The Beauty and the Beast: Covariance and Electromagnetic Media

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MINISTRY OF EDUCATION,
YOUTH AND SPORTS

UNIVERZITA KARLOVA
Matematicko-fyzikální
fakulta

The Menu

The Aperitif: The Goal

Goal

My aim will be threefold:

- Make electromagnetism in media beautiful[1]

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- Tell you about electromagnetic media in relativity.[2]

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[2] Hopefully self-contained and simple.

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My aim will be threefold:

- Make electromagnetism in media beautiful[1]
- Tell you about electromagnetic media in relativity.[2]
- Mention a nearly forgotten, neat effect from 1851.[3]

[1] For suitable choices of beauty.
[2] Hopefully self-contained and simple.
[3] That's 10 years before the Maxwell equations connected to this all. 54 before Einstein's *annus mirabilis*.

Entree: Warnings, Conventions, Introduction

Warning!

There once was a Sebastian in Ondřejov
Who's got biased opinions on matters of
  electrics historical,
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Conventions

- Signature: $-+++$
- $G = c = \hbar = 1$
- Sum convention: Same index once(!) up, once(!) down per term: Sum over it!
- Space-time indices: $abcd \cdots \in \{0, 1, 2, 3\}$
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The Beauty: Microscopic Electrodynamics?

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{\jmath} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The Beauty: Microscopic Electrodynamics with Forms?

$$*\,\mathrm{d}{*F} = \mu_0 J$$

$$\mathrm{d}F = 0$$

The Beauty: Covariant, Microscopic Electrodynamics!

$$-\nabla_a F^{ab} = \mu_0 J^b$$

$$\varepsilon^{abcd} \nabla_a F_{bc} = 0$$

The Beast: Macroscopic Electrodynamics

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Main Course: Macroscopic Electrodynamics Done Right

What's the Deal with F?

If we had done the derivation, we'd have seen:[4]

$$(F_{ab})_{a,b\in\{0,1,2,3\}} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

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- This is what *we* measure as $\vec{E}$ and $\vec{B}$
- Other observers will disagree. What do *they* see?

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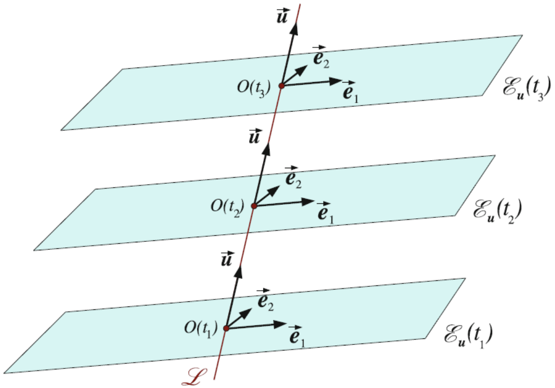
Observers and Four-Velocities

- We are all insignificant, infinitesimal point particles.



Source: Gourgoulhon—*Special Relativity in General Frames*, p.260

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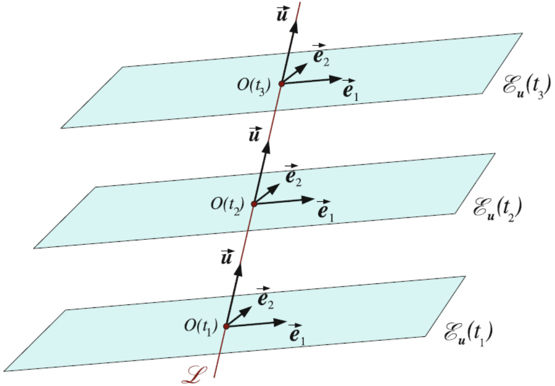
- Any observer follows their (time-like) curve in space-time



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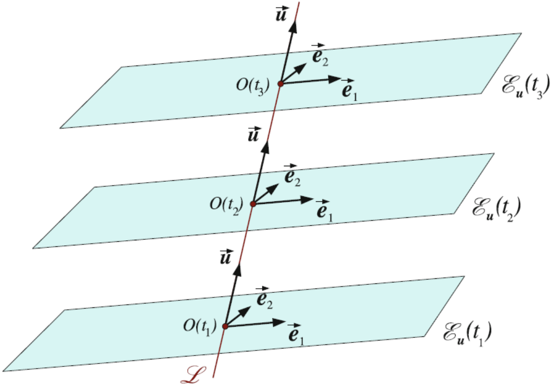
- This means, curve has tangent vector u^a s.t.

$$u^a u_a = u^a g_{ab} u^b = -1.$$



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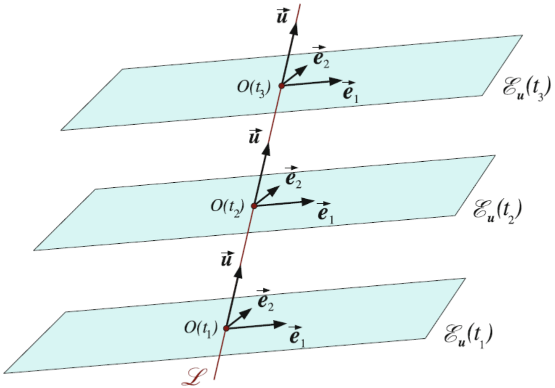

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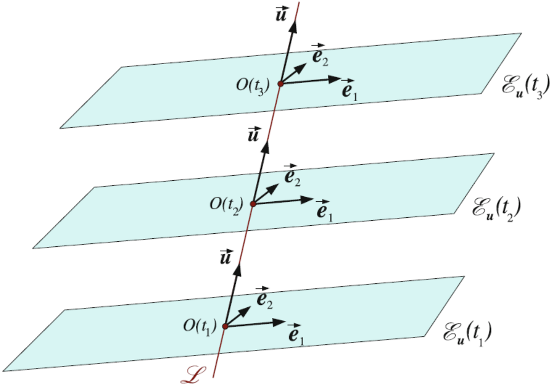
- At given time* t_α, there is (a) a time-like direction along the curve, (b) a spatial 3-space transverse to it.



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Main Course: Macroscopic Electrodynamics Done Right: A First $3+1$ Decomposition

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- What happens if we contract this with a second u?

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- Degrees of freedom:[6]
  - F: Antisymmetric 2-tensor $\rightarrow 6$
  - Two 4-vectors with one condition each $\rightarrow 3 + 3$

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[6] 3+1 dimensions are important here!

Main Course: Macroscopic Electrodynamics Done Right: The Macrocosmos

What Do We Need?

- Remember/Remind ourselves of the macroscopic Maxwell equations:

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- But what about $\vec{D}$ and $\vec{H}$?
- We need to link those to $\vec{E}$ and $\vec{B}$! $\Longrightarrow$ '**Constitutive relations**'

A Brief Word from Our Assumptions

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Constitutive Relations for Linear Electromagnetism

- The constitutive relations are:

$$\text{`fields'}\ (\vec{E}, \vec{B}) \quad \overset{\text{lin.}}{\mapsto} \quad \text{`excitations'}\ (\vec{D}, \vec{H})$$

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- Is this the most general, linear map $(\vec{E}, \vec{B}) \mapsto (\vec{D}, \vec{H})$?

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- Is this the most general, linear map $(\vec{E}, \vec{B}) \mapsto (\vec{D}, \vec{H})$?
- ζ is a new matrix, called 'magneto-electric tensor/matrix'.

Constitutive Relations for Linear Electromagnetism

- The constitutive relations are:

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- ζ is a new matrix, called 'magneto-electric tensor/matrix'.
- $\epsilon, \mu^{-1}, \zeta$ are all 3×3 matrices

Ingredients

- We look (at most) for a linear, invertible map $\mathbb{R}^6 \to \mathbb{R}^6$
- It is natural to assume a similar structure for $(\vec{D}, \vec{H})$, *i.e.*, collect them in a antisymmetric tensor G^{ab}
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- We will call this the '**excitation tensor**'
- So, we look for a map Z:

$$G^{ab} = Z^{abcd} F_{cd}.$$

Properties of the Constitutive Tensor

- Since both F and G are antisymmetric:

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- Hence, d.o.f. counting:

$$\#GL(6) = 36 \qquad \text{symmetric } \rightarrow 21$$

A Good Guess Is Good Enough

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- Note:

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- Finally: (Less trivial exercise ☺)

$$\varepsilon_{cd}{}^{af} \varepsilon^{ebcd} u_e u_f = -2(g^{ab} + u^a u^b) = -2h^{ab}$$
$$g^{b_1 c_1} \cdots g^{b_n c_n} \varepsilon_{c_1 \dots c_n} \varepsilon^{a_1 \dots a_n} = -n! g^{b_1 c_1} \cdots g^{b_n c_n} \delta^{a_1}{}_{[c_1} \cdots \delta^{a_n}{}_{c_n]}$$

The Constitutive *Tensor* and Its Orthonormal Decomposition[7]

- Define for any q^{ab} s.t. $u_a q^{ab} = q^{ab} u_b = 0$:

$$Q^{abcd}[q^{\circ\circ}] := u^a u^d q^{bc} + u^b u^c q^{ad} - u^b u^d q^{ac} - u^a u^c q^{bd}$$

- After *quite* some work (exercise) we finally get something along the lines of

$$Z^{abcd} = \frac{1}{2}\Big(Q^{abcd}_{q\to\epsilon_u} + (*Q^{abcd}_{q\to\mu_u^{-1}}*) + (*Q^{abcd}_{q\to\zeta_u}) + (Q^{abcd}_{q\to\zeta_u^T}*)\Big),$$

where $*M$, $M*$, $*M*$ are the left-, right-, and double-dual of M, respectively.

[7] **Warning!** The natural naming convention here is *different* from the Bel decompositon in GR or gravitomagnetism! And probably more defensible... ☹

Inversion

- The other way around: $\epsilon, \mu^{-1}, \zeta, \zeta^T$ in terms of Z^{abcd}
- This is another ... doable? exercise
- One gets:

$$\epsilon_v^{ab} = -2Z^{dacb} v_d v_c, \qquad \zeta_v^{ab} = 2(*Z)^{dacb} v_d v_c,$$

$$[\mu_v^{-1}]^{ab} = 2(*Z*)^{dacb} v_d v_c, \qquad \left[\zeta_v^{\dagger}\right]^{ab} = 2(Z*)^{dacb} v_d v_c.$$

Back to Square One: The Maxwell Equations

- So—what have we *done*? What horrors have we conjured? What evils unleashed? What atrocities committed? What sanity sacrificed?

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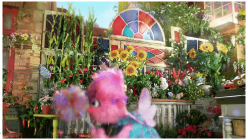

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- Note: This is actually shorter than the average macroscopic Maxwell's equations of experimental physics *and* more general!

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I promised something nice. This didn't always look nice. So, I'm gonna brainwash you into believing me!

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Dessert: The Fresnel–Fizeau Effect

Reminder on Boosts

Going from one inertial frame to a different inertial frame involves:

- Rotations in space
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$$x^a \quad \mapsto \quad x'^a = \Lambda^a{}_b x^b,$$

$$\omega_a \quad \mapsto \quad \omega'_a = \Lambda_a{}^b \omega_b = \left(\Lambda^{-1}\right)^b{}_a \, \omega_b,$$

where for a simple boost in x-direction (direction '1')

$$\Lambda^a{}_b = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \beta = v/c$$

Addenda on Boosts

- If inertial frames *not* collinear, Λ's quickly become a *mess*
- If two observer move inertially along u^a and v^a, respectively,

$$u^a v_a = \gamma$$

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  3. Decompose *this* expression w.r.t. v

Dessert: The Fresnel–Fizeau Effect: An Application!

An Application! An Application! The Relativist Has an Application!

- Say, a medium is homogeneous, isotropic, and has no magneto-electric effects in a given rest frame u.
- Then it is completely described for this observer by ϵ and μ^{-1}, *both scalars* now
- (Ok-ish exercise) We get:

$$Z^{abcd} = \frac{\mu^{-1}}{2}\left[(h^{ac} - \epsilon\mu u^a u^c)\left(h^{bd} - \epsilon\mu u^b u^d\right) - \left(h^{ad} - \epsilon\mu u^a u^d\right)\left(h^{bc} - \epsilon\mu u^b u^c\right)\right]$$

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- Trust me, this *will* prove useful! Swear!

Going through the Motions

- Let's perform the algorithm above
- This could be an exercise. I suggest [SV17; Sch18], however.[8]

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Going through the Motions

- Let's perform the algorithm above
- This could be an exercise. ☠ I suggest [SV17; Sch18], however.[8]
- We get with $h_v^{bd} := g^{bd} + v^b v^d$:

$$\epsilon_v^{bd} = \epsilon\, h_v^{bd} + (\epsilon - \mu^{-1})\Big[(u\cdot v)^2 h_v^{bd} - h_v^{be} h_{ef} h_v^{fd}\Big]$$

$$[\mu_v^{-1}]^{bd} = \frac{h_v^{bd}}{\mu} + (\mu^{-1} - \epsilon)\Big((u\cdot v)^2 h_v^{bd} - h_v^{be} h_{ef} h_v^{fd}\Big)$$

$$\zeta_v^{ac} = (\epsilon - \mu^{-1})(u\cdot v)\Big(\epsilon^{acef} v_e u_f\Big)$$

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What This Shows

- Isotropy gets broken
- Magneto-electric effects appear: $\vec{D} \neq \vec{D}(\vec{E})$, $\vec{D} = \vec{D}(\vec{E}, \vec{B})$
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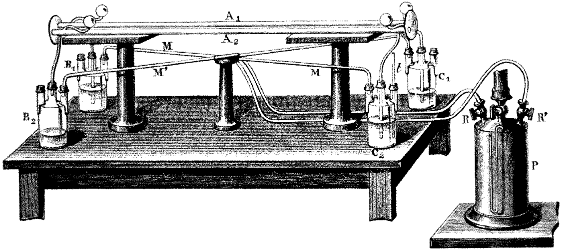
- And one more thing. . .

What This *IS*



- In 1851, Fizeau wanted to measure the dragging of light in a moving medium (water)
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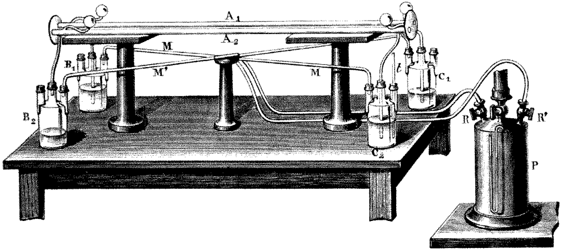
What This *IS*: The Fresnel–Fizeau Effect



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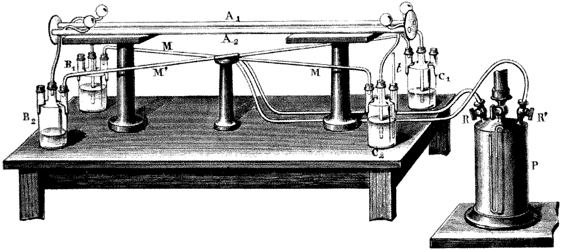

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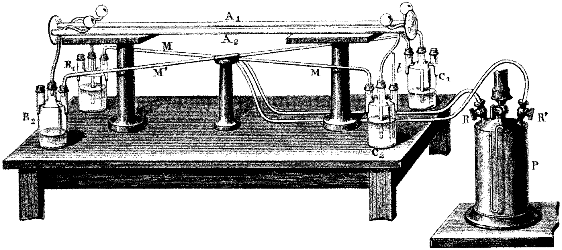

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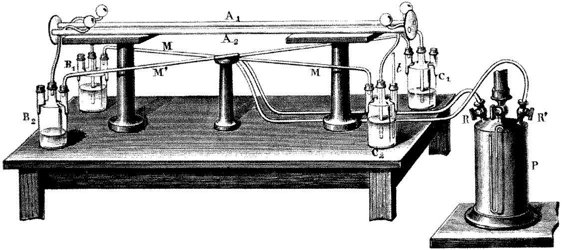

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- The above is exact and more general than just for inertial frames! YAY!

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Digestif: Bitters

Alas, the Bitters

- Conventions, conventions, conventions:[10]
  - The metric

[10] And not the fun ones involving cosplay.

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If we ſhadows have offended,
Think but this, and all is mended,
That you have but ſlumbered here
While theſe viſions did appear.

Thank you!

References I

Nothing above was new, but little is presented coherently in one place. Less so in a modern or legible form.

Cho09 Y. Choquet-Bruhat. *General Relativity and the Einstein Equations.* ISBN: 978-0-19-923072-3 (Oxford University Press, 2009).

FN14 L. Filipe O. Costa & J. Natário. Gravito-electromagnetic analogies. *General Relativity and Gravitation* **46,** 1792. doi:10.1007/s10714-014-1792-1. arviv: 1207.0465 (2014).

Gou13 É. Gourgoulhon. *Special Relativity in General Frames.* ISBN: 978-3-642-37275-9. doi:10.1007/978-3-642-37276-6 (Springer, 2013).

HO03 F. W. Hehl & Y. N. Obukhov. *Foundations of Classical Electrodynamics.* ISBN: 3-7643-4222-6 (Birkhäuser, 2003).

Jac75 J. D. Jackson. *Classical Electrodynamics.* 2nd ed. ISBN: 0-471-43132-X (John Wiley & Sons, 1975).

References II

LL84 L. D. Landau & E. M. Lifshitz. *Electrodynamics of Continuous Media.* 2nd ed. ISBN: 0-08-030276-9 (Pergamon Press, 1984).

MTW73 C. W. Misner, K. S. Thorne & J. A. Wheeler. *Gravitation.* ISBN: 0-7167-0334-3 (W. H. Freeman, 1973). Reprinted by Princeton University Press, ISBN: 978-0-691-17779-3.

Pos62 E. J. Post. *Formal Structure of Electromagnetics.* (North-Holland, 1962). Reprinted by Dover Publications, ISBN: 978-0-486-65427-0.

Sch18 S. Schuster. *Black Hole Evaporation: Sparsity in Analogue and General Relativistic Space-Times.* PhD thesis (Victoria University of Wellington, Dec. 2018). doi:10.26686/wgtn.17134433.v1. arXiv: 1901.05648.

SV17 S. Schuster & M. Visser. Effective metrics and a fully covariant description of constitutive tensors in electrodynamics. *Physical Review D* **96,** 124019. doi:10.1103/PhysRevD.96.124019. arXiv: 1706.06280 (Dec. 2017).

Fresnel Equation and Rubilar–Tamm Tensor

- The Tamm–Rubilar tensor (density)[11]

$$G^{abcd} \propto \frac{1}{4!} \varepsilon_{efgh} \varepsilon_{ijkl} Z^{efi(a} Z^{b|gj|c} Z^{d)hkl}$$

has interesting info!

- Contracted with four wave covectors k_a it gives a quartic Fresnel wave surface

- This describes all possible behaviour of linear electromagnetism: Birefringence, metrics, . . .

[11] Modulo missed factors of $\det g$, 2's, signs—the source was [HO03], which doesn't have the easiest notation.