)^{\nu}\Gamma\left(\frac{2\nu+1}{2}\right)K_\nu(\beta), \tag{19}$$

valid for $\mathrm{Re}(\beta)>0, \mathrm{Re}(\nu)>-1/2$. Applying these steps to (4) and (5)—for our chosen sparsities—results in the following sums of modified Bessel functions of the second kind $K_\nu(x)$:

$$\eta_{\text{peak},E,n/E} = \frac{(D-1)}{\sqrt{\pi}^{D-2}2^{D+3/2}}\frac{\Gamma\left(\frac{D-1}{2}\right)}{\Gamma\left(\frac{D+2}{2}\right)}\frac{\omega_{\text{peak},E,n/E}}{z^{\frac{D+1}{2}}} \times\left[\sum_{n=0}^{\infty}\frac{(-s)^n\,\mathrm{e}^{(n+1)\tilde{\mu}}}{(n+1)^{\frac{D-1}{2}}}K_{D+1/2}((n+1)z)\right]^{-1}\frac{\lambda_{\text{thermal}}^{D-1}}{g(D)c_{\text{eff}}A_{\text{H}}}, \tag{20a}$$

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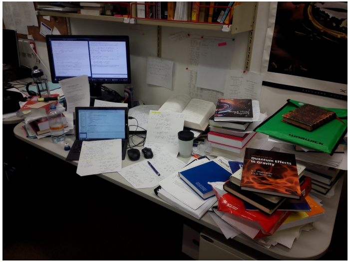

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Let's see how it goes—I'll aim for pictures! ;) (angel)

Conventions

Warning! Mostly for later convenience

Signature: $-+++$ (more in a moment!)

Units: $G = c = \hbar = 1$ (often)

Space-time indices: $abcd \cdots \in \{0, 1, 2, 3\}$

Spatial indices: $ijkl \cdots \in \{1, 2, 3\}$

Hatted indices: 'Frame indices' (undoes curvilinear messes—somewhat)

Einstein sum convention: Same index up, same index down = sum over this index

- Slides err on the side of verbosity in case of connection problems/distractions

Introduction:
General Relativity in Two Slides

Special Relativity

Special relativity:

- Distinguish past and present by the speed of light:
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- $\Longrightarrow$ Relativity of simultaneity,
  Lorentz boosts instead of Galileo 'boosts'

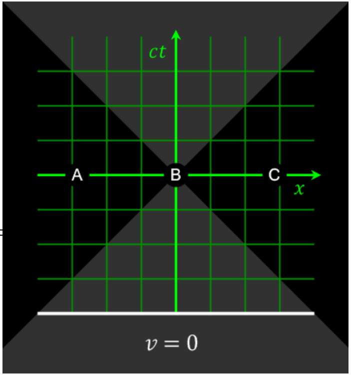

Image source: https://commons.wikimedia.org/wiki/File:Relativity_of_Simultaneity_Animation.gif

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- This only *looks* simple. It's only quasi-linear, and a coupled system for the ten components of g_{ab} with 2 physical d.o.f.

General Relativity [GR] Done Quick

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- The metric becomes a function of the space-time coordinates
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- Here it is as a PDE:

$\frac{1}{2}\partial_c g^{cf}[\partial_a g_{bf} + \partial_b g_{af} - \partial_f g_{ab}] - \frac{1}{2}\partial_b g^{cf}[\partial_a g_{cf}] + \frac{1}{4}g^{cg}[\partial_m g_{cg} + \partial_c g_{mg} - \partial_g g_{mc}]g^{mf}[\partial_a g_{bf} + \partial_b g_{af} - \partial_f g_{ab}] - \frac{1}{4}g^{cg}[\partial_m g_{bg} + \partial_b g_{mg} - \partial_g g_{mb}]g^{mf}[\partial_a g_{cf} + \partial_c g_{af} - \partial_f g_{ac}] - \frac{1}{2}g_{ab}g^{de}(\frac{1}{2}\partial_c g^{cf}[\partial_e g_{df} + \partial_d g_{ef} - \partial_f g_{ed}] - \frac{1}{2}\partial_d g^{cf}[\partial_e g_{cf} + \partial_c g_{ef} - \partial_f g_{ec}] + \frac{1}{4}g^{cf}[\partial_m g_{cf} + \partial_c g_{mf} - \partial_f g_{mc}]g^{mg}[\partial_e g_{dg} + \partial_d g_{eg} - \partial_g g_{ed}] - \frac{1}{4}g^{cf}[\partial_m g_{df} + \partial_d g_{mf} - \partial_f g_{md}]g^{mg}[\partial_e g_{cg} + \partial_c g_{eg} - \partial_g g_{ec}]) + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}$

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- This only *looks* simple. It's only quasi-linear, and a coupled system for the ten components of g_{ab} with 2 physical d.o.f.
- A moment of silence for numerical relativists. They need to discretize this. And then code the discretization. . .

Introduction:
Physics of GR

A Small Feature List of GR

- New features! Black holes! Gravitational waves! Big bangs!

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Schwarzschild:	Spherically symmetric, mass
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Chandrasekhar on the Kerr metric, [Cha83, p.529]

The treatment of the perturbations of the Kerr space-time in this chapter has been prolixious in its complexity. Perhaps, at a later time, the complexity will be unravelled by deeper insights. But mean time, the analysis has led us into a realm of the rococo: splendorous, joyful, and immensely ornate.

We Earned Us Some Illustration!

~~Shadow~~ Silhouette of Black Hole M87

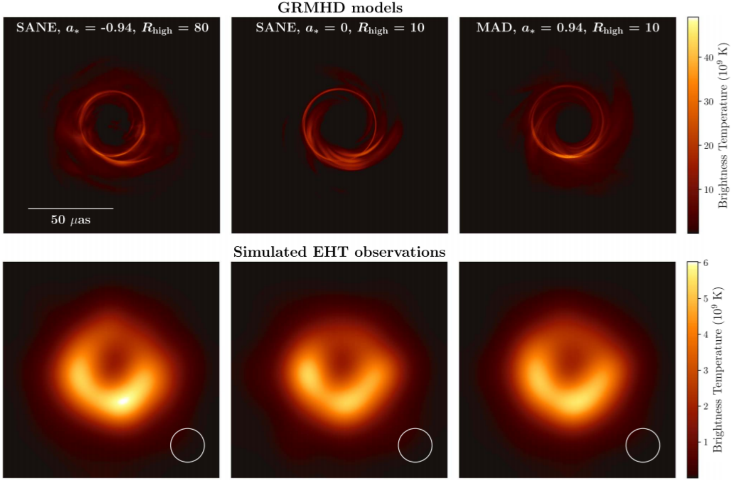

Image source: arXiv:1906.11238

We Earned Us Some Illustration!

Tipping Light-Cone Example (More Later)

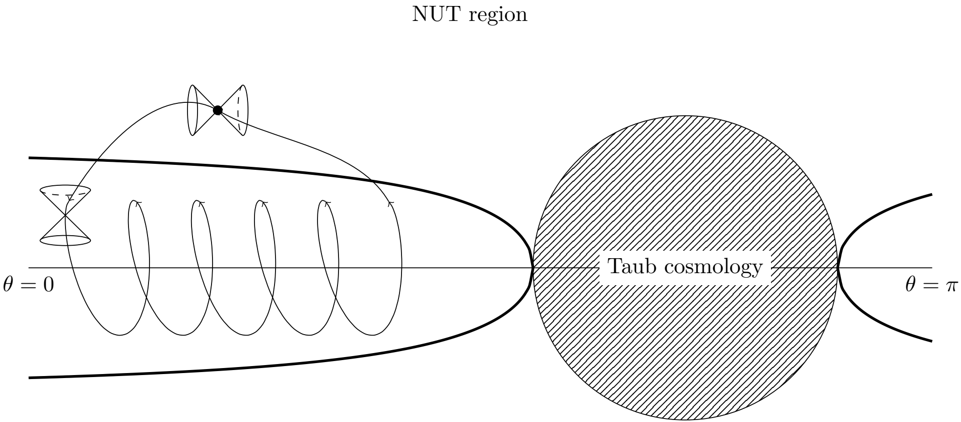

Image source: arXiv:1610.06135

We Earned Us Some Illustration!

Hollywood's visualization

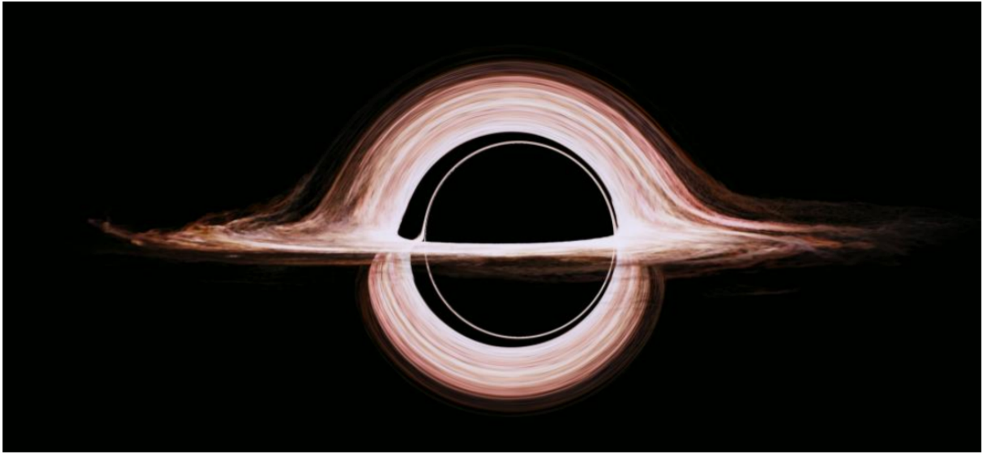

Image source: arXiv:1502.03808

We Earned Us Some Illustration!

Hollywood's visualization w/ grav. red/blueshift, Doppler shift, brightness as in Liouville's law

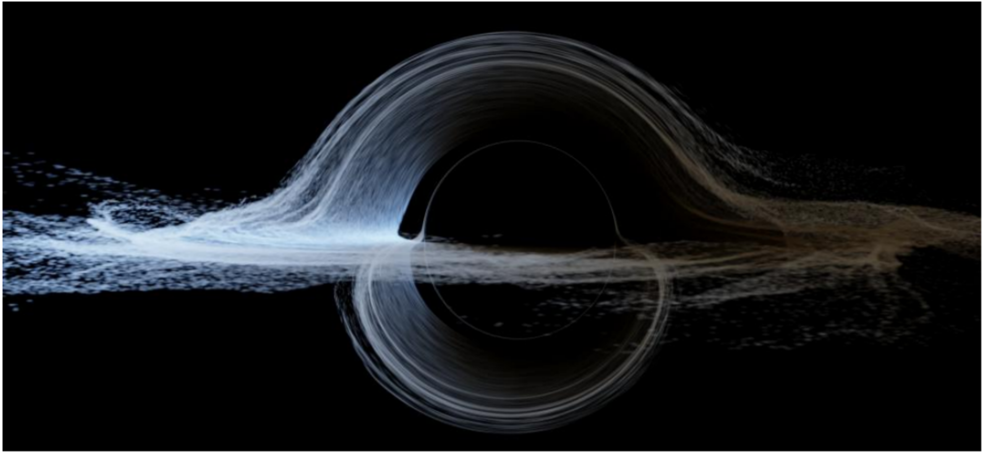

Image source: arXiv:1502.03808

Whence and Whither

- For various reasons, we expect things to be quite different at energies $\simeq 1.9561 \times 10^9\,\mathrm{J} = \sqrt{\frac{\hbar c^5}{G}} = E_{\text{Planck}}$ '$= 1$'
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  - This is just the tip of the iceberg
- Best hint so far? The Hawking effect.

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Quantum Field Theory: Flat, Curved, and In-Between

QFT—'All a physicists needs is a harmonic oscillator.'

- At every event a harm. osci., Fourier expansion of a field $\hat{\varphi}(x^a)$ using ladder operators $\hat{a}_{\mathbf{k}}$:

$$[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \qquad [\hat{a}^{\dagger}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0$$

- Vacuum $|0\rangle$ a Lorentz-invariant state of 'no particles', *i.e.*:

$$\hat{a}_{\mathbf{k}} |0\rangle = 0$$

- Multi-particle states:

$$\hat{a}^{\dagger}_{\mathbf{k}} |\ldots; N_{\mathbf{k}}; \ldots\rangle = \sqrt{N_{\mathbf{k}} + 1} \, |\ldots; N_{\mathbf{k}} + 1; \ldots\rangle \,,$$
$$\hat{a}_{\mathbf{k}} |\ldots; N_{\mathbf{k}}; \ldots\rangle = \sqrt{N_{\mathbf{k}}} \, |\ldots; N_{\mathbf{k}} - 1; \ldots\rangle$$

Summary: QFT for Particle Physics [Sch16a; PS95; Wei05a; Wei05b; Wei05c; Sre11]

$\underbrace{\text{Successful}}_{[\ldots]}$ $\underbrace{\text{relativistic}}_{\textit{Special}\text{ Relativity}}$ $\underbrace{\text{many-particle quantum physics}}_{\text{A Fock space for each particle type}}$ $\underbrace{\text{in inertial frames}}_{\text{SR, } \textit{i.e.}\text{, Lorentz frames}}$

Curved Space-Time QFT

- The unique vacuum needed its invariance under Poincaré transformations

3

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- Connected by Bogoliubov transformation:

$$\hat{a}'_i = \sum_j \alpha_{ji}\hat{a}_j + \beta^*_{ji}\hat{a}^\dagger_j \qquad \Longleftrightarrow \qquad \hat{a}_i = \sum_j \alpha^*_{ij}\hat{a}'_j - \beta^*_{ij}\hat{a}'^\dagger_j$$

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- In particular:

$$\left\langle 0' \middle| \hat{N}_i \middle| 0' \right\rangle = \sum_j |\beta_{ji}|^2, \qquad \text{where } \hat{N}_i := \hat{a}^\dagger_i \hat{a}_i$$

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Curved Space-Time QFT without Curvature: The Unruh Effect

- Normally, Minkowski has time translations generated by

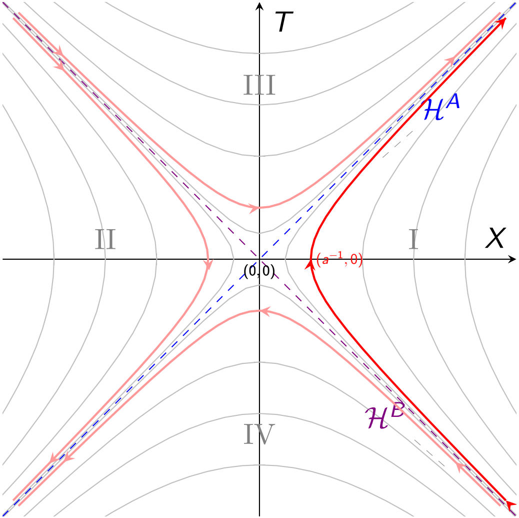
$$(\partial/\partial T)^a$$

- Take Minkowski space with different time-like (Killing) vector

$$b^a := a\left[X\left(\frac{\partial}{\partial T}\right)^a + T\left(\frac{\partial}{\partial X}\right)^a\right],$$

- This still defines time translations, but differently
- It also defines constant accelerated motion
- After long calculation [Tak86; Wal94; MW10], one gets

$$_{\text{Minkowski}}\langle 0|\hat{N}_i|0\rangle_{\text{Minkowski}} = \frac{1}{e^{\omega_i/T_{\text{Unruh}}} - 1}$$



Hawking Effect

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4

5

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Hawking Effect

Image source: https://web.archive.org/web/20130626081937/https://www.st-andrews.ac.uk/~ulf/fibre.html

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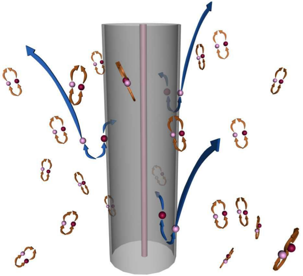

[4] arXiv:gr-qc/9406042, arXiv:gr-qc/9408003, arXiv:hep-th/9907001

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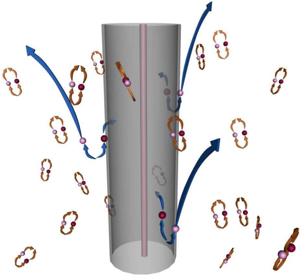
- $\Longrightarrow$ A static observer far away will see a freely falling observer's vacuum at the horizon at a temperature[5]



$$T_{\text{Hawking}} = \frac{\kappa}{2\pi} = \frac{1}{8\pi M} \frac{\hbar c^3}{G k_{\text{B}}}$$

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[5] A precise formulation takes much more technical effort.

Black Hole Thermodynamics

- Even before Hawking's result[6], similarities between thermodynamics and black holes were found[7]

[6] doi:10.1007/BF02345020
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  - 0th Surface gravity κ labels temperature, is constant across horizons
  - 1st $\mathrm{d}E = \frac{\kappa}{8\pi}\,\mathrm{d}A + \Omega\,\mathrm{d}J + \Phi\,\mathrm{d}Q$
  - 2nd $\frac{\mathrm{d}A_{\text{horizon}}}{\mathrm{d}t} \geq 0$ (Warning! Requires Einstein equation), $A \propto S_{\text{Bekenstein}}$
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- Warning! Even temperature is tricky to define in GR [8]

---

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[8] arXiv:1807.02915, arXiv:1805.05583, [RZ13]

Some 'Numbers'

- Unruh temperature:

$$T_{\text{Unruh}} = \frac{a}{2\pi}\frac{\hbar}{ck_{\text{B}}} \approx a\frac{1\,\mathrm{K}}{2.47 \times 10^{20}\,\mathrm{m\,s^{-2}}}$$

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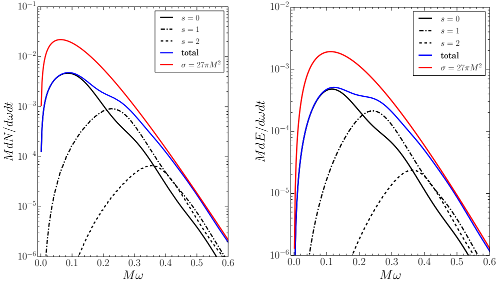
- Alas, all quite inaccessible

Usually ignored:

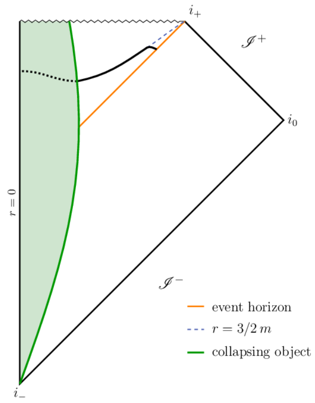
- UV cutoff/Backreaction: No particles emitted with $m_{\text{particle}} \geq M_{\text{black hole}}$
- Backscattering: Ignore gray body factors
- Adiabaticity/Backreaction: Shouldn't be *too* dynamical
- $A \ll \lambda_{\text{thermal}}$ or $\tau_{\text{emission gap}} \ll \tau_{\text{'thermal oscillation'}} \longrightarrow$ sparsity, different from 'normal' black body radiation[9]



[9] arXiv:1506.03975, arXiv:1512.05018 (image source)

Trouble

Purely classical, no CSTQFT

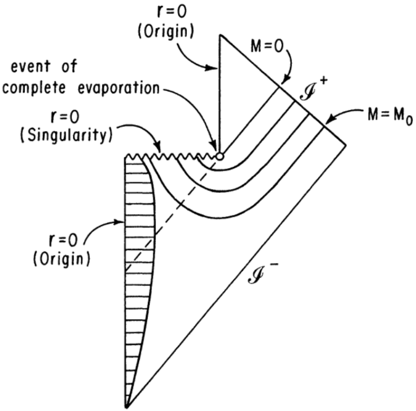


Source: arXiv:1607.07222

For more zoology, see: arXiv:2102.01105 and arXiv:1911.11200

Trouble

Formation of Cauchy horizon

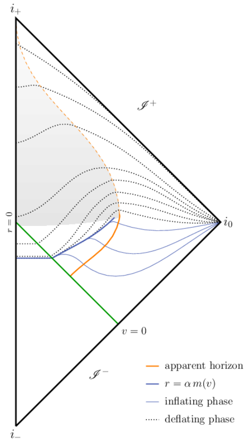


Source: [Wal94, p.178]

For more zoology, see: arXiv:2102.01105 and arXiv:1911.11200

Trouble

Only apparent horizons



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So: What can we do with just a metric besides astrophysical (C)ST(Q)FT?

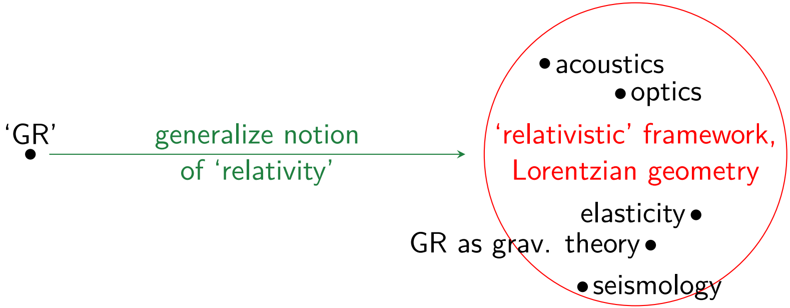
Analogues

The Physics of Metrics

Even classically, wave propagation on curved backgrounds is encountered elsewhere:

Image source: arXiv:1203.3018

The Analogue Space-Time Framework[10]

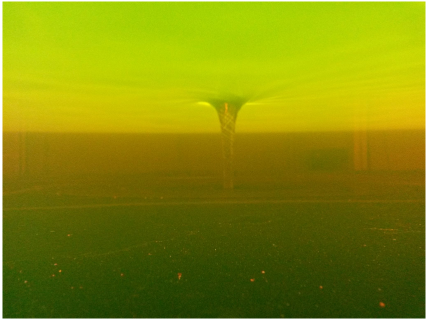


Relativity not just a ‘theory’ — it’s a framework!

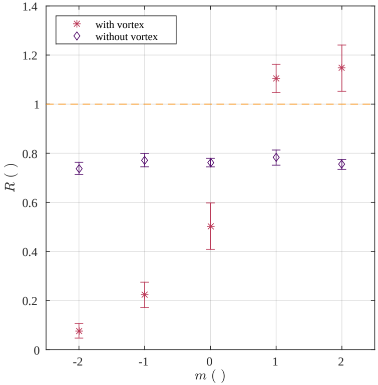
[10] arXiv:gr-qc/0201053

Current Progress—Superradiance

Nottingham Group, Surface Waves in Water[11]



Super-radiance:[12]



[11] Left picture by Jessica Santiago

[12] Right picture source, arXiv:1612.06180,

Current Progress—More Developments

- **Warning!** References might be skewed, biased, out-of-date, (very) incomplete!

---

a

b

c

d

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  - Transformation optics—'cloaks'; **Warning!** Not an analogue space-time in the strictest sense, 'analogue transformation' [b]

[a] arXiv:1710.05800, arXiv:gr-qc/0011026

[b] doi:10.1007/978-3-319-00266-8_10, arXiv:1006.3118, arXiv:1010.1587

[c]

[d]

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- Theoretical developments include
  - All of the above[c]
  - Graphene[d]
  - Sound waves
  - Superfluids

[a] arXiv:1710.05800, arXiv:gr-qc/0011026
[b] doi:10.1007/978-3-319-00266-8_10, arXiv:1006.3118, arXiv:1010.1587
[c] Me, me, me! Electromagnetic media! arXiv:1706.06280 arXiv:1808.07987
[d] arXiv:0709.1163, arXiv:1412.4554

Current Progress—More Developments

- **Warning!** References might be skewed, biased, out-of-date, (very) incomplete!
- Examples for analogues mostly measured in
  - Surface waves
  - Bose–Einstein condensates[a]
  - Electromagnetic media (*e.g.*, shock front of strong laser in medium changing its refractive index)
  - Transformation optics—'cloaks'; **Warning!** Not an analogue space-time in the strictest sense, 'analogue transformation' [b]
- Theoretical developments include
  - All of the above[c]
  - Graphene[d]
  - Sound waves
  - Superfluids
- Starting point for more: arXiv:gr-qc/0505065, 'Living Reviews in Relativity: Analogue Gravity'

[a] arXiv:1710.05800, arXiv:gr-qc/0011026
[b] doi:10.1007/978-3-319-00266-8_10, arXiv:1006.3118, arXiv:1010.1587
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Exotica

A Step Back—What Metrics Do We Care About?

Usually:

- Integrate the Einstein equation for a given source stress-energy T_{ab}
- Interpret it, work with it

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Instead—Metric Engineering:

- Pick a metric you want
- Calculate its stress-tensor by differentiating, test if T_{ab} sensible
- Bonus: Can reinterpret many alternative/modified theories of gravity's field equations like this:

$$G_{ab} = 8\pi T^{\text{eff}}_{ab}$$

[13]Einstein believed no solution would be found (soon/at all). Schwarzschild was better. [Sch16b]

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Let's have a look at pathological/weird space-times—'exotica'!

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Exotica:
Classics

Classic Exotica I: Gödel & Taub–NUT

- Gödel (1949):[14] GR doesn't fulfil Mach's principle. Proof: His Universe.
  - Homogeneous
  - Base manifold $\mathbb{R}^4$
  - At every point rotating about an axis

[14] doi:10.1103/RevModPhys.21.447
For more, see [GP12; Ste+03; HE74]

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Since, furthermore, R is a constant, the relativistic field equations (with the x_0-lines as world lines of matter), i.e., the equations[8]

$$R_{ik}-\tfrac{1}{2}g_{ik}R=8\pi\kappa\rho u_i u_k+\lambda g_{ik}$$

are satisfied (for a given value of ρ), if we put

$$1/a^2=8\pi\kappa\rho, \quad \lambda=-R/2=-1/2a^2=-4\pi\kappa\rho.$$

Image source: [Göd49, p.448]

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  - Base manifold $\mathbb{R}^4$
  - At every point rotating about an axis
  - An early example of metric engineering
  - Closed time-like curves (CTCs) everywhere

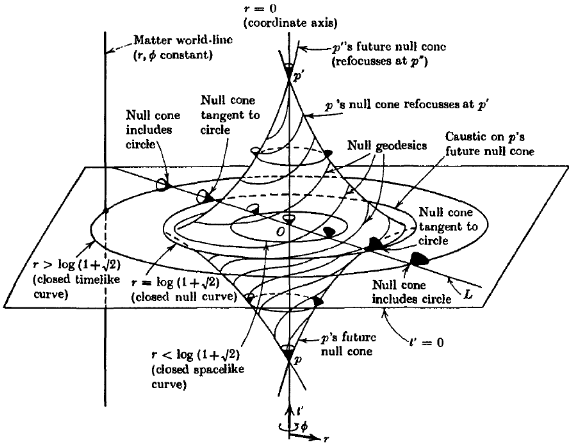

FIGURE 31. Gödel's universe with the irrelevant coordinate z suppressed. The space is rotationally symmetric about any point; the diagram represents correctly the rotational symmetry about the axis $r = 0$, and the time invariance. The light cone opens out and tips over as r increases (see line L) resulting in closed timelike curves. The diagram does not correctly represent the fact that *all* points are in fact equivalent.

[14] doi:10.1103/RevModPhys.21.447
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Classic Exotica I: Gödel & Taub–NUT

- Taub–NUT:
  - Vacuum(!) solution with
    - NUT parameter
    - Mass
    - Discrete $\varepsilon \in \{0, \pm 1\}$
  - 2 BHs connected by cosmology
  - BHs have string deficit, not asymptotically flat
  - *No* curvature singularities!

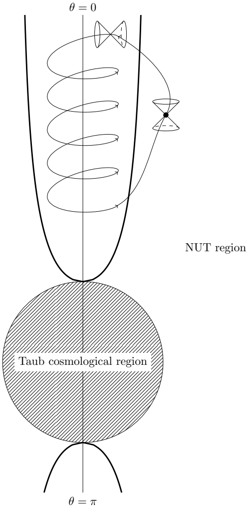

Image source: arXiv:1610.06135

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Classic Exotica II: Wormholes & Warp Drives

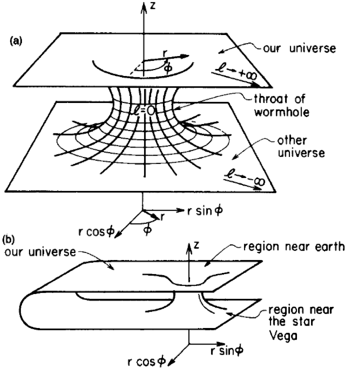
- Morris & Thorne, doi:10.1119/1.15620 and Morris, Thorne & Yurtsever, doi:10.1103/PhysRevLett.61.1446: Spherically symmetric, (possibly) traversible wormholes

with $l \in (-\infty, \infty)$:

$$ds^2 = -e^{2\phi(l)}\,dt^2 + dl^2 + r^2(l)\big(d\theta^2 + \sin^2\theta\,d\varphi^2\big),$$

with 2 patches, glued at throat:

$$= -e^{2\phi_\pm(r)}\,dt^2 + \frac{dr^2}{1 - b_\pm(r)/r} + r^2\big(d\theta^2 + \sin^2\theta\,d\varphi^2\big),$$



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- *Modified* theories of gravity can easily accommodate various wormholes
- Again visualized for Interstellar

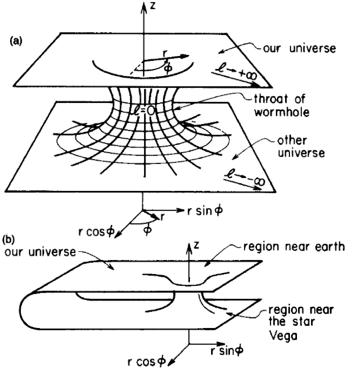


Image source: doi:10.1119/1.15620

Classic Exotica II: Wormholes & Warp Drives

- Warp drives: Alcubierre (1994) (arXiv:gr-qc/0009013) and Natário (2002) (gr-qc/0110086)

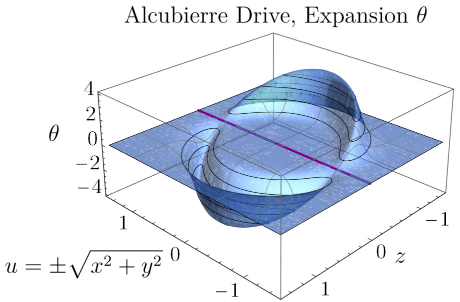
$$ds^2 = -\,dt^2 + \delta_{ij}\,\left(dx^i - v^i(x,y,z,t)\,dt\right)\,\left(dx^j - v\right.$$

Here:

  - $v_x = v_y = 0, v_z = \frac{dz_s(t)}{dt} f(r_s(t))$
  - $r_s = \sqrt{x^2 + y^2 + (z - z_s(t))^2}$
  - $f(r_s) = \dfrac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2\tanh(\sigma R)}$

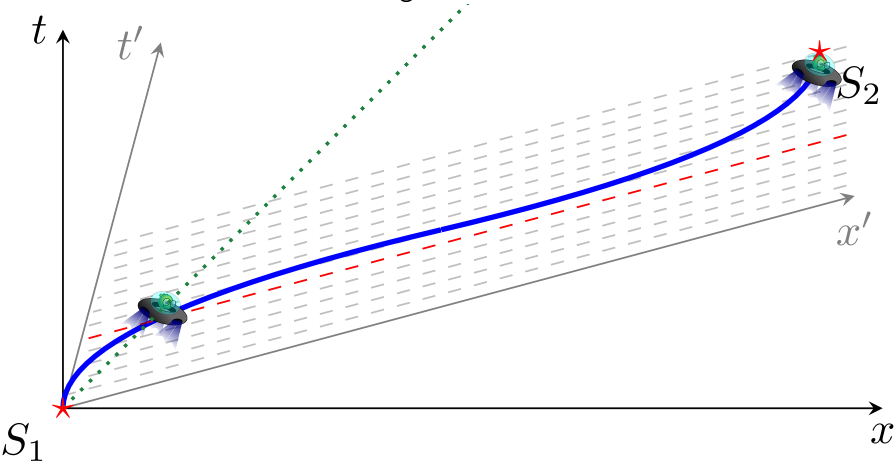
- Pull & push an empty, flat bubble through empty, flat space

Expansion of time-like curves:



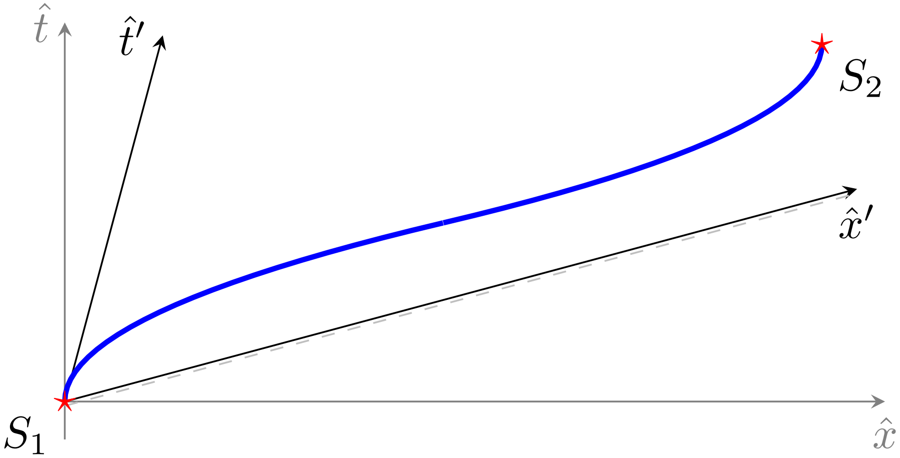
where $\sigma = 8$ and $\frac{dz_s(t)}{dt} = R = 1$

In the global rest frame:



Visualizing a Warp Drive

In a boosted frame:



Warp Drives for Time Travel[15]

- Take two bubbles, sufficiently separated, travelling in opposite direction b/w S_1 & S_2
- The bubble starting at S_2 is in a frame boosted towards S_1

[15] Following Everett, doi:10.1103/PhysRevD.53.7365

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- Travel back to S_1
- Boost to rest frame

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- The bubble starting at S_2 is in a frame boosted towards S_1
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- Boost into the other bubbles frame
- Travel back to S_1
- Boost to rest frame
- Arrive before you left

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Recent: Tractor Beams

There is more one can do.[16]

- Slightly modify the metric to:[17]

$$
\begin{aligned}
v_x(t,x,y,z) &= k(t,z)\, x\, h(x^2+y^2),\\
v_y(t,x,y,z) &= k(t,z)\, y\, h(x^2+y^2),\\
v_z(t,x,y,z) &= v(t,z)\, f(x^2+y^2).
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[16] arXiv:2106.05002

[17] Warning! This does not include the original Alcubierre metric!

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- **Warning!** This still violates the energy conditions, despite being relatively benign! (More below)

---

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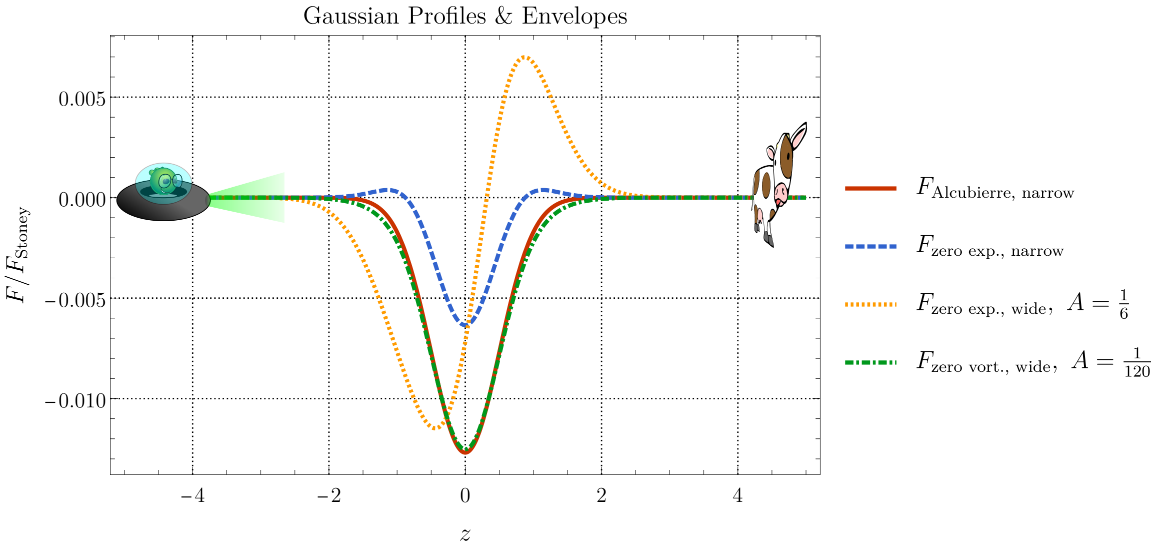
Tractor Beams: A Visualization



Exotica:
Energy Conditions

Classifying Weirdness: Energy Conditions

As the time travel suggests, there' issues.

- Time travel also easily doable with wormholes
- $\implies$ Issues with causality, stability, . . .

Classifying Weirdness: Energy Conditions

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Interpretation	WEC	SEC	NEC
'geometric'[a]	$\forall$ timelike V: $G_{ab}V^aV^b \geq 0$	$\forall$ timelike V: $R_{ab}V^aV^b \geq 0$	$\forall$ null k: $R_{ab}k^ak^b \geq 0$
physical	$\forall$ timelike V: $T_{ab}V^aV^b \geq 0$	$\forall$ timelike V: $(T_{ab} - \frac{1}{2}g_{ab})V^aV^b \geq 0$	$\forall$ null k: $T_{ab}k^ak^b \geq 0$
effective	$\rho \geq 0$ & $\forall \hat{a}$: $\rho + p_{\hat{a}} \geq 0$	$\rho + \sum_{\hat{a}} p_{\hat{a}} \geq 0$ & $\forall \hat{a}$: $\rho + p_{\hat{a}} \geq 0$	$\forall \hat{a}$: $\rho + p_{\hat{a}} \geq 0$
Interpretation	**DEC**	**†TEC†**	
'geometric'	$\forall$ timelike V, W: $G_{ab}V^aW^b \geq 0$	$\mathrm{tr}(G) \geq 0$	
physical	$\forall$ timelike V, W: $T_{ab}V^aW^b \geq 0$	$\mathrm{tr}(T) \geq 0$	
effective	$\rho \geq 0$ & $\forall \hat{a}$: $\rho \geq \lvert p_{\hat{a}} \rvert$	$\rho - \sum_{\hat{a}} p_{\hat{a}} \geq 0$	

[a] A.k.a. 'convergence conditions' (CC)

$$\text{DEC} \implies \text{WEC} \implies \text{NEC} \impliedby \text{SEC}$$

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As the name suggests—the NEC is the weakest.

Uses & Caveats

- Positive mass theorems
- Singularity theorems (cosmological and black holes)
- Cosmic no-hair theorem ($\Lambda > 0$ approaches de Sitter)
- *'Ruling out' exotic space-times*

There is an increasing list of physically viable violations of various kinds:

TEC
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SEC	• $\Lambda > 0$
	• Massive, minimally-coupled, non-tachyonic scalar fields (*e.g.*, inflatons)
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DEC	• [. . .]

Conclusion

Connecting the Dots

- Violation of ECs at best a warning sign
- Quantum gravity may play merry hell with our expectations

[18] Analogues are, again, part of this effort.

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- (Semi-)Classical effects in exotica *are* illuminating (arXiv:0904.0141, arXiv:1202.5708)
- Analogues may give clear insights into black holes—by necessity analogue black holes are regular!

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- (Semi-)Classical effects in exotica *are* illuminating (arXiv:0904.0141, arXiv:1202.5708)
- Analogues may give clear insights into black holes—by necessity analogue black holes are regular!
- There is an exciting sub-community, relativistic quantum information, trying to make relativity meet quantum foundations[18]

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My Own Trajectory

- More fun with warp drives

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- Detector models in astrophysical and analogue CSTQFT

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- Detector models in astrophysical and analogue CSTQFT
- Differential geometry in analogues
- Classical and semi-classical effects in pathological/exotic space-times

Thank you!

Kia Kaha!

For slides, see: `https://utf.mff.cuni.cz/~sschuster/`

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Black Hole Lifetime

- Stephan's Law: $\frac{\mathrm{d}E}{\mathrm{d}t} \simeq -\sigma A_{\mathrm{H}} T_{\mathrm{Hawking}}^4$, where Stefan–Boltzmann constant $\sigma = \frac{\pi^2 k_{\mathrm{B}}^4}{60 c^2 \hbar^3}$

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- Solve:

$$\tau_{\mathrm{lifetime}} \sim \frac{G^2}{\hbar c^4} M^3$$

Detectors

✓Detector:

$$\mathcal{L}_{\text{int}} = \left[\mu(\tau)\hat{\varphi}(X(\tau)) + \mu^{\dagger}(\tau)\hat{\varphi}^{\dagger}(\tau)\right] e^{-s|\tau|}$$

✗Not a detector:

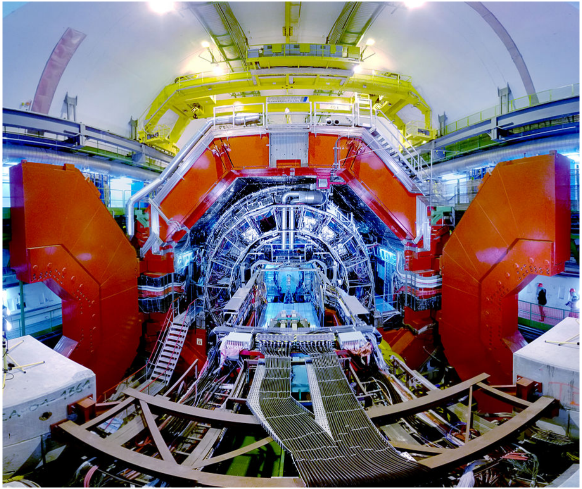

Image source: https://de.wikipedia.org/wiki/Datei:CERN_ALICE_Experiment.jpg

Talking about T_{ab} without T_{ab}

- In a given orthonormal frame, the components have an interpretation:

$$(T_{\hat{a}\hat{b}})_{\hat{a},\hat{b}} = \begin{pmatrix} \rho & \mathbf{S}^t \\ \mathbf{S} & \begin{pmatrix} p_{\hat{1}} & T_{\hat{1}\hat{2}} & T_{\hat{1}\hat{3}} \\ T_{\hat{1}\hat{2}} & p_{\hat{2}} & T_{\hat{2}\hat{3}} \\ T_{\hat{1}\hat{3}} & T_{\hat{2}\hat{3}} & p_{\hat{3}} \end{pmatrix} \end{pmatrix}$$

where ρ energy density, $\mathbf{S}$ energy flux, $p_{\hat{i}}$ pressures, $T_{\hat{i}\hat{j}}$ shear[19]

- In many contexts, one has relations between these components; 'equations of state'

[19] Assuming GR; hence $T_{ab} = T_{ba}$.

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- In many contexts, one has relations between these components; 'equations of state'—but GR does not have a lot

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A Word of Warning!

- There is some reliance on the Hawking–Ellis classification of stress-energy tensors[20]
- This is based on eigenvectors of $T^{\hat{a}}{}_{\hat{b}}$
- $T^{\hat{a}}{}_{\hat{b}}$ is *not* necessarily symmetric, even in GR!
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- Much.

[20] arXiv:1802.00865

Why the ∀ Is Important

Focus for a second on $\rho > 0$—this is *not* a full EC!

- For $\rho > 0$ and $\Gamma > 1$, fix a T to be

$$T_{\hat{a}\hat{b}} = \rho_0 \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & -\Gamma^2 & 0 & 0 \\ 0 & 0 & -\Gamma^2 & 0 \\ 0 & 0 & 0 & -\Gamma^2 \end{array}\right]_{\hat{a}\hat{b}}$$

- In the rest frame $V^{\hat{a}} = (1; 0, 0, 0)^{\hat{a}}$ we have:

$$\rho = T_{\hat{a}\hat{b}} V^{\hat{a}} V^{\hat{b}} = \rho_0 > 0$$

- Now pick observer in this frame with $\tilde{V}^{\hat{a}} = \gamma(1; v\ n^i)^{\hat{a}}$, where n^i any 3-direction
- Then:

$$\rho = T_{\hat{a}\hat{b}}\ \tilde{V}^{\hat{a}} \tilde{V}^{\hat{b}} = \rho_0 \gamma^2 (1 - \Gamma^2 v^2).$$

- If $|v| > 1/\Gamma$, the energy density for this observer will be negative

Extensions, Part I: Averaged Energy Conditions

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Still, especially (plausible) quantum matter can violate them.

Especially ANEC and AANEC found use, *e.g.*, in the topological censorship theorem, see arXiv:gr-qc/9305017

Extensions, Part II: Quantum Energy Inequalities[21]

- Instead of trying to guess the conditions, start from first principles.

[21] See arXiv:1208.5399

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- Choose a quantum field, compare possible (Hadamard) states with a reference state (*e.g.*, normal-ordered, . . .)

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- Choose a quantum field, compare possible (Hadamard) states with a reference state (*e.g.*, normal-ordered, ...)
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- Some averaged energy conditions can be regained sometimes
- Finally a definitive application of algebraic QFT

[21] See arXiv:1208.5399

Violation of the NEC in the Generic Case, Part I

- We have: NEC $\implies \rho + \bar{p} > 0$
- After some calculation in ADM decomposition:

$$\rho + \bar{p} = \frac{1}{24\pi}\left(-2\mathcal{L}_n K + K^2 - 3\mathrm{tr}(K^2)\right)$$

- Then, using $\mathrm{tr}(K^2) = \mathrm{tr}([K^{\mathrm{tf}}]^2) + \frac{1}{3}K^2$,

$$2\mathcal{L}_n K - K^2 + 3\mathrm{tr}(K^2) \leq 0$$

- With $\mathcal{L}_n K = \mathrm{d}K/\,\mathrm{d}\tau$, rearrange this several times

Violation of the NEC in the Generic Case, Part II

- Now have:

$$\text{NEC} \qquad \Longrightarrow \qquad 7\mathcal{L}_n K + 6\text{tr}K^2 \leq 0$$

- Hence:

$$\text{NEC} \qquad \Longrightarrow \qquad 7\frac{\mathrm{d}K}{\mathrm{d}\tau} + 2K^2 \leq 0$$

- Integrate back and forth in time

- Get finite-time singularities if NEC is fulfilled

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- NEC has to be violated

Warp Variations—And Recent Publicity

- Natário, a.k.a., zero expansion: Demand

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

- Zero vorticity (arXiv:2006.07125):

$$\boldsymbol{\nabla} \times \mathbf{v} = 0 \qquad \Longrightarrow \qquad \mathbf{v} = \boldsymbol{\nabla} \cdot \Phi$$

- **Warning!**
  - arXiv:2006.07125 does not provide an explicit example that can be checked; but zero-vorticity warp drives in general violate the NEC
  - arXiv:2104.06488 only uses metrics not fulfilling junction conditions
  - arXiv:2102.06824 only provides static, spherically symmetric metrics, no warp drives
  - arXiv:2102.05119, arXiv:2101.11467, arXiv:2008.06560 require conflicting assumptions, giving empty space, or use wrong index-placement
  - All six (and others before them) claim fulfilment of the energy conditions by finding one(!) observer, usually the Eulerian, to fulfil the necessary inequalities.
  - The '$\forall$' in the EC is not, and cannot be shown.

Engineering Metrics Violating Energy Conditions: A List

- Wormholes[22]
- Warp Drives[23] (recent publicity)
- New!New!New! Tractor Beams New!New!New!
- Krasnikov Hypertubes[24]
- [. . .]

[22] doi:10.1119/1.15620
[23] arXiv:gr-qc/0009013, arXiv:gr-qc/0110086
[24] arXiv:gr-qc/9511068

Stress-Energy Components and the ADM Split, Part I

- Energy density: $\rho = \dfrac{G_{nn}}{8\pi} = \dfrac{1}{16\pi}\Big(K^2 - \mathrm{tr}(K^2)\Big)$

- $\iff \qquad \rho = \dfrac{1}{16\pi}\Big\{\partial_i(v_i\, v_{j,j} - v_j\, v_{i,j}) - v_{[i,j]} v_{[i,j]}\Big\}$

- In terms of $\omega_i = \epsilon_{ijk}\, v_{[j,k]}$:

$$\iff \qquad \rho = \frac{1}{16\pi}\left\{\boldsymbol{\nabla} \cdot \{\vec{v}\, K - (\vec{v} \cdot \boldsymbol{\nabla})\vec{v}\} - \frac{1}{2}\,(\vec{\omega} \cdot \vec{\omega})\right\}$$

- Flux: $f_i = \dfrac{1}{16\pi}(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \vec{v}))_i$

Stress-Energy Components and the ADM Split, Part II

- Lastly,

$$T_{ij} = \frac{G_{ij}}{8\pi} = \frac{1}{8\pi}\left(\mathcal{L}_n K_{ij} + K K_{ij} - 2(K^2)_{ij} - \left(\mathcal{L}_n K + \frac{1}{2}K^2 + \frac{1}{2}\mathrm{tr}(K^2)\right)\delta_{ij}\right)$$

Stress-Energy Components and the ADM Split, Part II

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- This is why knowledge of the Hawking–Ellis type helps

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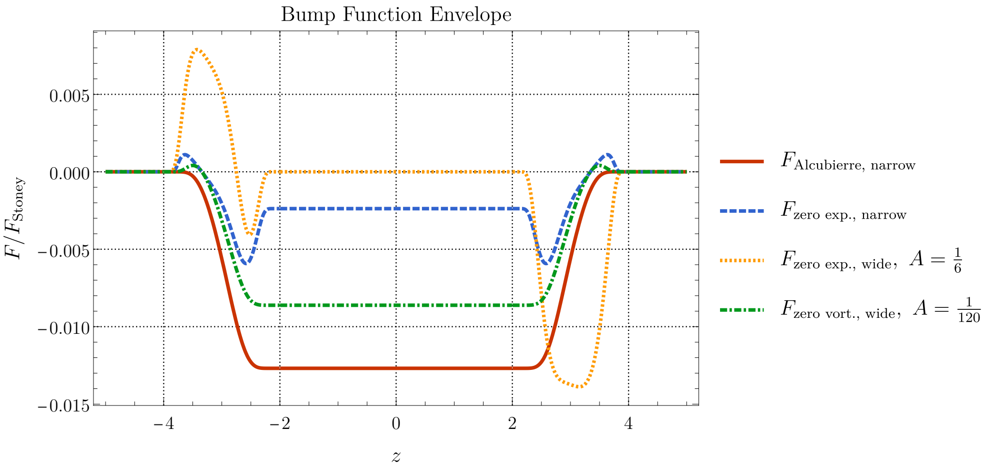
- This is why knowledge of the Hawking–Ellis type helps
- Define $\bar{p} = \frac{1}{3}\, T_{ij}\, \delta^{ij} = \frac{1}{24\pi}\left(-2\mathcal{L}_n K - \frac{1}{2}K^2 - \frac{3}{2}\mathrm{tr}(K^2)\right)$
- Then

$$\rho + \bar{p} = \frac{1}{24\pi}\left(-2\mathcal{L}_n K + K^2 - 3\mathrm{tr}(K^2)\right)$$
$$\rho + 3\bar{p} = -\frac{1}{4\pi}\left(\mathcal{L}_n K + \mathrm{tr}(K^2)\right)$$

Different Profiles, Quite Different Forces



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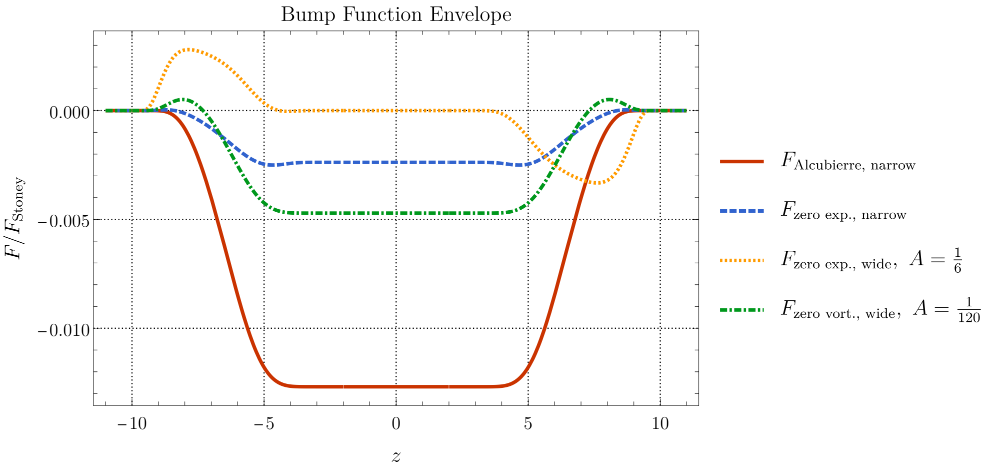
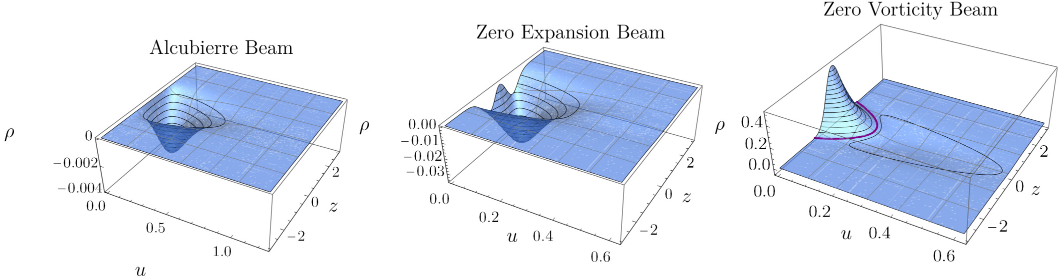
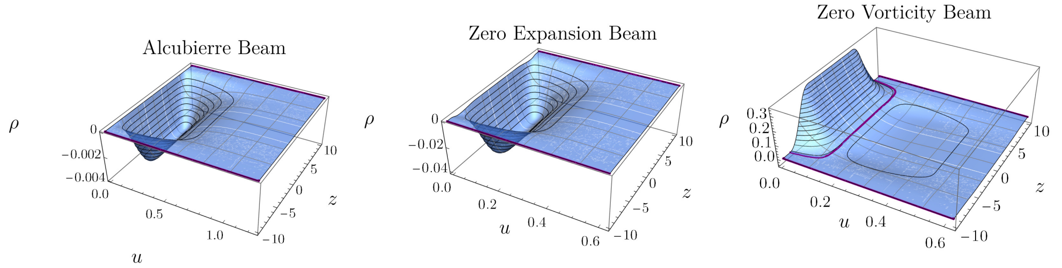


Energy Densities

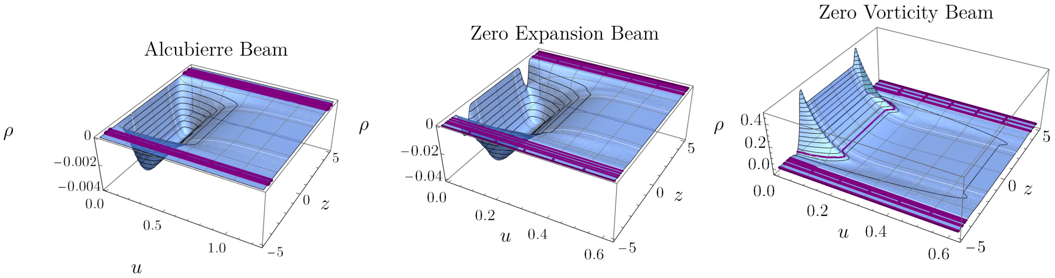
Gaussians



Bump Functions 1

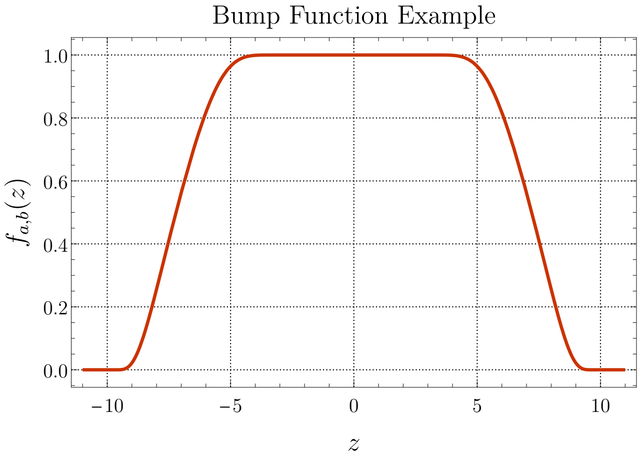


Bump Functions 2



Bump Functions

- $f_1(z) = \begin{cases} e^{-1/z} & z > 0 \\ 0 & \text{else.} \end{cases}$
- $f_2(z) = \dfrac{f_1(z)}{f_1(z) + f_1(1 - z)}$
- $f_{a,b}(z) = 1 - f_2\left(\dfrac{z^2 - a^2}{b^2 - a^2}\right)$
- This function is 0 for $z \in (-\infty, -b) \cup (b, \infty)$, is 1 in the interval $(-a, a)$, smoothly grows from 0 to 1 on $[-b, -a]$ and decays smoothly from 1 to 0 on $[a, b]$



Here: $a = 2$ and $b = 10$

Form of Profile Functions

- For Gaussian plots:
  - The functions:

$$f_{\text{Gauss, plot}} = h_{\text{Gauss, plot}} = e^{-u/A^2},$$
$$v_{\text{Gauss, plot}} = k_{\text{Gauss, plot}} = \Phi_{\text{Gauss, plot}} = e^{-z^2/B^2} e^{-t^2/C^2}$$

  - Here, $A = 0.5$, $B = C = 1.0$, and we evaluated the energy density and forces at $t = 1$
- For the bump functions instead:
  - $f_{a,b}(z)\, e^{-t^2/D^2}$ for v, k, or Φ, respectively.
  - First plots: $t = -1$, $a = 2$, $b = 10$, and $D = 1$
  - Second plots: $t = -1$, $a = 2$, $b = 4$, and $D = 1$

Facts from the ADM Decomposition

- Gauss–Codazzi equations:

$$R_{\hat{i}\hat{j}\hat{k}\hat{l}} = {}^{(3)}R_{\hat{i}\hat{j}\hat{k}\hat{l}} + K_{\hat{i}\hat{k}}K_{\hat{j}\hat{l}} - K_{\hat{i}\hat{l}}K_{\hat{j}\hat{k}}$$

$$R_{ijkl} \overset{\text{warp drive}}{=} K_{ik}K_{jl} - K_{il}K_{jk}$$

- Gauss–Mainardi equations:

$$R_{nijk} = R_{aijk}\,n^a = K_{ij,k} - K_{ik,j} = v_{(i,j),k} - v_{(i,k),j} = v_{[j,k],i}$$

- Furthermore:

$$R_{ninj} = R_{aibj}\,n^a\,n^b = -\mathcal{L}_n K_{ij} + (K^2)_{ij}$$

ADM-Decomposed Ricci Tensor

- For the Ricci tensor we find
$$R_{nn} = -\mathcal{L}_n K - \mathrm{tr}(K^2)$$
- Furthermore,
$$R_{ni} = K_{ij,j} - K_{,i}$$
- Finally,
$$R_{ij} = \mathcal{L}_n K_{ij} + K K_{ij} - 2(K^2)_{ij}.$$
- For the Ricci scalar
$$R = 2\mathcal{L}_n K + K^2 + \mathrm{tr}(K^2).$$

where $\mathrm{tr}(K^2) = (K^2)_{ij}\,\delta^{ij} = K_{ij}\,\delta^{ik}\,\delta^{jl}\,K_{kl}$

ADM-Decomposed Einstein Tensor

- $G_{nn} = \frac{1}{2}\Big(K^2 - \mathrm{tr}(K^2)\Big)$
- $G_{ni} = K_{ij,j} - K_{,i}$
- $G_{ij} = \mathcal{L}_n K_{ij} + K K_{ij} - 2(K^2)_{ij} - \left(\mathcal{L}_n K + \frac{1}{2}K^2 + \frac{1}{2}\mathrm{tr}(K^2)\right)\delta_{ij}$
- See: M. Alcubierre. *Introduction to 3+1 Numerical Relativity. International Series of Monographs on Physics* **140.** ISBN: 978-0-19-920567-7 (Oxford University Press, June 2008); É. Gourgoulhon. *3+1 Formalism and Bases of Numerical Relativity. Lect. Notes Phys.* **846.** ISBN: 978-3642245244. arXiv: gr-qc/0703035. http://arxiv.org/abs/gr-qc/0703035 (Springer-Verlag, Berlin Heidelberg, 2012)