We Don’t Have Time [Operators] for That! or— How to Perform Untaught Invocations to Summon Time into a Quantum Realm of Our Phyſical Exiſtence

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Limbus Poſtdoctōrum
Infernum Acadēmīæ

Ventūrus mox ad ūniverſitātēs prope vōs

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UNIVERZITA KARLOVA
Matematicko-fyzikální
fakulta

Motivation

QUID EST ERGO TEMPUS? SI NEMO EX ME QUAERAT, SCIO; SI QUAERENTI EXPLICARE VELIM, NESCIO:

WHAT IS TIME THEN? IF NOBODY ASKS ME, I KNOW: BUT IF I WERE DESIROUS TO EXPLAIN IT TO ONE THAT SHOULD ASK ME, PLAINLY I KNOW NOT.

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> *"What an entirely haunted time to be alive."*
>
> — Tamsyn Muir, *Nona the Ninth*

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Outline

Exordium: Prerequisites and Conventions

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- Physics is magic without magic.[1] The time is ripe for an appropriate preſentation.
- I consider humour $= \int \mathrm{d}\text{learning}$

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Technicalities

Required Background

Main part:

- Quantum mechanics
- Special relativity
- Fourier transformations

Only nice-to-have:

- Having heard of measure theory
- A bit of complex analysis

For the last *exciting* bit:

- Having heard of the 3+1 decomposition of GR

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- The Party Line: The 'Copenhagen interpretation'

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  - **Poſtulate II:** Measurements of state $|\Psi\rangle$ with an operator $\hat{A}$ yield an eigenvalue a of eigenstate $|a\rangle$ of $\hat{A}$ according to the Born rule with a probability

    $$\mathrm{d}P(a) = |\langle\Psi|a\rangle|^2\,\mathrm{d}a.$$

    After the measurement, the system is now in state $|a\rangle$.
  - **Poſtulate III:** A (closed) system evolves unitarily according to the Schrödinger equation:

    $$|\Psi(t)\rangle = U(t;t_0)\,|\Psi(t_0)\rangle\,.$$

    Extended to mixed states:

    $$\hat{\rho}(t) = U(t;t_0)\hat{\rho}(t_0)U^\dagger(t;t_0).$$

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- This either is a fully contradictory statement about Ψ's time evolution, or one needs a 'Heisenberg cut' clearly separating 'classical' measurements from 'quantum' evolution.

Repetitio: Of Times Classical

Detour: Some Results about Fourier Transformations

- **Theorem (Plancherel):** On the set of Schwartz functions[3] $\mathcal{S}$, the Fourier transform $\mathcal{F}$ is an isometry.

[3] $\mathcal{S}(\mathbb{R}^n) := \left\{ \phi \in C^\infty(\mathbb{R}^n) \,\middle|\, \forall \alpha, \beta \in \mathbb{N}_0^n : \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta \phi(x) \right| < \infty \right\}$

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- More importantly: There's physics here.

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Uncertainty Principles of Claſſical Phyſics[4]

- Roughly speaking:
  - Canonically conjugate variables $\Longrightarrow$ Fourier transform
  - Uncertainty + boundedness/asymtpotic conditions $\Longrightarrow$ Uncertainty 'principles'

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- Homework/Quiz for the musicians.

[4]See [FS97] and [Ben90]; Image source: Jared Smith (Archspire) https://www.youtube.com/watch?v=isznXyN104Q

Liber Non Ex Tempore—Time(?) in Quantum Mechanics

Redux: Time in Quantum Mechanics—Part I, Uncertainty Principles

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- There are subtleties regarding boundary conditions...

Redux: Time in Quantum Mechanics—Part II, Time–Energy Uncertainty

- We **also** have (things like) [MT45]:[5]

$$\Delta\tau\Delta E \geq \frac{\pi^2\hbar}{2}$$

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Time and The Clash— ‘Should I stay or ſhould I go?’

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- Bigger Problem: Assuming SR and QM even work simultaneously. Let's make it worse.

The Theorem—À la Schrödinger[6]

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- Let's combine this with (2), though with finite integral boundaries:

$$\int_{\omega_0}^{\omega_1} \left[e^{-i\omega t} \hat{T} c(x,\omega) - t e^{-i\omega t} c(x,\omega) \right] \mathrm{d}\omega = 0$$

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The Theorem—À la Schrödinger[7] (continued)

- One partial integration later:

$$\int_{\omega_0}^{\omega_1} \left[e^{-i\omega t} \hat{T} c(x,\omega) + i \left(\frac{\partial}{\partial \omega} e^{-i\omega t} \right) c(x,\omega) \right] \mathrm{d}\omega = i \left[e^{-i\omega_1 t} c(x,\omega_1) - e^{-i\omega_0 t} c(x,\omega_0) \right] \quad (3)$$

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- $\Longrightarrow$ Ψ's energy is either unbounded or $\Psi = 0$.

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$$\hat{T}c(x,\omega) - i\frac{\partial}{\partial\omega}c(x,\omega) = 0 \quad \Longleftrightarrow \quad \hat{T}^\dagger c^*(x,\omega) + i\frac{\partial}{\partial\omega}c^*(x,\omega) = 0$$

- Whence

$$\frac{\partial}{\partial\omega}\left|c(x,\omega)\right|^2 = i\left(c\hat{T}^\dagger c^* - c^*\hat{T}c\right) \overset{\text{Hermiticity}}{=} 0$$

- Drum roll, please!
- $\Longrightarrow$ Ψ's energy is either unbounded or $\Psi = 0$.
- We cannot have such a time operator $\hat{T}$.

[7] [Sch31]

The Theorem—À la Schrödinger[7] (continued)

- One partial integration later:

$$\int_{\omega_0}^{\omega_1} \left[e^{-i\omega t} \hat{T} c(x,\omega) + i \left(\frac{\partial}{\partial \omega} e^{-i\omega t} \right) c(x,\omega) \right] \mathrm{d}\omega = i \left[e^{-i\omega_1 t} c(x,\omega_1) - e^{-i\omega_0 t} c(x,\omega_0) \right] \quad (3)$$

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The Theorem—À la Unruh & Wald: The Statement

- Require $\exists$ a sequence $\{|t_i\rangle\}$ s.t.
  1. Each $|t_n\rangle$ is an eigenstate s.t. $t_0 < t_1 < \ldots$
  2. $\forall n : \exists t, m > n : |\langle t_m|U(t, t_m)|t_n\rangle| > 0$—Time *has* to progress
  3. $\forall n, t, m < n : |\langle t_m|U(t, t_m)|t_n\rangle| = 0$—Time does not run backwards

A No-Go Theorem

For $\hat{H}$ bounded from below, there are no $\hat{T}$ satisfying (1)—(3).

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- What they want: A time operator $\hat{T}$ that for (at least) some initial state $|t_0\rangle$ evolves monotonically to the future[8]

[8] 'Vorwärts immer, rückwärts nimmer!'

The Theorem—À la Unruh & Wald: The Proof

Proof:

- Pick $m > n$, define for $t \in \mathbb{C}$ (!)

$$f(t) := \langle t_n | \exp\left(-i\hat{H}t\right) | t_m \rangle$$

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Image source: https://tatertotsandjello.com/halloween-haunted-house-decor-ideas-free-printables-200-shutterfly-giveaway/

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Liber Non Ex Tempore—Time(?) in Quantum Mechanics:

The Forbidden Fruits—POVMs

Beauty Needs Imperfections: Rethinking Meaſurement

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- We still need to get a sense of probability and wave functions/density matrices
- The key insight: Think more probabilistic about the Born rule

The A and Ω of Positive Operator-Valued Measures

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- A POVM A is a map of outcomes to operators such that:
  (1) 'Positive:' $\forall \varphi \in \mathcal{H}, : \langle \varphi | A(X) | \varphi \rangle \geq 0 :\Leftrightarrow \forall X \in M \; : \; A(X) \geq 0.$

  (2) $A(\Omega) = \mathbb{1}_{\mathcal{H}}$

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- Let's make this a bit more familiar...

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- Meanwhile, POVMs are Hermitian operators O if and only if

$$O(X)^2 = O(X).$$

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[9] Not necessarily physically useful.

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- The eigenvectors of POVMs are overcomplete sets; property (2) overcounts a lot.
- If they are not overcomplete, they are PVMs (*i.e.*, Hermitian)
- Some things that now become possible
  - Phase operators
  - Coherent states[10]
  - Open quantum systems
  - Imprecise measurement (+coarse graining [Šaf+21])
  - Measurement problem in *Quantum Field Theory* [FV20]

[10] They have been around before—but POVM methods fit nicely.

Timely Confequences

- Earlier attempts for time operators—like ‘time of flight’

$$\hat{t}_{\text{t.o.f.}} := -\frac{m}{2}(\hat{p}^{-1}\hat{x} + \hat{x}\hat{p}^{-1})$$

  work.[BFH10]
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- Similarly, polar decomposition of ladder operator $\hat{a}$ with *non*-unitary $\hat{W}$:

$$\hat{a} = \hat{W}\widehat{|a|}, \qquad \text{with} \qquad \widehat{|a|} := \hat{n}^{1/2}$$

  having improper eigenstates $|\theta\rangle$

$$\hat{W}\,|\theta\rangle = e^{i\theta}\,|\theta\rangle\,, \qquad \text{with} \qquad |\theta\rangle = \sum_{n\geq 0} e^{in\theta}\,|n\rangle\,.$$

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- Get one of many possible time operators for $f(\theta) = \theta$ as 'first moment' of B_0:

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- Choosing two different $\hat{T}_{\theta^*}, \hat{T}_{\theta'}$ with $\theta^* \neq \theta' \mod 2\pi$:

$$[\hat{T}_{\theta^*}, \hat{T}_{\theta'}] \neq 0$$

Conclusio: Modern Times—From Clocks to Quantum Gravity

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Quantum Clocks and Gauge Theory

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Image source: https://en.wikipedia.org/wiki/File:Safetylast-1.jpg

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- Measure time evolution of an operator $\hat{A}$, stationary w.r.t. $\hat{H}_{\mathrm{C}}$, as

$$E(A|\tau) = \mathrm{tr}\left(\hat{A}\hat{P}_\tau\hat{\rho}\right) / \,\mathrm{tr}\left(\hat{P}_\tau\hat{\rho}\right),$$

where

$$\hat{P}_\tau = |\psi_C(\tau)\rangle\langle\psi_C(\tau)| \otimes \mathbb{1}_{\mathrm{R}}, \qquad \text{and} \qquad \hat{\rho} \in \mathcal{L}(\mathcal{H})$$

The Page–Wootters Formalifm: Pre-Cursors

A footnote in [Sch31] anticipates this:

"An interesting application of this is the following: if one knows of a system, composed of several, coupled subsystems, only the total energy, then it is impossible to know more about the distribution of energy across the subsystems than the statistical, time-independent data, which already follows from the knowledge of the total energy. Except for the case that individual subsystems are in truth fully decoupled, energetically isolated from the others."

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- Also, [DeW67] thinks along these lines. More on this later.

Definitely *Not* a Simple Solution: Gauge Theory[11]

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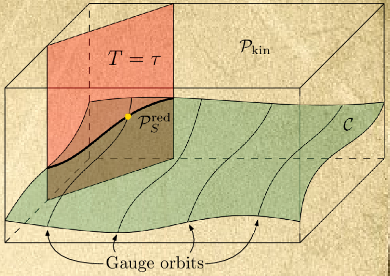

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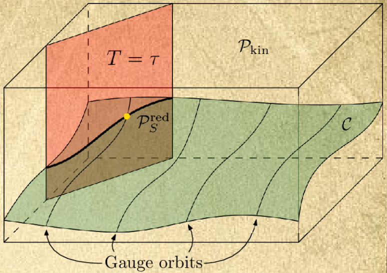
- 'Kinematical' Hilbert space has superfluous, 'gauge' info



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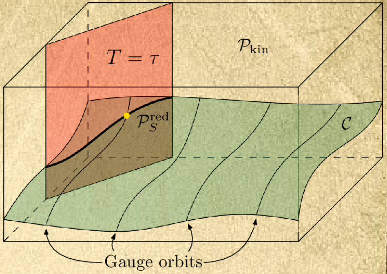

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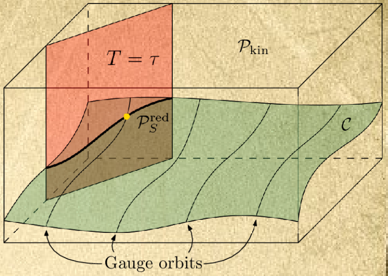

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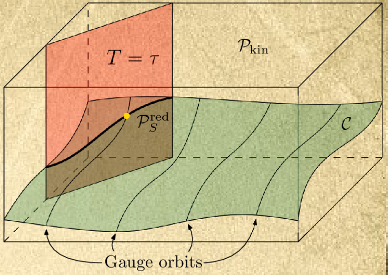

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Conclusio: Modern Times—From Clocks to Quantum Gravity:

Applications to the Physicist's Stone and More

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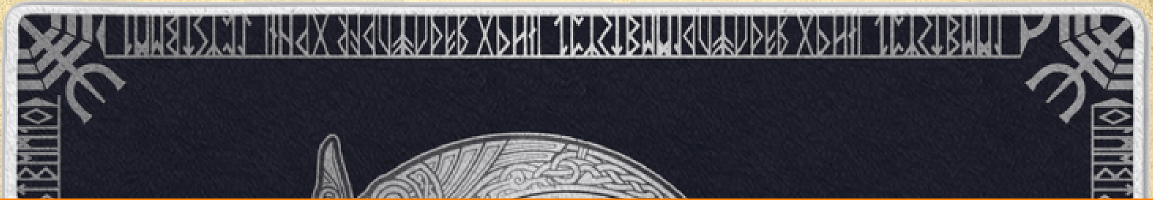

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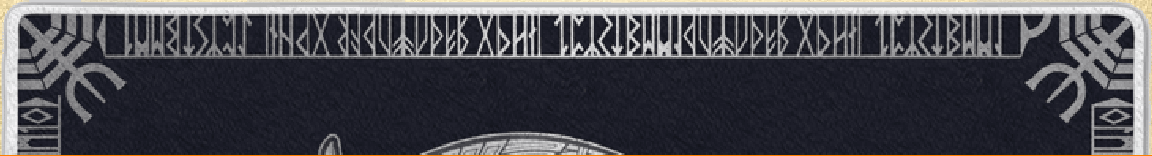

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  - Physics
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- Recent years have seen mesmerizing progress

þank you uerily·

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